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On the study of Brück conjecture and some non-linear complex differential equations

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Abstract. In this paper, we prove the following result: Let $f(z)$ and $\alpha(z)$ be two non-constant entire functions satisfying $\sigma(\alpha) < \mu(f)$ and $P(z)$ be a polynomial. If f is a non-constant entire solution of the differential equation $M[f] + \beta(z) - \alpha(z) = (f^{\gamma_M} - \alpha(z))e^{P(z)}$, where $\beta(z)$ is an entire function satisfying $\sigma(\beta) < \mu(f)$. Then $\sigma_2(f) = \deg P$. Our result generalizes the results due to Gundersen and Yang, Chang and Zhu and Li and Cao.

Keywords: Entire function; Nevanlinna theory; Differential equations in the complex domain

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1. Introduction

Let $f(z)$ be a non-constant meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory such as $T(r, f), m(r, f), N(r, f)$ (e.g. [3,5,12,13]). By $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to +\infty$, possibly outside a set of r with finite linear measure. A function $\alpha(z)$ is said to be small with respect to $f(z)$ if $\alpha(z)$ is a meromorphic function satisfying $T(r, \alpha) = S(r, f)$ i.e. $T(r, \alpha) = o(T(r, f))$ as $r \to +\infty$.

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Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. For a small function $a(z)$ with respect to both f and g, if the zeros of $f(z)-a(z)$ and $g(z)-a(z)$ coincide in locations and multiplicities we say that $f(z)$ and $g(z)$ share the function $a(z)$ CM (counting multiplicities) and if coincide in locations only we say that $f(z)$ and $g(z)$ share $a(z)$ IM (ignoring multiplicities). Note that $a(z)$ can be a polynomial or a value in $\mathbb{C}\cup\{\infty\}$.

For an entire function $f(z)=\sum_{n=0}^{\infty}a_nz^n$, the central index $\nu(r,f)$ is the greatest exponent m such that $|a_m|\,r^m=\mu(r,f)$, where $\mu(r,f)=\max_{n\geq 0}|a_n|\,r^n$ denote the maximum term of $f$on $|z|=r$. In this paper, we also need the following definitions.

Definition 1. Let $f(z)$ be a non-constant meromorphic function, then the order $\sigma(f)$ of $f(z)$ is defined by

$$\sigma(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to+\infty}\frac{\log\log M(r,f)}{\log r}$$

and the lower order $\mu(f)$ of $f(z)$ is defined by

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}=\liminf_{r\to+\infty}\frac{\log\log M(r,f)}{\log r}$$

where

$$M(r,f)=\max_{|z|=r}|f(z)|\,.$$

Definition 2 *([5])***.** The type $\tau(f)$ of an entire function $f(z)$ with $0<\sigma(f)=\sigma<+\infty$ is defined by

$$\tau(f)=\limsup_{r\to+\infty}\frac{\log M(r,f)}{r^{\sigma}}.$$

Following Yi an Yang [13] we define,

Definition 3. Let f be a non-constant meromorphic function, then the hyper order $\sigma_2(f)$ of $f(z)$ is defined as follows

$$\sigma_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}$$

and finally

Definition 4. Let f be a transcendental meromorphic function in the open complex plane $\mathbb{C}$. A differential monomial is an expression of the form

$$M[f]=(f)^{n_0}\left(f^{(1)}\right)^{n_1}\left(f^{(2)}\right)^{n_2}\cdots\left(f^{(k)}\right)^{n_k}$$

where $n_0,n_1,n_2,\ldots,n_k$ are non negative integers. The degree and weight of the differential monomial are respectively given by $\gamma_M=n_0+n_1+n_2+\cdots+n_k$ and $\Gamma_M=n_0+2n_1+\cdots+(k+1)n_k$.

Rubel and Yang [10] proved that if a non-constant entire function f and its derivative f' share two distinct finite complex numbers CM, then $f=f'$. What is the relation between

f and f', if an entire function f and its derivative f' share one finite complex number CM? Brück [1] made the conjecture that if f is a non-constant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer and if f and f' share one finite complex number a CM, then $f' - a = c(f - a)$ for some finite complex number $c \neq 0$. For the case $a = 0$, the above conjecture had been proved by Brück [1]. Under the assumptions $N(r, 1/f) = S(r, f)$, Brück [1] proved that the conjecture is true provided $a = 0$. In 1998, Gundersen and Yang [4] proved that the conjecture is true for entire functions of finite order. Later in 2009, Chang and Zhu [2] proved that conjecture remains valid if the value a is replaced by a function $a(z)$, provided $\sigma(a) < \sigma(f)$. Also, in 2008 Li and Cao [8] improved Brück's conjecture for entire function and its derivation sharing polynomials and obtained that if f is a non-constant entire solution of the equation $f^{(k)} - Q1 = e^P(f - Q2)$, where $Q1$ and $Q2$ are non-zero polynomials and $P(z)$ be any polynomial, then $\sigma_2(f) =$ degree of P.

Problem. In connection with Brück's conjecture the interesting question that presents itself is: what would happen if $f^{(k)}$ were replaced by a differential monomial and polynomials were replaced by entire functions?

In dealing with the above problem, we shall assume that f is a non-constant entire function such that $M[f]$ and f^{γ_M} share an entire function $\alpha(z)$ with $\sigma(\alpha) \leq \sigma(f)$ thereby improving and generalizing the results of Gundersen and Yang [4], Chang and Zhu [2] and Li and Cao [8].

2. Preparatory lemmas

In this section we state without proofs some lemmas needed in the sequel.

Lemma 1 (*[7]*). *Let $f(z)$ be a transcendental entire function, $\nu(r, f)$ be the central index of $f(z)$. Then there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r, f)$, we get*

$$\frac{f^j(z)}{f(z)} = \left\{\frac{\nu(r, f)}{z}\right\}^j (1 + o(1)), \quad \textit{for } j \in \mathbb{N}.$$

Lemma 2 (*[6]*). *Let $f(z)$ be an entire function of finite order $\sigma(f) = \sigma < +\infty$, and let $\nu(r, f)$ be the central index of f. Then*

$$\limsup_{r \to +\infty} \frac{\log \nu(r, f)}{\log r} = \sigma(f).$$

And if f is a transcendental entire function of hyper order $\sigma_2(f)$, then

$$\limsup_{r \to +\infty} \frac{\log \log \nu(r, f)}{\log r} = \sigma_2(f).$$

Lemma 3 (*[9]*). *Let $f(z)$ be a transcendental entire function and let $E \subset [1, +\infty)$ be a set having finite logarithmic measure. Then there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| =$*

$M(r_n, f)$, $\theta_n \in [0, 2\pi)$, $\lim_{n\to+\infty} \theta_n = \theta_0 \in [0, 2\pi)$, $r_n \notin E$ and if $0 < \sigma(f) < +\infty$, then for any given $\varepsilon > 0$ and sufficiently large r_n,

$$r_n^{\sigma(f)-\varepsilon} < \nu(r_n, f) < r_n^{\sigma(f)+\varepsilon}.$$

If $\sigma(f) = +\infty$, then for any given large $K > 0$ and sufficiently large r_n,

$$\nu(r_n, f) > r_n^K.$$

Lemma 4 ([7]). *Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$ with $b_n \neq 0$ be a polynomial. Then for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r = |z| > r_0$ the inequalities*

$$(1-\varepsilon)|b_n| r^n \leq |P(z)| \leq (1+\varepsilon)|b_n| r^n$$

hold.

Lemma 5 ([11]). *Let $f(z)$ and $A(z)$ be two entire functions with $0 < \sigma(f) = \sigma(A) = \sigma < +\infty$, $0 < \tau(A) = \tau(f) < +\infty$, then there exists a set $E \subset [1, +\infty)$ that has infinite logarithmic measure such that for all $r \in E$ and a positive number $\kappa > 0$, we have*

$$\frac{M(r, A)}{M(r, f)} < \exp\{-\kappa r^{\sigma}\}.$$

3. Statements and proofs of the main theorems

In this section we present the main results of the paper.

Theorem 1. *Let $f(z)$ and $\alpha(z)$ be two non-constant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$. Also, let $P(z)$ be a polynomial. If f is a nonconstant entire solution of the following differential equation*

$$M[f] - \alpha = (f^{\gamma_M} - \alpha) e^{P(z)}, \tag{1}$$

then $P(z)$ is a constant.

Proof. Suppose that $\deg P = m \geq 1$.
Let

$$P(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,$$

where $b_m, b_{m-1}, \ldots, b_0$ are constants and $b_m \neq 0, m \geq 1$. Thus, it follows from (1) and Lemma 4 that

$$|b_m| r^m (1 + o(1)) = |P(z)| = \left| \log \frac{\frac{M[f]}{f^{\gamma_M}} - \frac{\alpha}{f^{\gamma_M}}}{1 - \frac{\alpha}{f^{\gamma_M}}} \right|$$
$$= \left| \log \frac{\frac{M[f]}{f^{\gamma_M}} - \frac{\alpha}{f} \frac{1}{f^{\gamma_M - 1}}}{1 - \frac{\alpha}{f} \frac{1}{f^{\gamma_M - 1}}} \right|. \tag{2}$$

Since

$$
\begin{aligned}
M[f] &= (f)^{n_0}\left(f^{(1)}\right)^{n_1}\left(f^{(2)}\right)^{n_2}\cdots\left(f^{(k)}\right)^{n_k}\\
&= f^{\left(\sum_{j=0}^{k} n_j\right)}\prod_{j=1}^{k}\left(\frac{f^{(j)}}{f}\right)^{n_j}\\
&= f^{\gamma_M}\prod_{j=1}^{k}\left(\frac{f^{(j)}}{f}\right)^{n_j},
\end{aligned}
$$

and from Lemma 1, there exists a subset $E_1 \subset (1,+\infty)$ with finite logarithmic measure, such that for some point $|z| = re^{i\theta}(\theta \in [0,2\pi])$, $r \notin E_1$ and $M(r,f) = |f(z)|$, we have

$$
\frac{f^j(z)}{f(z)} = \left\{\frac{\nu(r,f)}{z}\right\}^j (1+o(1)), \quad 1 \le j \le k.
$$

Thus, it follows that

$$
\begin{aligned}
\frac{M[f]}{f^{\gamma_M}} &= \prod_{j=1}^{k}\left\{\frac{\nu(r,f)}{z}\right\}^{j.n_j}(1+o(1))\\
&= \left\{\frac{\nu(r,f)}{z}\right\}^{\left(\sum_{j=0}^{k} j.n_j\right)}(1+o(1)).
\end{aligned}
\tag{3}
$$

From Lemma 3, there exists $\left\{z_n = r_n e^{i\theta_n}\right\}$ such that $|f(z_n)| = M(r_n,f)$, $\theta_n \in [0,2\pi]$, $\lim_{n\to\infty}\theta_n = \theta_0 \in [0,2\pi]$, $r_n \notin E_1$, then for any given $\varepsilon > 0$ and sufficiently large r_n,

$$
r_n^{\sigma(f)-\varepsilon} < \nu(r_n,f) < r_n^{\sigma(f)+\varepsilon}. \tag{4}
$$

Then, from (3) and (4) we have

$$
\begin{aligned}
\frac{M[f]}{f^{\gamma_M}} &= \left\{\frac{\nu(r_n,f)}{r_n}\right\}^{\left(\sum_{j=0}^{k} j.n_j\right)}.(1+o(1))\\
&< r_n^{(\sigma(f)+\varepsilon-1)\left(\sum_{j=0}^{k} j.n_j\right)}.(1+o(1)).
\end{aligned}
\tag{5}
$$

Since $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$, using Lemma 5, there exists a set $E \subset [1,+\infty)$ that has infinite logarithmic measure such that for a sequence $\{r_n\}_{n=1}^{\infty} \in E_2 = E - E_1$, we have

$$
\frac{M(r_n,\alpha)}{M(r_n,f)} < \exp\left\{-\kappa r_n^{\sigma(f)}\right\} \to 0 \text{ as } n \to +\infty. \tag{6}
$$

From (2), (5) and (6) and Lemma 2, we get that

$$
|b_m|\, r^m (1+o(1)) = |P(z)| = O(\log r_n)
$$

which is impossible. Thus, $P(z)$ is not a polynomial, that is, $P(z)$ is a constant.

Hence, the proof.

Theorem 2. *Let $f(z)$ and $\alpha(z)$ be two non constant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$. Also, let $P(z)$ be a polynomial. If f is a non constant entire solution of the following differential equation.*

$$M[f] + \beta(z) - \alpha(z) = (f^{\gamma_M} - \alpha(z))\, e^{P(z)} \tag{7}$$

where $\beta(z)$ is an entire function satisfying $0 < \sigma(\beta) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\beta)$. Then $P(z)$ is a constant.

Proof. Rewriting (7) as

$$\frac{M[f] + \beta - \alpha}{f^{\gamma_M} - \alpha} = \frac{\frac{M[f]}{f^{\gamma_M}} + \frac{\beta}{f}\frac{1}{f^{\gamma_M - 1}} - \frac{\alpha}{f}\frac{1}{f^{\gamma_M - 1}}}{1 - \frac{\alpha}{f}\frac{1}{f^{\gamma_M - 1}}} = e^{P(z)}.$$

Our assumptions on τ and σ values give, using Lemma 5, that there exists a set $E \subset [1, +\infty)$ that has infinite logarithmic measure such that for a sequence $\{r_n\}_{n=1}^{\infty} \in E_3 = E - E_1$, we have

$$\frac{M(r_n, \alpha)}{M(r_n, f)} < \exp\left\{-\kappa r_n^{\sigma(f)}\right\} \to 0 \text{ as } n \to +\infty$$

and

$$\frac{M(r_n, \beta)}{M(r_n, f)} < \exp\left\{-\kappa r_n^{\sigma(f)}\right\} \to 0 \text{ as } n \to +\infty.$$

The arguments employed in the proof of Theorem 1 show that $P(z)$ is a constant.

Hence, the proof.

Theorem 3. *Let $f(z)$ and $\alpha(z)$ be two non constant entire functions satisfying $\sigma(\alpha) < \mu(f)$ and $P(z)$ be a polynomial. If f is a non constant entire solution of the following differential equation*

$$M[f] + \beta(z) - \alpha(z) = (f^{\gamma_M} - \alpha(z))\, e^{P(z)} \tag{8}$$

where $\beta(z)$ is an entire function satisfying $\sigma(\beta) < \mu(f)$. Then $\sigma_2(f) = \deg P$.

Proof. We will consider two cases (I) $\sigma(f) < +\infty$ and (II) $\sigma(f) = +\infty$.

Case I. Suppose that $\sigma(f) < +\infty$. Then $\sigma_2(f) = 0$. Since $\sigma(\alpha) < \mu(f)$ and $\sigma(\beta) < \mu(f)$, from Definitions of the order and the lower order, there exists infinite sequence $\{r_n\}_{n=1}^{\infty}$, we have

$$\frac{|\alpha(z_n)|}{|f(z_n)|} \to 0, \text{ and } \frac{|\beta(z_n)|}{|f(z_n)|} \to 0 \text{ as } n \to \infty.$$

Thus, by using the same argument as in Theorem 1, we get that $P(z)$ is a constant, that is, $\deg P = 0$. Therefore, $\sigma_2(f) = \deg P$.

Case II. Suppose that $\sigma(f) = +\infty$.

Rewriting (8), we have

$$\frac{\frac{M[f]}{f^{\gamma_M}}+\frac{\beta}{f}\frac{1}{f^{\gamma_M-1}}-\frac{\alpha}{f}\frac{1}{f^{\gamma_M-1}}}{1-\frac{\alpha}{f}\frac{1}{f^{\gamma_M-1}}}=e^{P(z)}.$$

From Lemma 1, there exists a subset $E_4 \subset [1,+\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0,1] \cup E_4$ and $|f(z)| = M(r, f)$, we get

$$\frac{f^j(z)}{f(z)}=\left\{\frac{\nu(r,f)}{z}\right\}^j(1+o(1)),1\le j\le k.$$

Thus, it follows that

$$\frac{M[f]}{f^{\gamma_M}}=\prod_{j=1}^{k}\left\{\frac{\nu(r,f)}{z}\right\}^{j.n_j}(1+o(1))=\left\{\frac{\nu(r,f)}{z}\right\}^{\left(\sum_{j=0}^{k}j.n_j\right)}(1+o(1)) \tag{9}$$

since $\sigma(f) = +\infty$, then it follows from Lemma 3 that there exists $\left\{z_n = r_n e^{i\theta_n}\right\}$ with $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi]$, $\lim_{n\to\infty}\theta_n = \theta_0 \in [0, 2\pi]$, $r_n \notin E_5 \subset [1,+\infty)$, such that for any large constant K and for sufficiently large r_n, we have

$$\nu(r_n,f)\ge r_n^K. \tag{10}$$

Since $\sigma(\alpha) < \mu(f)$ and $\sigma(\beta) < \mu(f)$, from definitions of order and lower order, there exists infinite sequence $\{r_n\}_{n=1}^{\infty}$, we have

$$\frac{|\alpha(z_n)|}{|f(z_n)|}\to 0,\ \text{and}\ \frac{|\beta(z_n)|}{|f(z_n)|}\to 0,\ \text{and as } n\to\infty. \tag{11}$$

Thus, it follows from (8) to (11) that

$$\left\{\frac{\nu(r_n,f)}{z_n}\right\}^{\left(\sum_{j=0}^{k}j.n_j\right)}(1+o(1))=e^{P(z_n)}. \tag{12}$$

Let

$$P(z)=b_m z^m+b_{m-1}z^{m-1}+\cdots+b_0,$$

where $b_m, b_{m-1}, \ldots, b_0$ are constants and $b_m \neq 0$, $m \geq 1$. From Lemma 4, there exists sufficiently large positive number r_0 and $n_0 \in N_+$, such that for sufficiently large positive integer $n > n_0$ satisfying $|z_n| = r_n > r_0$, we have for every $\varepsilon' > 0$

$$\log|b_m|+m\log|z_n|+\log\left|1-\varepsilon'\right|\le\log|P(z_n)|\le\left|\log\log e^{P(z_n)}\right|. \tag{13}$$

It follows from (12) that

$$\begin{aligned}&\left|\log\log e^{P(z_n)}\right|\le\log\log\nu(r_n,f)+\log\log r_n+O(1)\\&\le\log\log\nu(r_n,f)+O(\log\log r_n).\end{aligned} \tag{14}$$

Thus, we have from (13) and (14) and Lemma 2 that

$$m=\deg P\le\sigma_2(f). \tag{15}$$

Also, it follows from (12) that

$$M(r_n, e^{P(z_n)}) \geq \left\{ \frac{\nu(r_n, f)}{r_n} \right\}^{\left(\sum_{j=0}^{k} j.n_j\right)}.$$

Then we have

$$\{\nu(r_n, f)\}^{\left(\sum_{j=0}^{k} j.n_j\right)} \leq (r_n)^{\left(\sum_{j=0}^{k} j.n_j\right)} M\left(r_n, e^{P(z_n)}\right). \tag{16}$$

Thus, it follows from (16) and Lemma 2 that

$$\begin{aligned} \sigma_2(f) &= \limsup_{r_n \to +\infty} \frac{\log\log \nu(r_n, f)}{\log r_n} \\ &= \limsup_{r_n \to +\infty} \frac{\log\log (\nu(r_n, f))^{\left(\sum_{j=0}^{k} j.n_j\right)}}{\log r_n} \\ &\leq \limsup_{r_n \to +\infty} \frac{\log\log (r_n)^{\left(\sum_{j=0}^{k} j.n_j\right)} M\left(r_n, e^{P(z_n)}\right)}{\log r_n} = \sigma\left(e^P\right). \end{aligned} \tag{17}$$

Since $P(z)$ is a polynomial, then $\sigma\left(e^P\right) = \deg P = m$. By combining (15) and (17), we have $\sigma_2(f) = \deg P$.

Corollary 1. *Let $f(z)$ and $\alpha(z)$ be two non constant entire functions and satisfy $\sigma(\alpha) < \mu(f)$. Also, let $P(z)$ be a polynomial. If f is a non constant entire solution of the following differential equation*

$$M[f] - \alpha = (f^{\gamma_M} - \alpha)\, e^{P(z)},$$

then $\sigma_2(f) = \deg P$.

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