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Oscillation criteria for a class of third order damped differential equations

OSAMA MOAAZ[a,*], ELMETWALLY M. ELABBASY[a], EBTESAM SHAABAN[b]

[a] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
[b] Department of Mathematics, Faculty of Education, AL Asmarya Islamic University, Zliten, Libya

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Abstract. The present study concerns the oscillation of a class of third-order nonlinear delay differential equations with middle term. We offer a new description of oscillation of the third-order equations in terms of oscillation of a related well studied second-order linear differential equation without damping. By using the integral averaging technique, we establish new oscillation results for this equation. Some examples are provided to illustrate the main results.

Keywords: Oscillation; Third-order; Functional delay; Differential equations

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1. INTRODUCTION

This paper is concerned with the oscillation and the asymptotic behavior of the third-order nonlinear functional differential equations with delayed argument

$$\left(r_2(\tau)\left(r_1(\tau)\left(y'(\tau)\right)^{\alpha}\right)'\right)' + \phi\left(\tau, y'(\delta(\tau))\right) + q(\tau) f(y(\sigma(\tau))) = 0, \tag{1.1}$$

* Corresponding author.
E-mail addresses: o_moaaz@mans.edu.eg (O. Moaaz), emelabbasy@mans.edu.eg (E.M. Elabbasy), ebtesamshaaban@asmarya.edu.ly (E. Shaaban).
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for $\tau \geq \tau_0$. Assume that the following conditions are satisfied:

(C_1) $r_1, r_2, q \in C\left(I, \mathbb{R}^+\right), \delta, \sigma \in C(I, \mathbb{R})$ such that $I = [\tau_0, \infty), \delta(\tau) \leq \tau, \sigma(\tau) \leq \tau$, $\sigma'(\tau) > 0$ and $\lim_{\tau\to\infty}\delta(\tau) = \lim_{\tau\to\infty}\sigma(\tau) = \infty$.

(C_2) There is a real function $p(t)$, $p(t) > 0$ such that $\phi(\tau, u) \geq k_1 p(\tau) u^\alpha$ and $\phi(\tau, -u) = -\phi(\tau, u)$.

(C_3) $f \in C(\mathbb{R}, \mathbb{R})$ such that $f(x)/x^\beta \geq k_2 > 0$, where α and β are quotients of odd positive integers.

A function y is called a solution of (1.1), if $y(\tau)$ satisfies (1.1) and if y, $r_1\left(y'\right)^\alpha$ and $r_2\left(r_1\left(y'\right)^\alpha\right)' \in C^1\left(\left[\tau_y, \infty\right), \mathbb{R}\right)$ for some $\tau_y \geq \tau_0$. We only consider those solutions of (1.1) which satisfy $\sup\{|y(\tau)| : \tau_1 \leq \tau < \infty\} > 0$ for any $\tau_1 \in I$ and exist on I. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory.

Determining oscillation and nonoscillation of functional differential equations received deal of a great interest in recent years, see the papers [1–24]. Special cases of Eq. (1.1) include the equation

$$\left(r_2(\tau)\left(r_1(\tau)y'(\tau)\right)'\right)' + p(\tau)y'(\tau) + q(\tau)f(y(g(\tau))) = 0. \tag{1.2}$$

The oscillatory behavior of solutions of (1.2) has been discussed in a number of studies, see for example the papers by Tiryaki et al. [23], Aktas et al. [3], Grace [17] and Padhi et al. [20].

In this paper, we study the oscillation and asymptotic behavior of solutions of Eq. (1.1). Our results improve and unify the results in Tiryaki et al. [23] and Elabbasy et al. [11], and to extend and generalize the earlier ones presented in Bohner et al. [7].

2. Some lemmas

In this section, we state and prove the following lemmas which we will use in the proof of our main results. For simplicity, we introduce the following notation:

$$E_0y = y,\ E_1y = r_1\left((E_0y)'\right)^\alpha,\ E_2y = r_2(E_1y)',\ E_3 = (E_2y)',$$

$$R_1(\tau, \tau_1) = \int_{\tau_1}^{\tau} \frac{ds}{(r_1(s))^{\frac{1}{\alpha}}},\ R_2(\tau, \tau_1) = \int_{\tau_1}^{\tau} \frac{ds}{r_2(s)},$$

and

$$R_{12}(\tau, \tau_1) = \int_{\tau_1}^{\tau} \left(\frac{R_2(s, \tau_1)}{r_1(s)}\right)^{\frac{1}{\alpha}} ds,$$

for $\tau_0 \leq \tau_1 \leq \tau < \infty$. We suppose that

$$R_1(\tau, \tau_0) \to \infty \text{ as } \tau \to \infty$$

and

$$R_2(\tau, \tau_0) \to \infty \text{ as } \tau \to \infty.$$

Lemma 2.1. *[Grace [17]] Assume that $x(\tau)$ is a bounded solution of equation*

$$\left(r_2(\tau)x'(\tau)\right)' = G(\tau)x(h(\tau)). \tag{2.1}$$

Suppose that

$$\lim_{\tau\to\infty}\sup\int_{h(\tau)}^{\tau} G(s)\,R_2(h(\tau),h(s))\,ds > 1 \tag{2.2}$$

or

$$\lim_{\tau\to\infty}\sup\int_{h(\tau)}^{\tau}\left(\frac{1}{r_2(u)}\int_u^{\tau} G(s)\,ds\right)du > 1, \tag{2.3}$$

where $r_2(\tau)$ *is as in* (C_1), $G(\tau)\in C\left(I,\mathbb{R}^+\right)$, $h\in C^1(I,\mathbb{R})$ *such that* $h(\tau)\le\tau$, $h'(\tau)\ge 0$ *for* $\tau\ge\tau_0$ *and* $\lim_{\tau\to\infty}h(\tau)=\infty$. *Then the solution* $x(\tau)$ *is oscillatory.*

Lemma 2.2. *Assume that the second-order differential equation*

$$\left(r_2(\tau)\,v'(\tau)\right)' + \frac{k_1 p(\tau)}{r_1(\delta(\tau))}v(\tau)=0 \tag{2.4}$$

is nonoscillatory. If y *is a positive solution of* Eq. (1.1) *on* $[\tau_1,\infty)$, *then there exists a* $\tau_1\ge\tau_0$ *such that* $y(\tau)$ *has only the following two cases*

(i) $E_1 y(\tau)>0,\ E_2 y(\tau)>0,$

(ii) $E_1 y(\tau)<0,\ E_2 y(\tau)>0.$

Proof. Let y be a positive solution of Eq. (1.1). We assume that there exists a $\tau_1\ge\tau_0$ such that $y(\tau)>0$ and $y(\sigma(\tau))$ for $\tau\ge\tau_1$. We note that $z(\tau)=-E_1 y(\tau)$ is a solution of the second-order nonhomogeneous delay differential inequality

$$\left(r_2(\tau)\,z'(\tau)\right)' + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right)z(\delta(\tau)) \ge q(\tau)\,f(y(\sigma(\tau))),\ \ \tau\ge\tau_1. \tag{2.5}$$

Now, let v be a solution of Eq. (2.4) and $v(\tau)>0$ for $\tau\ge\tau_1\ge\tau_0$. The case when $v(\tau)$ is ultimately negative can similarly be dealt with. Let z be an oscillatory solution of (2.5) with consecutive zeros at a and b, such that

$$\tau_1 < a < b$$

and

$$z'(a)\ge 0 \text{ and } z'(b)\le 0.$$

By using Rolle's Theorem, there exists a point c in (a,b) such that $z'(c)=0$, and hence

$$z'(\tau)>0 \text{ for all } \tau\in[a,c) \text{ and } z'(\tau)<0 \text{ for all } \tau\in(c,b].$$

Thus, we find

$$\begin{aligned}
0 &< \int_a^b v(\tau)\,q(\tau)\,f(y(\sigma(\tau)))\,d\tau\\
&\le \int_a^b \left(\left(r_2(\tau)\,z'(\tau)\right)' + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right)z(\delta(\tau))\right)v(\tau)\,d\tau\\
&= r_2(\tau)\,z'(\tau)\,v(\tau)\Big|_a^b - \int_a^b r_2(\tau)\,z'(\tau)\,v'(\tau)\,d\tau\\
&\quad + \int_a^b \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right)z(\delta(\tau))\,v(\tau)\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&= r_2(\tau) z'(\tau) v(\tau)\Big|_a^b - r_2(\tau) z(\tau) v'(\tau)\Big|_a^b + \int_a^b \big(r_2(\tau) v'(\tau)\big)' z(\tau)\, d\tau \\
&\quad + \int_a^b \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) z(\delta(\tau))\, v(\tau)\, d\tau \\
&= r_2(\tau) z'(\tau) v(\tau)\Big|_a^b + \int_a^b \left(\big(r_2(\tau) v'(\tau)\big)' z(\tau) + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) z(\delta(\tau))\, v(\tau)\right) d\tau \\
&= r_2(\tau) z'(\tau) v(\tau)\Big|_a^b + \int_a^c \left(\big(r_2(\tau) v'(\tau)\big)' z(\tau) + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) z(\delta(\tau))\, v(\tau)\right) d\tau \\
&\quad + \int_c^b \left(\big(r_2(\tau) v'(\tau)\big)' z(\tau) + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) z(\delta(\tau))\, v(\tau)\right) d\tau \\
&\le r_2(\tau) z'(\tau) v(\tau)\Big|_a^b + \int_a^c \left(\big(r_2(\tau) v'(\tau)\big)' + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) v(\tau)\right) z(\tau)\, d\tau \\
&\quad + \int_c^b \left(\big(r_2(\tau) v'(\tau)\big)' + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) v(\tau)\right) z(\delta(\tau))\, d\tau \\
&= r_2(\tau) z'(\tau) v(\tau)\Big|_a^b \le 0,
\end{aligned}
$$

which is a contradiction. Then, we have either $E_1 y(\tau) > 0$ or $E_1 y(\tau) < 0$ for $\tau \ge \tau_2$. Next, we assume that $E_1 y(\tau) > 0$ on $[\tau_1, \infty)$. From (1.1), $E_2 y(\tau)$ is strictly decreasing and hence

$$
\begin{aligned}
E_1 y(\tau) &= E_1 y(\tau_2) + \int_{\tau_2}^{\tau} (E_1 y(s))' ds \\
&= E_1 y(\tau_2) + \int_{\tau_2}^{\tau} \frac{E_2 y(s)}{r_2(s)} ds \\
&\le E_1 y(\tau_2) + \int_{\tau_2}^{\tau} \frac{E_2 y(\tau_2)}{r_2(s)} ds \\
&= E_1 y(\tau_2) + E_2 y(\tau_2)\, R_2(\tau, \tau_2).
\end{aligned}
$$

So $E_2 y(\tau_2) > 0$ as otherwise $E_1 y(\tau) \to -\infty$ as $\tau \to \infty$, a contradiction to the positivity of $E_1 y$. Altogether, $E_2 y > 0$ on $[\tau_1, \infty)$. Now, we assume that $E_1 y < 0$ on $[\tau_1, \infty)$. The case $E_2 y(\tau) < 0$ cannot hold for all large τ, say $\tau \ge \tau_2 \ge \tau_1$. Then, by integration of

$$
y'(\tau) = \left(\frac{E_1 y(\tau)}{r_1(\tau)}\right)^{\frac{1}{\alpha}} \le \left(\frac{E_1 y(\tau_2)}{r_1(\tau)}\right)^{\frac{1}{\alpha}} \text{ for } \tau \ge \tau_2,
$$

we obtain $y(\tau) < 0$ for all large τ, which is a contradiction. The proof is complete. □

Lemma 2.3. *Assume that $y(\tau)$ is a positive solution of (1.1) with case* (**i**) *for $\tau \ge \tau_1 \ge \tau_0$. Then*

$$
E_1 y(\tau) \ge R_2(\tau, \tau_1)\, E_2 y(\tau) \ \textit{for}\ \tau \ge \tau_1 \tag{2.6}
$$

and

$$
y(\tau) \ge R_{12}(\tau, \tau_1)\, E_2^{\frac{1}{\alpha}} y(\tau) \ \textit{for}\ \tau \ge \tau_1. \tag{2.7}
$$

Proof. Let $y(\tau)$ be a positive solution of (1.1). We assume that there exists a $\tau_1 \geq \tau_0$ such that $y(\tau) > 0$ and $y(\sigma(\tau))$ for $\tau \geq \tau_1$. Now, from (1.1) we have

$$E_3 y(\tau) = -\left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) E_1 y(\delta(\tau)) - k_2 q(\tau) y^{\beta}(\sigma(\tau)) \leq 0.$$

Since $E_2 y$ is nonincreasing on $[\tau_1, \infty)$, we obtain

$$\begin{aligned} E_1 y(\tau) &\geq \int_{\tau_1}^{\tau} (E_1 y(s))' ds = \int_{\tau_1}^{\tau} \frac{E_2 y(s)}{r_2(s)} ds \\ &\geq R_2(\tau, \tau_1) E_2 y(\tau). \end{aligned}$$

This implies that

$$y'(\tau) \geq \left(\frac{R_2(\tau, \tau_1)}{r_1(\tau)}\right)^{\frac{1}{\alpha}} E_2^{\frac{1}{\alpha}} y(\tau).$$

By integrating this inequality from τ_1 to τ and using $E_3 y \leq 0$, we obtain

$$\begin{aligned} y(\tau) &= y(\tau_1) + \int_{\tau_1}^{\tau} y'(s) ds \geq \int_{\tau_1}^{\tau} y'(s) ds \\ &\geq \int_{\tau_1}^{\tau} \left(\frac{R_2(s, \tau_1)}{r_1(s)}\right)^{\frac{1}{\alpha}} E_2^{\frac{1}{\alpha}} y(s) ds \\ &\geq \left[\int_{\tau_1}^{\tau} \left(\frac{R_2(s, \tau_1)}{r_1(s)}\right)^{\frac{1}{\alpha}} ds\right] E_2^{\frac{1}{\alpha}} y(\tau) \\ &= R_{12}(\tau, \tau_1) E_2^{\frac{1}{\alpha}} y(\tau), \text{ for } \tau \geq \tau_1. \end{aligned}$$

This completes the proof. □

3. OSCILLATION–COMPARISON METHOD

In this section, we are ready to establish the main results for (1.1). We present the following comparison result for the delay case. For $\tau \geq \tau_1 \geq \tau_0$, we set

$$C(\tau) = \frac{k_1 p(\tau)}{r_1(\delta(\tau))} R_2(\delta(\tau), \tau_1), \; G_1(\tau) = k_2 q(\tau) R_{12}^{\beta}(\sigma(\tau), \tau_1),$$

and

$$\eta(\tau) = \exp\left(\int_{\tau_1}^{\tau} C(s) ds\right).$$

Theorem 3.1. *Assume that (2.4) is nonoscillatory and $\alpha \geq \beta$. Suppose there exists a function $h \in C^1(I, \mathbb{R}^+)$ such that*

$$\sigma(\tau) \leq h(\tau) \leq \delta(\tau) \leq \tau, \; h'(\tau) \geq 0 \textit{ for } \tau \geq \tau_0,$$

and (2.2) or (2.3) holds with

$$G(\tau) = \left(bk_2 q(\tau) R_1^{\beta}(h(\tau), \sigma(\tau)) - \frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) \geq 0, \textit{ for } \tau \geq \tau_1. \tag{3.1}$$

Moreover, suppose that every solution of the first-order delay equation

$$x'(\tau) + \eta^{1-\frac{\beta}{\alpha}}(\sigma(\tau))\, G_1(\tau)\, x^{\frac{\beta}{\alpha}}(\sigma(\tau)) = 0 \tag{3.2}$$

is oscillatory. Then every solution of Eq. (1.1) *or* $E_2 y(\tau)$ *is oscillatory.*

Proof. Let $y(\tau)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that there exists a $\tau_1 \geq \tau_0$ such that $y(\tau) > 0$ and $y(\sigma(\tau))$ for $\tau \geq \tau_1$. From Lemma 2.2, there exists a $\tau_2 \geq \tau_1$ such that either $y(\tau)$ has the property (**i**) or (**ii**) for $\tau \geq \tau_2$. Now, let $y(\tau)$ has the property (**i**). Since $\sigma(\tau) \to \infty$ as $\tau \to \infty$, we can choose $\tau_2 \geq \tau_1$ such that $\sigma(\tau) \geq \tau_1$ for $\tau \geq \tau_2$ and hence (2.7) gives

$$y(\sigma(\tau)) \geq R_{12}(\sigma(\tau), \tau_1)\, E_2^{\frac{1}{\alpha}} y(\sigma(\tau)) \ \text{ for } \tau \geq \tau_1. \tag{3.3}$$

Using (2.6) and (3.3) in (1.1), we obtain

$$\begin{aligned}(E_2 y(\tau))' &+ \frac{k_1 p(\tau)}{r_1(\delta(\tau))} E_2 y(\delta(\tau))\, R_2(\delta(\tau), \tau_1) \\ &+ k_2 q(\tau)\, E_2^{\beta/\alpha} y(\sigma(\tau))\, R_{12}^{\beta}(\sigma(\tau), \tau_1) \leq 0.\end{aligned}$$

Since $E_2 y$ is decreasing, we get

$$\begin{aligned}(E_2 y(\tau))' &+ \frac{k_1 p(\tau)}{r_1(\delta(\tau))} E_2 y(\tau)\, R_2(\delta(\tau), \tau_1) \\ &+ k_2 q(\tau)\, E_2^{\frac{\beta}{\alpha}} y(\sigma(\tau))\, R_{12}^{\beta}(\sigma(\tau), \tau_1) \leq 0.\end{aligned}$$

Let $\psi(\tau) = E_2 y(\tau)$, we have

$$\psi'(\tau) + C(\tau)\, \psi(\tau) + G_1(\tau)\, \psi^{\frac{\beta}{\alpha}}(\sigma(\tau)) \leq 0.$$

Therefore,

$$(\eta(\tau)\, \psi(\tau))' + \eta(\tau)\, G_1(\tau)\, \psi^{\frac{\beta}{\alpha}}(\sigma(\tau)) \leq 0.$$

Next, letting $x = \eta\psi > 0$ in the above inequality and noting that $\eta(\sigma(\tau)) \leq \eta(\tau)$, we have

$$x'(\tau) + \eta^{1-\frac{\beta}{\alpha}}(\sigma(\tau))\, G_1(\tau)\, x^{\frac{\beta}{\alpha}}(\sigma(\tau)) \leq 0.$$

Then, this inequality has a positive solution. Also, by [[2]. Corollary 2.3.5], we see that (3.2) has a positive solution, which is a contradiction. Moreover, if $y(\tau)$ has the property (**ii**), then, for $v \geq u \geq \tau_3$, we find

$$\begin{aligned} y(u) - y(v) &= -\int_u^v \frac{\left(r_1(s)\left(y'(s)\right)^{\alpha}\right)^{1/\alpha}}{r_1^{1/\alpha}(s)} ds \\ &\geq -E_1^{1/\alpha} y(v) \left(\int_u^v r_1^{-1/\alpha}(s)\, ds\right) \\ &= R_1(v, u)\left(-E_1^{1/\alpha} y(v)\right).\end{aligned}$$

Letting $u = \sigma(\tau)$ and $v = h(\tau)$, we get

$$\begin{aligned} y(\sigma(\tau)) &\geq R_1(h(\tau), \sigma(\tau))\left(-E_1^{1/\alpha} y(h(\tau))\right) \\ &= R_1(h(\tau), \sigma(\tau))\, x(h(\tau)),\end{aligned}$$

where $x(\tau) = -E_1^{1/\alpha} y(\tau) > 0$. From (1.1), we have that x is decreasing and

$$\sigma(\tau) \leq h(\tau) \leq \delta(\tau) \leq \tau.$$

Let $z = x^\alpha$, we find

$$\left(r_2(\tau) z'(\tau)\right)' + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) z(\delta(\tau)) \\ \geq k_2 q(\tau) R_1^\beta(h(\tau), \sigma(\tau)) z(h(\tau)) z^{\frac{\beta}{\alpha}-1}(h(\tau)).$$

Since z is decreasing and $\alpha \geq \beta$, there exists a constant b such that $z^{\frac{\beta}{\alpha}-1} \geq b$ for $\tau \geq \tau_2$. Thus, we obtain

$$\left(r_2(\tau) z'(\tau)\right)' \geq \left(bk_2 q(\tau) R_1^\beta(h(\tau), \sigma(\tau)) - \frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) z(h(\tau)). \tag{3.4}$$

Now, let z be a bounded nonoscillatory solution of (1.1), say $z(\tau) > 0$, $z(h(\tau)) > 0$ for all $\tau \geq \tau_1$. By (3.4), we have $r_2(\tau) z'(\tau)$ is strictly increasing. So, for all $\tau_2 > \tau_1$, we get

$$z(\tau) > z(\tau_2) + r_2(\tau_2) z'(\tau_2) \int_{\tau_2}^{\tau} \frac{ds}{r_2(s)} \\ = z(\tau_2) + r_2(\tau_2) z'(\tau_2) R_2(\tau, \tau_2).$$

Hence, $z'(\tau_2) < 0$ as otherwise $z(\tau) \to \infty$ as $\tau \to \infty$, a contradiction to the boundedness of z. So, we find

$$z > 0,\ z' < 0 \text{ and } \left(r_2 z'\right)' > 0. \tag{3.5}$$

Now, for $v \geq u \geq \tau_1$, we have

$$z(u) > -\int_u^v z'(s)\, ds \\ \geq -r_2(v) z'(v) \int_u^v \frac{1}{r_2(s)} ds \\ = -R_2(v, u) r_2(v) z'(v), \tag{3.6}$$

for $\tau \geq s \geq \tau_1$. Setting $u = h(s)$ and $v = h(\tau)$ in (3.6), we obtain

$$z(h(s)) > -R_2(h(\tau), h(s)) r_2(h(\tau)) z'(h(\tau)). \tag{3.7}$$

Integrating (3.4) from $h(\tau)$ to τ, we get

$$-r_2(h(\tau)) z'(h(\tau)) > r_2(\tau) z'(\tau) - r_2(h(\tau)) z'(h(\tau)) \\ = \int_{h(\tau)}^{\tau} \left(r_2(s) z'(s)\right)' ds \\ \geq \int_{h(\tau)}^{\tau} G(s) z(h(s))\, ds.$$

From (3.7), we have

$$-r_2(h(\tau)) z'(h(\tau)) > -\left[\int_{h(\tau)}^{\tau} G(s) R_2(h(\tau), h(s))\, ds\right] r_2(h(\tau)) z'(h(\tau)).$$

Therefore,

$$1 > \int_{h(\tau)}^{\tau} G(s) R_2(h(\tau), h(s))\, ds. \tag{3.8}$$

Taking lim sup as $\tau \to \infty$ on both sides of (3.8) yields a contradiction to (2.2), and completes the proof. $\square$

Corollary 3.1. *Assume that (2.4) is nonoscillatory and $\alpha \geq \beta$. Suppose there exists a function $h \in C^1(I, \mathbb{R})$ such that*

$$\sigma(\tau) \leq h(\tau) \leq \delta(\tau) \leq \tau, \ \ h'(\tau) \geq 0 \text{ for } \tau \geq \tau_0,$$

and (2.2) or (2.3) holds with G as in Theorem 3.1. If

$$\lim_{\tau\to\infty} \inf \int_{\sigma(\tau)}^{\tau} G_1(s)\, ds > \frac{1}{e} \text{ when } \alpha = \beta$$

and

$$\int^{\infty} \eta^{1-\frac{\beta}{\alpha}}(\sigma(s))\, G_1(s)\, ds = \infty \text{ when } \alpha > \beta,$$

then every solution y of (1.1) or $E_2 y$ is oscillatory.

The proof of this corollary is immediate, and hence is omitted.

Theorem 3.2. *Assume that*

$$\left(r_2(\tau)\, x'(\tau)\right)' + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) x(\delta(\tau)) = 0$$

is oscillatory and $\alpha \geq \beta$. Suppose that (2.2) or (2.3) holds with G as in Theorem 3.1, then every solution of (1.1) or y' is oscillatory.

Proof. Let $y(\tau)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that there exists a $\tau_1 \geq \tau_0$ such that $y(\tau) > 0$ and $y(\sigma(\tau))$ for $\tau \geq \tau_1$. From Lemma 2.2, there exists a $\tau_2 \geq \tau_1$ such that either $y(\tau)$ has the property (**i**) or (**ii**) for $\tau \geq \tau_2$. Next, let $y(\tau)$ has the property (**i**). From (1.1), we obtain

$$\left(r_2(\tau)\, x'(\tau)\right)' + \left(\frac{k_1 p(\tau)}{r_1(\delta(\tau))}\right) x(\delta(\tau)) \leq 0,$$

where $x = E_1 y > 0$. By [[16], Lemma 2.6], we get that (2.4) has a positive solution which is a contradiction. On the other hand, if $y(\tau)$ has the property (**ii**), the proof in this case is similar to that of Theorem 3.1 and this completes the proof. $\square$

4. Oscillation-integral averaging method

This section is concerned with some new oscillation results for (1.1) by using the integral average technique. We introduce the class of functions X, which will be used in this section. Let

$$D_0 = \{(\tau, s) : \tau > s > \tau_0\} \text{ and } D = \{(\tau, s) : \tau \geq s > \tau_0\}.$$

A function $H \in C(D, \mathbb{R})$ is said to belong to the class X, if

$$H(\tau, s) > 0 \text{ for } (\tau, s) \in D_0, \ \ H(\tau, \tau) = 0$$

and H has a continuous and nonpositive partial derivative on D_0 with respect the second variable, and for a positive continuous function h^*

$$-\frac{\partial H(\tau,s)}{\partial s} = h^*(\tau,s)\sqrt{H(\tau,s)}, \text{ for } (\tau,s) \in D_0.$$

For the choice $H(\tau,s) = (\tau-s)^n$, $n \in \mathbb{N}$, the Philos-type conditions reduce to the Kamenev-type ones.

Theorem 4.1. *Assume that (2.4) is nonoscillatory and $\alpha \geq \beta$. Suppose that there exist two functions $\rho, h \in C^1(I,\mathbb{R})$, such that*

$$\sigma(\tau) \leq h(\tau) \leq \delta(\tau) \leq \tau, \; h'(\tau) \geq 0 \text{ and } \rho(\tau) > 0 \text{ for } \tau \geq \tau_0,$$

and a function $H \in X$ satisfying

$$\limsup_{\tau\to\infty} \frac{1}{H(\tau,\tau_1)} \int_{\tau_1}^{\tau} \left[k_2\rho(s)q(s)H(\tau,s) - \frac{P^2(\tau,s)}{4B(s)} \right] ds = \infty, \tag{4.1}$$

for all large $\tau \geq \tau_1$, where

$$P(\tau,s) = h^*(\tau,s) - A(s)\sqrt{H(\tau,s)},$$

with

$$A(\tau) = \frac{\rho'(\tau)}{\rho(\tau)} - \frac{k_1 p(\tau)}{r_1(\delta(\tau))} R_2(\delta(\tau),\tau_1) \tag{4.2}$$

and

$$B(\tau) = \beta b_2^{\beta-\alpha}\rho^{-1}(\tau)\sigma'(\tau)R_{12}^{\beta-1}(\sigma(\tau),\tau_1)\left(\frac{R_2(\sigma(\tau),\tau_1)}{r_1(\sigma(\tau))}\right)^{1/\alpha}. \tag{4.3}$$

If (2.2) or (2.3) holds with G as in Theorem 3.1, then every solution of (1.1) or E_2y is oscillatory.

Proof. Let $y(\tau)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that there exists a $\tau_1 \geq \tau_0$ such that $y(\tau) > 0$ and $y(\sigma(\tau))$ for $\tau \geq \tau_1$. From Lemma 2.2, there exists a $\tau_2 \geq \tau_1$ such that either $y(\tau)$ has the property (**i**) or (**ii**) for $\tau \geq \tau_2$. Next, let $y(\tau)$ has the property (**i**). Define the function

$$\omega(\tau) = \rho(\tau)\frac{E_2y(\tau)}{y^\beta(\sigma(\tau))}, \tag{4.4}$$

on $[\tau_1,\infty)$. Then $\omega(\tau) > 0$ for $\tau \geq \tau_1$. From (2.7) and $E_3y(\tau) < 0$, we have

$$\begin{aligned}\omega(\tau) &= \rho(\tau)\frac{E_2y(\tau)}{y^\beta(\sigma(\tau))} \leq \rho(\tau)\frac{E_2y(\sigma(\tau))}{y^\beta(\sigma(\tau))} \\ &\leq \rho(\tau)R_{12}^{-\alpha}(\sigma(\tau),\tau_1)\,y^{\alpha-\beta}(\sigma(\tau)).\end{aligned} \tag{4.5}$$

Also, since $E_3y(\tau) < 0$, we have

$$0 < E_2y(\tau) \leq E_2y(\tau_1) = b_1,$$

for $\tau \geq \tau_1$. From this we get

$$r_2(\tau)(E_1y(\tau))' = E_2y(\tau) \leq b_1.$$

Thus, for all $\tau \geq \tau_2$, we obtain

$$\begin{aligned} r_1(\tau)\left(y'(\tau)\right)^{\alpha} &= E_1 y(\tau_1) + \int_{\tau_1}^{\tau} (E_1 y(s))' ds \\ &\leq E_1 y(\tau_1) + b_1 \int_{\tau_1}^{\tau} \frac{ds}{r_2(s)} \\ &= E_1 y(\tau_1) + b_1 R_2(\tau, \tau_1) \\ &= \left(\frac{E_1 y(\tau_1)}{R_2(\tau, \tau_1)} + b_1\right) R_2(\tau, \tau_1) \\ &\leq \left(\frac{E_1 y(\tau_1)}{R_2(\tau_2, \tau_1)} + b_1\right) R_2(\tau, \tau_1) = b_1^* R_2(\tau, \tau_1), \end{aligned}$$

where

$$b_1^* = \frac{E_1 y(\tau_1)}{R_2(\tau_2, \tau_1)} + b_1.$$

This implies that

$$\begin{aligned} y(\tau) &= y(\tau_2) + \int_{\tau_2}^{\tau} y'(s)\, ds \leq y(\tau_2) + \int_{\tau_2}^{\tau} \left(\frac{b_1^* R_2(s, \tau_1)}{r_1(s)}\right)^{1/\alpha} ds \\ &= y(\tau_2) + \left(b_1^*\right)^{1/\alpha} R_{12}(\tau, \tau_1) \\ &= \left(\frac{y(\tau_2)}{R_{12}(\tau, \tau_1)} + \left(b_1^*\right)^{1/\alpha}\right) R_{12}(\tau, \tau_1) \\ &\leq \left(\frac{y(\tau_2)}{R_{12}(\tau_2, \tau_1)} + \left(b_1^*\right)^{1/\alpha}\right) R_{12}(\tau, \tau_1) \\ &= b_2 R_{12}(\tau, \tau_1) \end{aligned} \tag{4.6}$$

where

$$b_2 = \left(b_1^*\right)^{1/\alpha} + \frac{y(\tau_2)}{R_{12}(\tau_2, \tau_1)}.$$

Therefore, we have

$$y^{\frac{\beta}{\alpha}-1}(\sigma(\tau)) \geq b_2^{\frac{\beta}{\alpha}-1} R_{12}^{\frac{\beta}{\alpha}-1}(\sigma(\tau), \tau_1), \text{ for } \tau \geq \tau_2. \tag{4.7}$$

By using (4.6) in (4.5), we obtain

$$\begin{aligned} \omega(\tau) &\leq \rho(\tau) R_{12}^{-\alpha}(\sigma(\tau), \tau_1) b_2^{\alpha-\beta} R_{12}^{\alpha-\beta}(\sigma(\tau), \tau_1) \\ &\leq b_2^{\alpha-\beta} \rho(\tau) R_{12}^{-\beta}(\sigma(\tau), \tau_1), \end{aligned}$$

and hence

$$\omega^{\frac{1}{\alpha}-1}(\tau) \geq b_2^{(\alpha-\beta)\left(\frac{1}{\alpha}-1\right)} \rho^{\frac{1}{\alpha}-1}(\tau) R_{12}^{-\beta\left(\frac{1}{\alpha}-1\right)}(\sigma(\tau), \tau_1). \tag{4.8}$$

By differentiating (4.4), we obtain

$$\omega'(\tau) = \frac{\rho'(\tau)}{\rho(\tau)}\omega(\tau) + \frac{(E_2 y(\tau))'}{E_2 y(\tau)}\omega(\tau) - \beta\sigma'(\tau)\frac{y'(\sigma(\tau))}{y(\sigma(\tau))}\omega(\tau).$$

From (1.1), (C_2) and (C_3), we find

$$\omega'(\tau) \leq \frac{\rho'(\tau)}{\rho(\tau)}\omega(\tau) - \frac{k_1 p(\tau)\left(y'(\delta(\tau))\right)^{\alpha}}{E_2 y(\tau)}\omega(\tau) - \frac{k_2 q(\tau) y^{\beta}(\sigma(\tau))}{E_2 y(\tau)}\omega(\tau) - \beta\sigma'(\tau)\frac{y'(\sigma(\tau))}{y(\sigma(\tau))}\omega(\tau).$$

Since $E_1 y(\tau) > 0$, we get

$$\omega'(\tau) \leq \frac{\rho'(\tau)}{\rho(\tau)}\omega(\tau) - \frac{\frac{k_1 p(\tau)}{r_1(\delta(\tau))}E_1 y(\delta(\tau))}{E_2 y(\tau)}\omega(\tau) - k_2\rho(\tau) q(\tau) - \beta\sigma'(\tau)\frac{y'(\sigma(\tau))}{y(\sigma(\tau))}\omega(\tau).$$

Using (2.6) implies that

$$\begin{aligned}\omega'(\tau) &\leq \frac{\rho'(\tau)}{\rho(\tau)}\omega(\tau) - \frac{k_1 p(\tau)}{r_1(\delta(\tau))}R_2(\delta(\tau),\tau_1)\omega(\tau) - k_2\rho(\tau) q(\tau) - \beta\sigma'(\tau)\frac{y'(\sigma(\tau))}{y(\sigma(\tau))}\omega(\tau) \\ &\leq A(\tau) - k_2\rho(\tau) q(\tau) - \beta\sigma'(\tau)\frac{y'(\sigma(\tau))}{y(\sigma(\tau))}\omega(\tau). \end{aligned} \tag{4.9}$$

From (2.6) and definition $E_1 y(\tau)$, we find

$$\begin{aligned} y'(\sigma(\tau)) &= \left(\frac{1}{r_1(\sigma(\tau))}E_1 y(\sigma(\tau))\right)^{1/\alpha} \\ &\geq \left(\frac{R_2(\sigma(\tau),\tau_1)}{r_1(\sigma(\tau))}\right)^{1/\alpha} E_2^{1/\alpha} y(\sigma(\tau)) \\ &\geq \left(\frac{R_2(\sigma(\tau),\tau_1)}{r_1(\sigma(\tau))}\right)^{1/\alpha} E_2^{1/\alpha} y(\tau). \end{aligned}$$

Hence,

$$\begin{aligned}\frac{y'(\sigma(\tau))}{y(\sigma(\tau))} &\geq \left(\frac{R_2(\sigma(\tau),\tau_1)}{\rho(\tau) r_1(\sigma(\tau))}\right)^{1/\alpha} \frac{\rho^{1/\alpha}(\tau) E_2^{1/\alpha} y(\tau)}{y^{\frac{\beta}{\alpha}}(\sigma(\tau))} y^{\frac{\beta}{\alpha}-1}(\sigma(\tau)) \\ &= \left(\frac{R_2(\sigma(\tau),\tau_1)}{\rho(\tau) r_1(\sigma(\tau))}\right)^{1/\alpha} \omega^{1/\alpha}(\tau) y^{\frac{\beta}{\alpha}-1}(\sigma(\tau)). \end{aligned} \tag{4.10}$$

By using (4.10) in (4.9), we get

$$\omega'(\tau) \leq A(\tau)\omega(\tau) - \beta\sigma'(\tau)\left(\frac{R_2(\sigma(\tau),\tau_1)}{\rho(\tau) r_1(\sigma(\tau))}\right)^{1/\alpha} y^{\frac{\beta}{\alpha}-1}(\sigma(\tau))\omega^{1/\alpha}(\tau) - k_2\rho(\tau) q(\tau). \tag{4.11}$$

Using (4.7) and (4.8), we have

$$\begin{aligned}\omega'(\tau) &\leq A(\tau)\omega(\tau) - \beta b_2^{\beta-\alpha}\rho^{-1}(\tau)\sigma'(\tau) R_{12}^{\beta-1}(\sigma(\tau),\tau_1)\left(\frac{R_2(\sigma(\tau),\tau_1)}{r_1(\sigma(\tau))}\right)\omega^2(\tau) - k_2\rho(\tau) q(\tau) \\ &\leq A(\tau)\omega(\tau) - k_2\rho(\tau) q(\tau) - B(\tau)\omega^2(\tau), \end{aligned} \tag{4.12}$$

where A and B are as in (4.2) and (4.3). Now, by integrating the last inequality from τ_1 to τ, we have

$$\begin{aligned}
\int_{\tau_1}^{\tau} k_2\rho(s)\, q(s)\, H(\tau,s)\, ds &\le \int_{\tau_1}^{\tau} H(\tau,s)\left[-\omega'(s) + (A\omega)(s) - \left(B\omega^2\right)(s)\right] ds \\
&= \int_{\tau_1}^{\tau}\left[\frac{\partial H(\tau,s)}{\partial s}\omega(s) + H(\tau,s)\left(\left(A\omega - B\omega^2\right)(s)\right)\right] ds \\
&\quad - H(\tau,s)\,\omega(s)|_{\tau_1}^{\tau} + \\
&= H(\tau,\tau_1)\,\omega(\tau_1) - \int_{\tau_1}^{\tau}\left[H(\tau,s)\, B(s)\, \omega^2(s)\right. \\
&\quad \left. + \omega(s)\left(h^*(\tau,s)\sqrt{H(\tau,s)} - H(\tau,s)\, A(s)\right)\right] ds \\
&= -\int_{\tau_1}^{\tau}\left(\sqrt{H(\tau,s)}\sqrt{B(s)}\omega(s) + \frac{P(\tau,s)}{2\sqrt{B(s)}}\right)^2 ds \\
&\quad + H(\tau,\tau_1)\,\omega(\tau_1) + \int_{\tau_1}^{\tau}\frac{P^2(\tau,s)}{4B(s)} \\
&\le H(\tau,\tau_1)\,\omega(\tau_1) + \int_{\tau_1}^{\tau}\frac{P^2(\tau,s)}{4B(s)}.
\end{aligned}$$

Thus, we get

$$\frac{1}{H(\tau,\tau_1)}\int_{\tau_1}^{\tau}\left[k_2\rho(s)\, q(s)\, H(\tau,s) - \frac{P^2(\tau,s)}{4B(s)}\right] ds \le \omega(\tau_1),$$

which is a contradiction. The rest of the proof is similar to that of Theorem 3.1. □

Theorem 4.2. *Suppose that the hypotheses of Theorem 4.1 hold, except (4.1). Assume that for every $\tau_1 > \tau_0$,*

$$0 < \inf_{s\ge\tau_1}\left[\liminf_{\tau\to\infty}\frac{H(\tau,s)}{H(\tau,\tau_1)}\right] < \infty,$$

$$\limsup_{\tau\to\infty}\frac{1}{H(\tau,\tau_1)}\int_{\tau_1}^{\tau}\frac{d_3 P^2(\tau,s)}{B(s)} ds < \infty,$$

and there exists $\varphi \in C(I)$ such that

$$\int_{\tau_1}^{\infty}\frac{1}{d_3}\varphi_+^2(s)\, B(s)\, ds = \infty,\ \ \varphi_+(s) = \max\{0, \varphi(s)\},$$

and

$$\limsup_{\tau\to\infty}\frac{1}{H(\tau,\tau_1)}\int_{\tau_1}^{\tau}\left[k_2\rho(s)\, q(s)\, H(\tau,s) - \frac{P^2(\tau,s)}{4B(s)}\right] ds \ge \varphi(\tau_1). \tag{4.13}$$

Then every solution of (1.1) or $E_2 y$ is oscillatory.

Proof. Let y be a positive solution of (1.1) on $[\tau_1, \infty)$ and $y(\tau)$ has the property (**i**). Proceeding as in the proof of Theorem 4.1, we get

$$\int_{\tau_1}^{\tau} k_2 \rho(s) q(s) H(\tau, s) ds \leq -\int_{\tau_1}^{\tau} \left[\sqrt{H(\tau, s) B(s)} \omega(s) + \frac{P(\tau, s)}{2\sqrt{B(s)}} \right]^2 ds + H(\tau, \tau_1) \omega(\tau_1) + \int_{\tau_1}^{\tau} \frac{P^2(\tau, s)}{4B(s)} ds.$$

Using (4.13), we find

$$\begin{aligned} \varphi(\tau_1) &\leq \limsup_{\tau \to \infty} \frac{1}{H(\tau, \tau_1)} \int_{\tau_1}^{\tau} \left[k_2 \rho(s) q(s) H(\tau, s) - \frac{P^2(\tau, s)}{4B(s)} \right] ds \\ &\leq \omega(\tau_1) - \liminf_{\tau \to \infty} \frac{1}{H(\tau, \tau_1)} \int_{\tau_1}^{\tau} \left[\sqrt{H(\tau, s) B(s)} \omega(s) + \frac{P(\tau, s)}{2\sqrt{B(s)}} \right]^2 ds, \end{aligned}$$

and hence

$$\liminf_{\tau \to \infty} \frac{1}{H(\tau, \tau_1)} \int_{\tau_1}^{\tau} \left[\sqrt{H(\tau, s) B(s)} \omega(s) + \frac{P(\tau, s)}{2\sqrt{B(s)}} \right]^2 ds < \infty. \tag{4.14}$$

Now, we define

$$C_1(\tau) = \frac{1}{H(\tau, \tau_1)} \int_{\tau_1}^{\tau} H(\tau, s) B(s) \omega^2(s) ds$$

and

$$C_2(\tau) = \frac{1}{H(\tau, \tau_1)} \int_{\tau_1}^{\tau} \sqrt{H(\tau, s)} P(\tau, s) \omega(s) ds.$$

It follows from (4.14) that

$$\liminf_{\tau \to \infty} [C_1(\tau) + C_2(\tau)] < \infty.$$

The remainder of this proof is similar to that of [21, Theorem 2], and hence is omitted. On the other hand, if $y(\tau)$ has the property (**ii**), the proof is similar to that of the proof of Theorem 3.1. □

5. Examples

Example 5.1. For $\tau \geq 1$, consider the equation

$$y'''(\tau) + 3y'(\tau) + \frac{3}{d^*} y\left(\frac{\tau}{3}\right) = 0. \tag{5.1}$$

We note that $\alpha = \beta = 1$ and $f(y) = y$. Furthermore, we choose $k_1 = k_2 = 1$, $\rho(\tau) = 1$ and $H(\tau, s) = (\tau - s)^2$. Then $h^*(\tau, s) = 2$, $P(\tau, s) = 2 + (\tau - 3)(\tau - s)$, $A(\tau) = -(\tau - 3)$, $B(\tau) = d^*/3(\tau/3 - 1)$ and

$$\limsup_{\tau \to \infty} \frac{1}{(\tau - 1)^2} \int_1^{\tau} \left[\frac{3}{d^*}(\tau - s)^2 - \frac{(2 + (\tau - 3)(\tau - s))^2}{4d^*/3(\tau/3 - 1)} \right] ds = \infty.$$

Altogether, all hypotheses of Theorem 4.1 are satisfied, so every solution of (5.1) or $E_2 y$ is oscillatory.

Example 5.2. Consider the equation

$$y'''(\tau) + y'(\tau - \pi) + 2y\left(\tau - \frac{3\pi}{2}\right) = 0, \tag{5.2}$$

where $\tau > \frac{3}{2}\pi$. We note that $\alpha = \beta = 1$ and $f(y) = y$. It easy to check that all hypotheses of Theorem 3.2 are satisfied with $k_1 = k_2 = 1$. So every solution of (5.2) is oscillatory or y' is oscillatory. One oscillatory solution is $y(\tau) = \sin \tau$.

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