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On the range of the generalized Fourier transform associated with a Cherednick type operator on the real line

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Abstract. In this work, we establish the real Paley–Wiener theorem for the generalized Fourier transform on $\mathbb{R}$. Therefore, we study the connection between the real Paley–Wiener theorem and local spectral theory. Finally, we generalize Roe's theorem.

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1. Introduction

In this paper, we consider the first-order singular differential-difference operator on $\mathbb{R}$

$$\Lambda f(x) = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x) - f(-x)}{2}\right) - \rho f(-x), \tag{1.1}$$

where

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > \frac{-1}{2},$$

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B being a positive C^{∞} even function on $\mathbb{R}$, with $B(0) = 1$ and $\rho > 0$. In addition we suppose that

(i) For all $x \geqslant 0$, $A(x)$ is increasing and $\lim_{x\to+\infty} A(x) = +\infty$.
(ii) For all $x > 0$, $\frac{A'(x)}{A(x)}$ is decreasing and $\lim_{x\to+\infty} \frac{A'(x)}{A(x)} = 2\rho$.

For

$$A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1},\ \rho = \alpha+\beta+1, \quad \alpha \geqslant \beta > \frac{-1}{2},$$

we regain the differential-difference operator

$$T_{(\alpha,\beta)}f(x) = f'(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\frac{f(x)-f(-x)}{2} - \rho f(-x).$$

In Cherednik's notation, $T_{(\alpha,\beta)}$ is written as

$$T_{(\alpha,\beta)}f(x) = f'(x) + \left[\frac{2k_1}{1-e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\right] [f(x) - f(-x)] - (k_1+2k_2)f(x),$$

with $\alpha = k_1 + k_2 - \frac{1}{2}$ and $\beta = k_2 - \frac{1}{2}$. For recent results and more details in this direction we refer to [6,7,9,10,13,15].

For each $\lambda \in \mathbb{C}$, the differential-difference equation

$$\Lambda u = i\lambda u, \quad u(0) = 1, \tag{1.2}$$

admits a unique C^{∞} solution on $\mathbb{R}$, denoted by Φ_λ and given by

$$\Phi_\lambda(x) = \begin{cases} \varphi_\lambda(x) + \frac{1}{i\lambda-\rho}\frac{d}{dx}\varphi_\lambda(x) & \text{if} \quad \lambda \neq -i\rho, \\ 1 + \frac{2\rho}{A(x)}\int_0^x A(t)\,dt & \text{if} \quad \lambda = -i\rho, \end{cases}$$

where φ_λ is the eigenfunction of the second order singular differential operator Δ on $]0,+\infty[$:

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}. \tag{1.3}$$

The function φ_λ satisfies the following properties:

(i) For every $x \in [0,+\infty[$, the function $\lambda \to \varphi_\lambda(x)$ is even and entire on $\mathbb{C}$.
(ii) For all $x \in [0,+\infty[$ and $\lambda \in \mathbb{C}$ such that $|Im\lambda| \leqslant \rho$, we have $|\varphi_\lambda(x)| \leqslant 1$.
(iii) We have

$$\forall x \geqslant 0, \forall \lambda \in \mathbb{R}, |\varphi_\lambda(x)| \leqslant \varphi_0(x), \tag{1.4}$$

and there exists a positive constant c_0 such that

$$\forall x \geqslant 0, \quad 0 < \varphi_0(x) \leqslant c_0(1+x)e^{-\rho x}. \tag{1.5}$$

Furthermore, for every $x \in \mathbb{R}$, the function $\lambda \to \Phi_\lambda(x)$ is entire on $\mathbb{C}$, and there exists a positive constant M such that

$$|\Phi_\lambda(x)| \leqslant M(1+|x|)(1+|\lambda|)e^{-\rho|x|}, \quad \forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}. \tag{1.6}$$

For more details see [8,16].

Remark. If $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $\alpha \geqslant \beta \geqslant \frac{-1}{2}$ and $\alpha \neq \frac{-1}{2}$ then the functions $\Phi_\lambda^{(\alpha,\beta)}$ are closely related to the Jacobi functions. Specifically

$$\Phi_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{(\alpha,\beta)}(x) - \frac{1}{\rho - i\lambda}\frac{d}{dx}\varphi_\lambda^{(\alpha,\beta)}(x), \tag{1.7}$$

where $\varphi_\lambda^{(\alpha,\beta)}(x) = F_{21}\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x\right)$, (see [3, p. 3]) and references there.

The Paley–Wiener theorem [11] for functions is one of most useful theorems in harmonic analysis. This theorem has as aim to characterize functions with compact support through the properties of the analytic extensions of their classical Fourier transform. Recently there has been a great interest to characterize the space of functions who's Fourier transform has compact support in several situations. They have become known as real Paley Wiener theorems, in which the adjective real expresses that information about the support of f comes from growth rates associated to the function $\mathcal{F}f$ on $\mathbb{R}^d$. We refer to the survey of Andersen and de Jeu [2] and references there.

The set-up is as follows. Let f be a C^∞-function on $\mathbb{R}$ such that for all $n \in \mathbb{R}$, the function $\frac{d^n f}{dx^n}$ belongs to the Lebesgue space $L^p(\mathbb{R})$, then the limit $R_f := \lim_{n\to\infty}\left\|\frac{d^n f}{dx^n}\right\|_p^{\frac{1}{n}}$ exists in $[0, +\infty]$ and we have

$$R_f = \sup\{|\lambda|, \lambda \in \operatorname{supp}\mathcal{F}(f)\},$$

where $\mathcal{F}(f)$ is the classical Fourier transform of f. This result is due to Bang [4]. It was established for many other integral transforms in [1,2].

This paper is organized as follows. In Section 2, the necessary notations and previous results are given. Section 3, we prove the real Paley–Wiener theorem for the generalized Schwartz spaces and for the L^p_A-functions. Finally, in the last section we give the analogue of Roe's theorem for generalized Fourier transform.

2. Notations and backgrounds

In this section we give an introduction to the harmonic analysis associated with the differential-difference operator and some notations. In the following we denote by:

- $\mathcal{D}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$ with compact support. We have

$$\mathcal{D}(\mathbb{R}) = \bigcup_{a>0}\mathcal{D}_a(\mathbb{R}),$$

where $\mathcal{D}_a(\mathbb{R})$ is the space of C^∞ functions on $\mathbb{R}$ with support in $[-a, a]$. The topology on $\mathcal{D}_a(\mathbb{R})$ is defined by the semi-norms

$$p_n(\psi) = \sup_{k\leqslant n, x\in[-a,a]} |\psi^{(k)}(x)|, \quad n \in \mathbb{N}.$$

The space $\mathcal{D}(\mathbb{R})$ is equipped with the inductive limit topology.

- $\mathcal{E}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$. Its topology is defined by the semi-norms

$$q_{n,k}(\phi) = \sup_{k \leqslant n, x \in K} |\phi^{(k)}(x)|, \quad n \in \mathbb{N},$$

where K is a compact subset of $\mathbb{R}$ and $n \in \mathbb{N}$.

- $\mathcal{S}(\mathbb{R})$ the classical Schwartz space on $\mathbb{R}$. The topology of this space is given by the semi-norms

$$v_{l,n}(f) = \sup_{k \leqslant n, x \in \mathbb{R}} (1 + |x|)^l |f^{(k)}(x)|.$$

- $\mathcal{S}_2(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$, such that for all $l, n \in \mathbb{N}$,

$$\sigma_{l,n}(f) = \sup_{k \leqslant n, x \in \mathbb{R}} (1 + |x|)^l \varphi_0^{-1}(x) |f^{(k)}(x)| < +\infty.$$

Its topology is defined by the semi-norms $\sigma_{l,n}$, $l, n \in \mathbb{N}$.

- $Pw_a(\mathbb{C})$ the space consists of all entire functions h on $\mathbb{C}$ which satisfy

$$\forall m \in \mathbb{N},\ P_m(h) = \sup_{\lambda \in \mathbb{C}} (1 + |\lambda|)^m e^{-a|Im\,\lambda|} |h(\lambda)| < +\infty.$$

The topology on $Pw_a(\mathbb{C})$ is defined by the semi-norms $P_m, m \in \mathbb{N}$.We set

$$Pw(\mathbb{C}) = \bigcup_{a>0} Pw_a(\mathbb{C}).$$

This space called Paley–Wiener space is equipped with the inductive limit topology.

- $\mathcal{D}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$. It is the topological dual space of $\mathcal{D}(\mathbb{R})$.
- $\mathcal{S}'_2(\mathbb{R})$ the topological dual space of $\mathcal{S}_2(\mathbb{R})$.

In the following we give some properties and we recall some results associated with the operator Λ.

Lemma 2.1. *For all f in $\mathcal{E}(\mathbb{R})$ and all g in $\mathcal{D}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \Lambda f(x) g(x) A(x)\, dx = -\int_{\mathbb{R}} f(x)[\Lambda g(x) + 2\rho Sg(x)]\, A(x)\, dx, \tag{2.1}$$

where S is the operator defined by $Sg(x) = g(-x)$. Moreover, for all $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} \Lambda^n f(x) g(x) A(x)\, dx = (-1)^n \int_{\mathbb{R}} f(x)[\Lambda^n g(x) + 2\bar{n}\rho \Lambda^{n-1} Sg(x)]\, A(x)\, dx, \tag{2.2}$$

where $\bar{n} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Notation. We denote by

$$T := \Lambda + 2\rho S.$$

Remark. We can see by a direct calculation that for all n in $\mathbb{N}$, we have $T^{2n} = \Lambda^{2n}$.

For all $1 \leqslant p \leqslant \infty$, we denote by $L^p_A(\mathbb{R})$, the space of measurable functions f is such that

$$\begin{cases} \|f\|_{p,A}^p = \int_{\mathbb{R}} |f(x)|^p A(x)dx < \infty, & \text{if } 1 \leqslant p < \infty, \\ \|f\|_\infty = \underset{x\in\mathbb{R}}{\text{ess sup}}|f(x)|, & \text{if } p = \infty. \end{cases}$$

In the sequel, we give some results about the generalized Fourier transform.

Definition 2.1. [8] The generalized Fourier transform of a function f in $\mathcal{D}(\mathbb{R})$ is defined by

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-\lambda}(x)A(x)\,dx, \quad \lambda \in \mathbb{C}. \tag{2.3}$$

Theorem 2.2. [8]

(i) *For all f in $\mathcal{D}(\mathbb{R})$,*

$$\mathcal{F}Tf(\lambda) = i\lambda\mathcal{F}f(\lambda). \tag{2.4}$$

(ii) *For all f in $\mathcal{D}(\mathbb{R})$,*

$$\mathcal{F}f(\lambda) = \mathcal{F}_\Delta(f_e)(\lambda) + (i\lambda - \rho)\mathcal{F}_\Delta Jf_o(\lambda), \tag{2.5}$$

where f_e (resp f_o) denotes the even (resp odd) part of f, $\mathcal{F}_\Delta$ stands for the Fourier transform related to the differential operator Δ as donated by (1.3), defined on the subspace of $\mathcal{D}(\mathbb{R})$ consisting of even functions by

$$\mathcal{F}_\Delta h(\lambda) = \int_{\mathbb{R}} h(x)\varphi_\lambda(x)A(x)\,dx, \quad \lambda \in \mathbb{C},$$

and

$$Jf_o(x) := \int_{-\infty}^{x} f_o(t)\,dt.$$

Theorem 2.3. [8] *For all $f \in \mathcal{D}(\mathbb{R})$,*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(\lambda)\Phi_{-\lambda}(-x)d\sigma_1(\lambda) + \int_{-\rho}^{\rho} \mathcal{F}f(i\lambda)\Phi_{-i\lambda}(-x)d\sigma_2(\lambda), \tag{2.6}$$

where

$$d\sigma_1(\lambda) = \left(1 - \frac{i\rho}{\lambda}\right) d\mu_1(\lambda) \,\text{and}\, d\sigma_2(\lambda) = \left(1 - \frac{\rho}{\lambda}\right) d\mu_2(\lambda), \tag{2.7}$$

where μ_1 is an even positive tempered measure on $\mathbb{R}$ and μ_2 is positive measure on $\mathbb{R}$ with support in $[-\rho, \rho]$.

Remark. [8]

(i) The pair (μ_1, μ_2) is called the spectral measure associated with the differential operator Δ.

(ii) For $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}, \quad \alpha \geqslant \beta > \frac{-1}{2}$, we have

$$\mu_1(d\lambda) = \frac{d\lambda}{|c(\lambda)|^2} \text{ and } \mu_2 = 0,$$

where $c(\lambda) = \frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\left(\frac{\rho+i\lambda}{2}\right)\Gamma\left(\alpha+1-\frac{\rho-i\lambda}{2}\right)}$, $\lambda \in \mathbb{C} \setminus \{i\mathbb{Z}\}$.

Theorem 2.4. *The generalized Fourier transform $\mathcal{F}$ is a topological isomorphism from*

(i) $\mathcal{D}(\mathbb{R})$ *onto* $Pw(\mathbb{C})$, *([8]).*
(ii) $\mathcal{S}_2(\mathbb{R})$ *onto* $\mathcal{S}(\mathbb{R})$, *([16]).*

In the sequel we give the essential properties of the generalized convolution product which are developed in the paper [8].

Definition 2.2. The generalized translation operators ${}^t\tau^x$, $x \in \mathbb{R}$, are defined on $\mathcal{S}_2(\mathbb{R})$, by the relation

$$\mathcal{F}({}^t\tau^x f)(\lambda) = \Phi_{-\lambda}(x)\mathcal{F}f(\lambda), \quad \lambda \in \mathbb{C}.$$

Remark.

(i) From the relation (2.6) we deduce that for all x, y in $\mathbb{R}$, we have
$${}^t\tau^x f(y) = {}^t\tau^{-y} f(-x).$$
(ii) From the definition above and the relations (1.6) and (2.6), there exists a positive constant C such that for all $x, y \in \mathbb{R}$ and all $\lambda \in \mathbb{C}$, we have
$$|{}^t\tau^x f(y)| \leqslant C(1+|x|)(1+|y|)e^{-\rho|x|}e^{-\rho|y|}. \tag{2.8}$$

Theorem 2.5. [8] *For f in $\mathcal{S}_2(\mathbb{R})$, the function $v(x, y) = {}^t\tau^{-x}(f)(y)$ is the unique solution of the problem*

$$\begin{cases} T_x v(x, y) = T_y v(x, y), \\ v(0, y) = f(y). \end{cases}$$

Proposition 2.6. *For all f in $\mathcal{S}_2(\mathbb{R})$ and y in $\mathbb{R}$, we have ${}^t\tau^y f$ belongs to $\mathcal{S}_2(\mathbb{R})$ and for all integer n, we have*

$$T^n\, {}^t\tau^y f = {}^t\tau^y\, T^n f. \tag{2.9}$$

Proof. From the Definition 2.2, the relation (1.6) and (ii) of the Theorem 2.4, we conclude that ${}^t\tau^y f$ belongs to $\mathcal{S}_2(\mathbb{R})$ for all f in $\mathcal{S}_2(\mathbb{R})$ and all y in $\mathbb{R}$. Moreover, for all y in $\mathbb{R}$ and λ in $\mathbb{C}$, we have

$$\mathcal{F}(T\,{}^t\tau^y f)(\lambda) = i\lambda\Phi_{-\lambda}(y)\mathcal{F}f(\lambda) = \mathcal{F}({}^t\tau^y\, Tf)(\lambda),$$

then the relation (2.9) follows easily from the injectivity of the generalized fourier transform. □

Definition 2.3. For f in $\mathcal{D}(\mathbb{R})$ and g in $\mathcal{E}(\mathbb{R})$, the generalized convolution product $f\#g$ is defined by

$$f\#g(x) = \int_{\mathbb{R}} ({}^t\tau^y f)(x)g(y)A(y)dy, \quad x \in \mathbb{R}.$$

Proposition 2.7. *Let f and g in $\mathcal{S}_2(\mathbb{R})$ then*

$$\mathcal{F}(f\#g) = \mathcal{F}(f)\mathcal{F}(g). \tag{2.10}$$

Proposition 2.8. *For all f, g in $\mathcal{D}(\mathbb{R})$ (resp $\mathcal{S}_2(\mathbb{R})$) we have*

$$\int_{\mathbb{R}} f(-x)g(x)A(x)dx = \int_{\mathbb{R}} \mathcal{F}f(\lambda)\mathcal{F}g(\lambda)\sigma_1(d\lambda) + \int_{-\rho}^{\rho} \mathcal{F}f(i\lambda)\mathcal{F}g(i\lambda)\sigma_2(d\lambda). \tag{2.11}$$

Proof. We have,

$$f\#g(0) = \int_{\mathbb{R}} {}^t\tau^y f(0)g(y)A(y)dy = \int_{\mathbb{R}} f(-y)g(y)A(y)dy.$$

Moreover, using the relations (2.6) and (2.10), we get

$$f\#g(0) = \int_{\mathbb{R}} \mathcal{F}f(\lambda)\mathcal{F}g(\lambda)\sigma_1(d\lambda) + \int_{-\rho}^{\rho} \mathcal{F}f(i\lambda)\mathcal{F}g(i\lambda)\sigma_2(d\lambda). \quad \square$$

Lemma 2.9. *For all $f, g, h \in \mathcal{D}(\mathbb{R})$ (resp $\mathcal{S}_2(\mathbb{R})$), we have*

(i) $$\int_{\mathbb{R}} (f\#g)(x)\, h(-x)A(x)dx = \int_{\mathbb{R}} f(x)\,(g\#h)(-x)A(x)dx, \tag{2.12}$$

(ii) $$T(f\#g)(x) = (Tf)\#g(x) = f\#(Tg)(x). \tag{2.13}$$

Proof.

(i) The result follows from Proposition 2.8.

(ii) We have

$$\begin{aligned}\mathcal{F}(T(f\#g))(\lambda) &= i\lambda\mathcal{F}(f\#g)(\lambda) = i\lambda\mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda), = \mathcal{F}(Tf)(\lambda)\mathcal{F}(g)(\lambda)\\ &= \mathcal{F}(f\#(Tg))(\lambda). \quad \square\end{aligned}$$

Lemma 2.10. *For $f, g \in \mathcal{D}$ (resp. $\mathcal{S}_2$), for all $n \in \mathbb{N}$, we have*

$$T^n(f\#g) = (T^nf)\#g = f\#(T^ng), \tag{2.14}$$

$$T^nf = \mathcal{F}^{-1}(\mathcal{F}T^nf) = \mathcal{F}^{-1}(P_n\mathcal{F}f), \tag{2.15}$$

where $P_n(x) = (ix)^n$.

Definition 2.4.

(i) We define the distributional generalized Fourier transform $\mathcal{F}_d$ on $L^p_A(\mathbb{R})$ for all $1 \leqslant p \leq 2$, by transposition

$$\langle \mathcal{F}_d f, \phi\rangle := \langle f, \mathcal{F}^{-1}\phi\rangle = \int_{\mathbb{R}} f(x)\mathcal{F}^{-1}\phi(x)A(x)dx, \ \forall \phi \in \mathcal{S}(\mathbb{R}), \ \forall f \in L^p_A(\mathbb{R}). \tag{2.16}$$

In other words,

$$\langle \mathcal{F}_d f, \mathcal{F}\phi\rangle = \langle f, \phi\rangle. \tag{2.17}$$

(ii) The generalized Fourier transform of a distribution S in $\mathcal{S}'_2(\mathbb{R})$ is defined by

$$< \mathcal{F}(S), \psi > = < S, \mathcal{F}^{-1}\psi >, \quad \psi \in \mathcal{S}(\mathbb{R}). \tag{2.18}$$

Remark. We have from [5], $\mathcal{S}_2(\mathbb{R}) \subset L^p_A(\mathbb{R})$, for all $p \geqslant 2$.

3. The real Paley–Wiener theorem

We will now consider the real Paley–Wiener theorem for L^p-functions in the spirit of Bang [4]. We define the real Paley–Wiener space $PW_R(\mathbb{R})$ as the space of all f in $\mathcal{E}(\mathbb{R})$ such that, for all N in $\mathbb{N}$,

$$\sup_{x\in\mathbb{R}, n\in\mathbb{N}} R^{-n}n^{-M}e^{\rho|x|}(1+|x|)^N|T^nf(x)| < \infty, \tag{3.1}$$

where $M = M(N)$ is a positive integer depending on N.

Theorem 3.1. *Let* $R > 0$. *The generalized Fourier transform* $\mathcal{F}$ *is a bijection from* $PW_R(\mathbb{R})$ *onto* $\mathcal{D}_R(\mathbb{R})$.

Proof. Let $f \in PW_R(\mathbb{R})$ and consider λ outside $[-R, R]$. Then, we have with $N \geqslant 2$,

$$\begin{aligned}|\mathcal{F}f(\lambda)| &= |\lambda^{-n}\mathcal{F}(T^nf)(\lambda)| \\ &\leqslant |\lambda^{-n}| \textstyle\int_{\mathbb{R}} |T^nf(x)|(1+|\lambda|)(1+|x|)e^{-\rho|x|}A(x)dx \\ &\leqslant C\left(\tfrac{R}{|\lambda|}\right)^n n^M(1+|\lambda|)\textstyle\int_{\mathbb{R}}(1+|x|)^{1-N}dx \to 0, \quad \text{as } n\to\infty,\end{aligned}$$

and thus $\mathrm{supp}\mathcal{F}f \subset [-R, R]$.

Assume conversely that $\mathcal{F}f$ has compact support in $[-R, R]$, for $f \in \mathcal{S}_2(\mathbb{R})$. We fix $N \in \mathbb{N}$, then we have

$$|e^{\rho|x|}(1+|x|)^N T^n f(x)| = |e^{\rho|x|}(1+|x|)^N \mathcal{F}^{-1}(P_n \mathcal{F} f)(x)|$$
$$\leqslant \sup_x e^{\rho|x|}(1+|x|)^N |\mathcal{F}^{-1}(P_n \mathcal{F} f)(x)|.$$

Using the fact that $\mathcal{F}^{-1}$ is a topological isomorphism from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}_2(\mathbb{R})$, we obtain

$$|e^{\rho|x|}(1+|x|)^N T^n f(x)| \leqslant c \sup_\lambda \sum_{1\leqslant k,l\leqslant M} (1+|\lambda|)^k \left| \frac{d^l}{d\lambda^l}(\lambda^n \mathcal{F} f(\lambda)) \right|,$$

for a positive constant c and a positive integer M, depending only on N (independent of n). Leibniz's rule yields

$$\frac{d^l}{d\lambda^l}(\lambda^n \mathcal{F} f(\lambda)) = \sum_{j=0}^{l} \binom{l}{j} \frac{n!}{(n-j)!} \lambda^{n-j} \left(\frac{d}{d\lambda}\right)^{l-j} \mathcal{F} f(\lambda). \tag{3.2}$$

Using the estimates

$$\sum_{j=0}^{l} \binom{l}{j} \frac{n!}{(n-j)!} \leqslant M M! n^M \quad \text{and} \quad R^{n-j} \leqslant \left(1+\frac{1}{R}\right)^M R^n,$$

we get

$$|e^{\rho|x|}(1+|x|)^N T^n f(x)| \leqslant c n^M M M! \left(1+\frac{1}{R}\right)^M R^n,$$

for a positive constant c, independent of n, and we see that f belongs to $PW_R(\mathbb{R})$. □

For f in $L^p_A(\mathbb{R})$, $1 \leqslant p \leq 2$, we define R_f the radius of distributional support of $\mathcal{F}_d f$, as

$$R_f := \sup\{|\lambda| : \lambda \in \operatorname{supp} \mathcal{F}_d f\}.$$

We note that for all $f \in \mathcal{S}_2(\mathbb{R})$, $\operatorname{supp} \mathcal{F}_d f = \operatorname{supp} \mathcal{F} f$.

Theorem 3.2. *Let f in $\mathcal{E}(\mathbb{R})$ be such that $T^n f$ in $L^p_A(\mathbb{R})$, for all $n \in \mathbb{N}$ and all $1 \leqslant p \leq 2$. Then we have*

$$\lim_{n\to\infty} \|T^n f\|_{p,A}^{\frac{1}{n}} = \sup\{|\lambda| : \lambda \in \operatorname{supp} \mathcal{F}_d f\}. \tag{3.3}$$

Proof. Let f be as in the theorem and such that supp $\mathcal{F}_d f \subset [-R, R]$, for some finite $R > 0$. To this end, choose $\varepsilon > 0$ and fix a function $\phi \in \mathcal{S}_2(\mathbb{R})$ such that $\mathcal{F}\phi = 1$ on $[-R_f, R_f]$ and $\mathcal{F}\phi = 0$ outside $[-R_f - \varepsilon, R_f + \varepsilon]$. From the Proposition 2.7 and Remark 2, we have for $\psi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}
\langle T^{2n} f, \psi \rangle &= \langle \mathcal{F}_d(T^{2n} f), \mathcal{F}\psi \rangle = \langle \mathcal{F}_d(T^{2n} f), \mathcal{F}\phi \mathcal{F}\psi \rangle, \\
&= \langle T^{2n} f, \phi \# \psi \rangle = \langle f, T^{2n} \phi \# \psi \rangle, \\
&= \int_{\mathbb{R}} f(x) \left(\int_{\mathbb{R}} ({}^t\tau^y T^{2n} \phi)(x) \psi(y) A(y) dy \right) A(x) dx, \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) ({}^t\tau^{-x} T^{2n} \phi)(-y) A(x) dx \right) \psi(y) A(y) dy,
\end{aligned}$$

As supp $\mathcal{F}\phi \subset [-R_f - \varepsilon, R_f + \varepsilon]$, then from the Theorem 3.1, the Definition 2.2, the inversion formula for the generalized fourier transform and the relations (1.6) and (2.8), we deduce that there exists a positive constant M such that for all n, N in $\mathbb{N}$ and $x, y \in \mathbb{R}$, we have

$$|({}^t\tau^{-x} T^{2n}\phi)(-y)| \leqslant M(1+|y|)^{-N} e^{-\rho|x|} e^{-\rho|y|} (R_f + \varepsilon)^{2n} \tag{3.4}$$

Therefore, using Hölder's inequality and the last relation, we have for $\psi \in \mathcal{D}(\mathbb{R})$, $|\langle T^{2n}f, \psi\rangle| \leqslant | \int_{\mathbb{R}}$

$$\begin{aligned}
&[\int_{\mathbb{R}} f(x)({}^t\tau^{-x}T^{2n}\phi)(-y)A(x)dx]\psi(y)A(y)dy| \\
&\leqslant \|f\|_{p,A} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |({}^t\tau^{-x}T^{2n}\phi)(-y)|^{p'} A(x)\,dx\right]^{\frac{1}{p'}} |\psi(y)|\,A(y)dy \\
&\leqslant M\|f\|_{p,A} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-\rho p'(|x|+|y|)} (R_f+\varepsilon)^{2np'} A(x)\,dx\right]^{\frac{1}{p'}} |\psi(y)|\,A(y)dy \\
&\leqslant M'(R_f+\varepsilon)^{2n} \|f\|_{p,A} \int_{\mathbb{R}} e^{-\rho|y|}|\psi(y)|\,A(y)dy \\
&\leqslant M''(R_f+\varepsilon)^{2n} \|f\|_{p,A} \|\psi\|_{p',A}.
\end{aligned}$$

where p' is the conjugate exponent of p. Moreover

$$\begin{aligned}
\langle T^{2n+1}f, \psi\rangle &= \langle \mathcal{F}_d(T^{2n+1}f), \mathcal{F}\psi\rangle = \langle \mathcal{F}_d(T^{2n+1}f), \mathcal{F}\phi\mathcal{F}\psi\rangle \\
&= \langle T^{2n+1}f, \phi\#\psi\rangle = \langle Tf, T^{2n}\phi\#\psi\rangle = \langle (\check{(Tf)}\#T^{2n}\phi)\check{}, \psi\rangle.
\end{aligned}$$

Therefore, using Hölder's inequality and the relation (3.4), we have for $\psi \in \mathcal{D}(\mathbb{R})$,

$$|\langle T^{2n+1}f, \psi\rangle| \leqslant M'(R_f+\varepsilon)^{2n} \|Tf\|_{p,A} \|\psi\|_{p',A}.$$

we conclude that

$$\limsup_{n\to\infty} \|T^n f\|_{p,A}^{\frac{1}{n}} \leqslant R_f + \varepsilon.$$

Therefore,

$$\limsup_{n\to\infty} \|T^n f\|_{p,A}^{\frac{1}{n}} \leqslant R_f. \tag{3.5}$$

Now, consider an arbitrary $f \in \mathcal{E}(\mathbb{R})$ such that for all $1 \leqslant p \leq 2$ and all $n \in \mathbb{N}$, $T^n f \in L^p_A(\mathbb{R})$ and let $0 \neq \lambda_0 \in \operatorname{supp} \mathcal{F}_d f$. Choose $\varepsilon > 0$ such that $0 < 2\varepsilon < |\lambda_0|$, $\psi \in \mathcal{D}(\mathbb{R})$ with supp $\psi \subset [|\lambda_0| - \varepsilon, |\lambda_0| + \varepsilon]$ and $\langle \mathcal{F}_d f, \psi\rangle \neq 0$. Define

$$\lambda^{2n}\psi_{2n}(\lambda) := (\lambda_0 - 2\varepsilon)^{2n}\psi(\lambda).$$

Using Hölder's inequality, we get

$$\begin{aligned}
(\lambda_0 - 2\varepsilon)^{2n}|\langle \mathcal{F}_d f, \psi\rangle| &= |\langle \mathcal{F}_d f, \mathcal{F}(T^{2n}\mathcal{F}^{-1}(\psi_{2n}))\rangle| = |\langle f, T^{2n}(\mathcal{F}^{-1}\psi_{2n})\rangle| \\
&= |\langle T^{2n}f, \mathcal{F}^{-1}\psi_{2n}\rangle| \leqslant \|T^{2n}f\|_{p,A}\|\mathcal{F}^{-1}\psi_{2n}\|_{p',A} \leqslant c\|T^{2n}f\|_{p,A}.
\end{aligned}$$

However, we have

$$(|\lambda_0| - 2\varepsilon) \leqslant |\lambda_0 - 2\varepsilon| = \lim_{n\to\infty}[(\lambda_0 - 2\varepsilon)^{2n}|\langle \mathcal{F}_d f, \psi\rangle|]^{\frac{1}{2n}}$$
$$\leqslant \liminf_{n\to\infty} c^{\frac{1}{2n}}\|T^{2n}f\|_{p,A}^{\frac{1}{2n}}$$
$$= \liminf_{n\to\infty}\|T^{2n}f\|_{p,A}^{\frac{1}{2n}},$$

where p' is the conjugate exponent of p. We conclude that, for any $\lambda_0 \in \mathrm{supp}\mathcal{F}_d f$,

$$|\lambda_0| \leqslant \liminf_{n\to\infty} \|^{2n}f\|_{p,A}^{\frac{1}{2n}}.$$

Now, choose $\varepsilon > 0$ such that $0 < 2\varepsilon < |\lambda_0|$, and choose $\psi \in \mathcal{D}(\mathbb{R})$ such that supp $\psi \subset [|\lambda_0| - \varepsilon, |\lambda_0| + \varepsilon]$ and $\langle \mathcal{F}_d(Tf), \psi\rangle \neq 0$. Define

$$\lambda^{2n}\psi_{2n}(\lambda) := (\lambda_0 - 2\varepsilon)^{2n}\psi(\lambda).$$

We have,

$$(\lambda_0 - 2\varepsilon)^{2n}|\langle \mathcal{F}_d\ (Tf), \psi\rangle| = |\langle \mathcal{F}_d(Tf), \mathcal{F}(T^{2n}\mathcal{F}^{-1}(\psi_{2n}))\rangle|$$
$$= |\langle Tf, T^{2n}(\mathcal{F}^{-1}\psi_{2n})\rangle| = |\langle T^{2n+1}f, \mathcal{F}^{-1}\psi_{2n}\rangle|$$
$$\leqslant \|T^{2n+1}f\|_{p,A}\|\mathcal{F}^{-1}\psi_{2n}\|_{p',A} \leqslant C\|T^{2n+1}f\|_{p,A}.$$

We conclude that, for any $\lambda_0 \in \mathrm{supp}\mathcal{F}_d f$,

$$|\lambda_0| \leqslant \liminf_{n\to\infty} \|T^{2n+1}f\|_{p,A}^{\frac{1}{2n+1}}.$$

These two estimates together yield

$$|\lambda_0| \leqslant \liminf_{n\to\infty} \|T^n f\|_{p,A}^{\frac{1}{n}}$$

and the theorem follows. $\square$

4. Roe's theorem associated with the differential-difference operator

Studying the classical Fourier transform, J. Roe has proved in [12], the following main result.

Theorem 4.1. *If a function and all its derivatives and integrals are absolutely uniformly bounded, then the function is a sine function with period* 2π.

In [14], the author looks at a theorem of Roe in n-dimensional setting, and in place of derivatives and anti derivatives, he uses powers of the Laplacian.

Inspired by this work, we shall prove in this section an analogue result for the differential-difference operator T.

Theorem 4.2. *Suppose* $P(\xi) = \sum_n a_n \xi^n$ *be a polynomial with complex values. Let* $a \geqslant 0$, $\delta > 2$ *and let* $\{f_j\}_{-\infty}^{+\infty}$ *be a sequence of complex-valued functions on* $\mathbb{R}$ *satisfying the two following conditions*

$$P(-iT)f_j = f_{j+1}, \tag{4.1}$$

$$|f_j(x)| \leqslant M_j(1+|x|)^a e^{-\delta\rho|x|}, \tag{4.2}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the sublinear growth condition*

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{4.3}$$

Then $f_0 = f_+ + f_-$ *where* $P(-iT)f_+ = f_+$ *and* $P(-iT)f_- = -f_-$. *If 1 (or (-1)) is not in the range of* P *then* $f_+ = 0$ *(or* $f_- = 0$*).*

To prove this theorem we need the following Lemmas.

Lemma 4.3. *Let* $(f_j)_{j\in\mathbb{Z}}$ *be a sequence of functions on* $\mathbb{R}$ *satisfying*

$$P(-iT)f_j = f_{j+1}, \tag{4.4}$$

$$|f_j(x)| \leqslant M_j(1+|x|)^a e^{-\delta\rho|x|}, \tag{4.5}$$

and

$$\lim_{j\to\infty} \frac{M_j}{(1+\varepsilon)^j} = 0, \tag{4.6}$$

for all $\varepsilon > 0$, *then*

$$\text{supp}(\mathcal{F}_d\ (f_0)) \subset S := \{\xi, |P(\xi)| = 1\}.$$

Proof. From the relation (4.5), we see that f_0 belongs to $L^p_A(\mathbb{R})$ for all $1 \leqslant p \leqslant 2$. At first we show that $\mathcal{F}_d f_0$ is supported in $\{\xi, |P(\xi)| \leqslant 1\}$. To do this we need to show that $< \mathcal{F}_d\ (f_0), \phi >= 0$ if $\phi \in D(\mathbb{R})$ and supp $\phi \cap \{\xi, |P(\xi)| \leqslant 1\} = \emptyset$. Since supp$(\phi)$ is compact, there is some $r < 1$ so that $\frac{1}{|P(\xi)|} \leqslant r$, for all $\xi \in$supp(ϕ). Then

$$\begin{aligned} < \mathcal{F}_d f_0, \phi > &=< P^j \mathcal{F}_d\ (f_0), \tfrac{\phi}{P^j} >, \\ &=< \mathcal{F}_d(P(-iT^j f_0)), \tfrac{\phi}{P^j} >, \\ &=< P(-iT^j f_0), \mathcal{F}^{-1}\left(\tfrac{\phi}{P^j}\right) > . \end{aligned}$$

Choose an integer m with $2m \geqslant a+1$. A direct calculation, using the hypothesis of the Lemma and Cauchy-Schwartz inequality, implies

$$\begin{aligned} | < \mathcal{F}_d f_0, \phi > | &\leqslant \textstyle\int_{\mathbb{R}} |P(-iT)^j f_0(x)||\mathcal{F}^{-1}\left(\tfrac{\phi}{P^j}\right)(x)|A(x)dx, \\ &\leqslant CM_j \left\| e^{\rho|x|}(1+x^2)^m \mathcal{F}^{-1}\left(\tfrac{\phi}{P^j}\right)(x)\right\|_\infty. \end{aligned}$$

From the continuity of $\mathcal{F}^{-1}$ and the fact that ϕ is supported in $\{\xi, |P(\xi)| \geqslant 1+\varepsilon\}$ for some fixed $\varepsilon > 0$, it is not hard to prove that the right-hand side of this goes to zero as $j \to \infty$ and so $< \mathcal{F}_d f_0, \phi >= 0$. To complete the proof we need to show that $\mathcal{F}_d f_0$ is also supported in $\{\xi, |P(\xi)| \geqslant 1\}$, which means $< \mathcal{F}_d f_0, \phi >= 0$ if ϕ is supported in the set $\{\xi, |P(\xi)| < 1\}$. Here we use (4.4) to obtain

$$< \mathcal{F}_d f_0, \phi >=< P^j \mathcal{F}_d(f_{-j}), \phi >=< \mathcal{F}_d(f_{-j}), P^j \phi >,$$

and we proceed as previously. □

The next step in the proof is we assume firstly that (-1) is not a value of $P(\xi)$, and show that $P(-iT)f_0 = f_0$.

Lemma 4.4. *There exists an integer N such that*

$$(P-1)^{N+1}\mathcal{F}_d f_0 = 0. \tag{4.7}$$

Proof. From the growth conditions on the sequence $(f_j)_{j\in\mathbb{Z}}$, Lemma 4.3 and the assumption that $P(\xi)\neq -1$, we obtain

$$\operatorname{supp}(\mathcal{F}_d(f_0)) \subset \{\xi, P(\xi)=1\}.$$

As $\mathcal{F}_d(f_0)$ is a continuous linear functional on $\mathcal{S}(\mathbb{R})$, there is a constant C and integers m and N so that

$$| < \mathcal{F}_d(f_0), \phi > | \leqslant C v_{N,m}(\phi),$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Thus the distribution $\mathcal{F}_d(f_0)$ is of order $\leqslant N$. For this N we want to prove that

$$(P-1)^{N+1}\mathcal{F}_d f_0 = 0.$$

To simplify notation set $Q := P-1$. Then we need to show that for any compactly supported function ϕ in $\mathcal{D}(\mathbb{R})$, the

$$< Q^{N+1}\mathcal{F}_d f_0, \phi > = < \mathcal{F}_d f_0, Q^{N+1}\phi > = 0.$$

Let $g : \mathbb{R} \to [0,1]$ be in $\mathcal{D}(\mathbb{R})$ such that $g=1$ on $\left[\frac{-1}{2}, \frac{1}{2}\right]$ and $g=0$ outside $[-1,1]$.

Set $g_r(t) := g\left(\frac{t}{r}\right)$, $Q_r = g_r(Q)Q^{N+1}\phi$. Then $Q_r = Q^{N+1}\phi$ in a neighborhood of

$$\operatorname{supp}\mathcal{F}_d f_0 \subset \{\xi, Q(\xi)=0\} = \{\xi, P(\xi)=1\}.$$

Thus, we have

$$| < \mathcal{F}_d f_0, Q^{N+1}\phi > | = | < \mathcal{F}_d f_0, Q_r > | \leqslant c v_{N,m}(Q_r).$$

We prove that $v_{N,m}(Q_r) \to 0$ as $r \to 0$. Thus (4.7) is proved.

Inverting the generalized Fourier transform in (4.7) yields that

$$(P(-iT)-1)^{N+1} f_0 = 0. \tag{4.8}$$

This equation implies

$$\begin{aligned}\operatorname{span}\{f_0, f_1, \ldots\} &= \operatorname{span}\{f_0, P(-iT)f_0, P(-iT)^2 f_0, \ldots\}\\ &= \operatorname{span}\{f_0, P(-iT)f_0, \ldots, P(-iT)^N f_0\}.\end{aligned}$$

We shall now prove that we can take $N=0$ in (4.8). If not then $(P(-iT)-1)f_0 \neq 0$. Let p be the largest positive integer so that $(P(-iT)-1)^p f_0 \neq 0$. Clearly $p \leqslant N$. Thus

$$f := [P(-iT)-1]^{p-1} f_0 \in \operatorname{span}\{f_0, f_1, \ldots, f_N\},$$

will satisfy

$$[P(-iT)-1]^2 f = 0 \quad \text{and} \quad [P(-iT)-1]f \neq 0. \tag{4.9}$$

Write

$$f = a_0 f_0 + \cdots + a_N f_N,$$

for constants $a_0, \ldots, a_N$. Then

$$P^j(-iT)f = a_0 f_j + \cdots + a_N f_{N+j}.$$

If $c_j = |a_0|M_j + \cdots + |a_N|M_{j+N}$, then the previous relation and the relation (4.2) imply that

$$|P^j(-iT)f(x)| \leqslant c_j(1+|x|)^a e^{-\delta\rho|x|}.$$

By (4.3) the constant c_j satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{c_j}{j} = 0. \tag{4.10}$$

An induction using (4.9) implies that for $j \geqslant 2$ we have

$$P^j(-iT)f = jP(-iT)f - (j-1)f = j((P(-iT)-1)f + f).$$

Thus

$$|(P(-iT)-1)f(x)| \leqslant \frac{1}{j}|P^j(-iT)f(x)| + \frac{|f(x)|}{j} \leqslant \frac{c_j}{j}(1+|x|)^a e^{-\delta\rho|x|} + \frac{|f(x)|}{j}.$$

Letting $j \to \infty$ and using (4.10) implies $[P(-iT)-1]f = 0$. But this contradicts (4.9). Consequently, $N = 0$ in (4.8). This completes the proof in the case that (-1) is not in the range of P.

In the case that 1 is not in the range of P we apply the same argument to $-P(-iT)$ to conclude $P(-iT)f_0 = -f_0$.

In the general case, let $\mathcal{L} = P(-iT)^2$. Then

$$\mathcal{F}(\mathcal{L}f)(\xi) = P(\xi)^2\mathcal{F}(f)(\xi).$$

$\mathcal{L}f_{2p} = f_{2(p+1)}$ and $P(\xi)^2 \neq -1$. Thus we can (as before) conclude, for the sequence $(f_{2p})_{p\in\mathbb{Z}}$ that

$$\mathcal{L}f_0 = P(-iT)^2 f_0 = f_0.$$

Set $f_+ = \frac{1}{2}(f_0 + P(-iT)f_0)$ and $f_- = \frac{1}{2}(f_0 - P(-iT)f_0)$. Then $f = f_+ + f_-$, $P(-iT)f_+ = f_+$ and $P(-iT)f_- = -f_-$.

This completes the proof of Theorem 4.2. □

Remark. If we take $P(y) = -|y|^2$, then $P(-iT) = \Delta$ and Theorem 4.2 gives $\Delta f_0 = -f_0$. This characterizes eigenfunctions f of generalized Laplace operator Δ with polynomial growth in terms of the size of the powers $\Delta^j f$, $-\infty < j < +\infty$. It also generalizes results of Roe [12].

4.1. The heat kernel

Proposition 4.5. *The heat kernel $u_t(x)$ defined for $t > 0$ by*

$$u_t(x) = \mathcal{F}^{-1}(e^{-t(.)^2})(x). \tag{4.11}$$

belongs to $\mathcal{S}_2(\mathbb{R})$ *for all* $t > 0$, *and satisfies the equation*

$$\Lambda_x^2 u_t(x) - \partial_t u_t(x) = 0, \quad \text{for all } (t,x) \in]0,+\infty[\times\mathbb{R}. \tag{4.12}$$

Proof. For all $t > 0$, as the function $\lambda \mapsto e^{-t\lambda^2}$ belongs to $\mathcal{S}(\mathbb{R})$, we deduce that the function u_t belongs to $\mathcal{S}_2(\mathbb{R})$ for all $t > 0$. Now from the definition of u_t, we have

$$u_t(x) = \int_{\mathbb{R}} e^{-\lambda^2 t}\Phi_{-\lambda}(-x)d\sigma_1(\lambda) + \int_{-\rho}^{\rho} e^{\lambda^2 t}\Phi_{-i\lambda}(-x)d\sigma_2(\lambda).$$

Deriving the last equation by report t, we obtain

$$\partial_t u_t(x) = -\int_{\mathbb{R}} \lambda^2 e^{-\lambda^2 t}\Phi_{-\lambda}(-x)d\sigma_1(\lambda) + \int_{-\rho}^{\rho} \lambda^2 e^{\lambda^2 t}\Phi_{-i\lambda}(-x)d\sigma_2(\lambda).$$

Moreover, we have

$$\Lambda^2 f = \Delta f(x) + \frac{d}{dx}\left(\frac{A'}{A}\right)\frac{f(x)-f(-x)}{2} + \rho^2 f(x).$$

with

$$\Delta f = \frac{d^2}{dx^2}f + \frac{A'}{A}\frac{d}{dx}f.$$

This gives,

$$\Lambda^2 u_t(x) = \int_{\mathbb{R}} e^{-\lambda^2 t}\Lambda^2\Phi_{-\lambda}(-x)d\sigma_1(\lambda) + \int_{-\rho}^{\rho} e^{\lambda^2 t}\Lambda^2\Phi_{-i\lambda}(-x)d\sigma_2(\lambda).$$

However, we get

$$(\Lambda^2 f)(-x) = \Lambda^2(f(-x)), \quad \text{for all } f. \tag{4.13}$$

Using (1.2) and (4.13), this gives

$$\Lambda^2\Phi_{-\lambda}(-x) = -\lambda^2\Phi_{-\lambda}(-x),$$

and

$$\Lambda^2\Phi_{-i\lambda}(-x) = \lambda^2\Phi_{-i\lambda}(-x).$$

This yields

$$\Lambda^2 u_t(x) - \partial_t u_t(x) = 0. \qquad \square$$

As an application of the above Theorem we have the following Corollary.

Corollary 4.6. *If in Theorem* 4.2, *we replace* (4.2) *by*

$$\|f_j\|_{p,A} \leqslant M_j e^{-\delta\rho|x|}, \quad 1 \leqslant p \leq 2, \tag{4.14}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the sublinear growth condition*

$$\lim_{j\to\infty}\frac{M_{|j|}}{j}=0. \tag{4.15}$$

Then $f=f_+ + f_-$ *where* $P(-iT)f_+=f_+$ *and* $P(-iT)f_- = -f_-$. *If 1 (or (-1)) is not in the range of* P *then* $f_+=0$ *(or* $f_-=0$).

Proof. Let $n\in\mathbb{N}$. Consider the functions $F_{j,n}(x)=(f_j\#u_t)(x)$ where u_t is defined in (4.11). Using Hölder's inequality gives

$$\forall x\in\mathbb{R},\ |F_{j,n}(x)|\leqslant \|f_j\|_{p,A}\|{}^t\tau^x u_t\|_{p',A},$$

where p' is the conjugate exponent of p. On the other hand, we have

$$P(-iT)F_{j,n}=F_{j+1,n},\ j\in\mathbb{Z}.$$

Thus $\{F_{j,n}\}_{j\in\mathbb{Z}}$ verifies the relations (4.2) and (4.3) of Theorem 4.2 and the result follows immediately. $\square$

In the space of distributions $\mathcal{D}'(\mathbb{R})$, we use the regularization of distributions to obtain the analogue of Theorem 4.2.

Theorem 4.7. *Let* $P(\xi)=\sum_n a_n\xi^n$ *be a polynomial with complex values in* ξ *and let*

$$P(-iT)=\sum_n(-i)^n a_n T^n. \tag{4.16}$$

Let $u_j\in\mathcal{D}'(\mathbb{R}), j\in\mathbb{Z}$. *Suppose that for every compact subset* K *of* $\mathbb{R}$, *there exist a non-negative integer* N *and a positive constants* $M_j:=M_j(k,N)$ *such that*

(i) $P(-iT)u_j=u_{j+1}$,
(ii) $\|u_j\#\phi\|_\infty\leqslant M_j\sum_{n\leqslant N}\sup_{x\in K}|T^n\phi(x)|$ *for all* $j\in\mathbb{Z}$ *and* $\phi\in\mathcal{D}(\mathbb{R})$, *where* $(M_j)_{j\in\mathbb{Z}}$ *satisfies the sublinear growth condition*

$$\lim_{j\to\infty}\frac{M_{|j|}}{j}=0.$$

Then $u_0=u_+ + u_-$ *where* $P(-iT)u_+=u_+$ *and* $P(-iT)u_-=-u_-$. *If 1 (or* (-1)*) is not in the range of* P *then* $u_+=0$ *(or* $u_-=0$).

Proof. Let $\chi\in\mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}}\chi(x)A(x)\,dx=1$ and set

$$\chi_n(x)=\frac{A(nx)}{nA(x)}\chi(nx),\ n\in\mathbb{N}^*.$$

Then $\chi_n\to\delta$ in $\mathcal{D}'(\mathbb{R})$ and support $\chi_n\subset$ support χ for all n. For each $j\in\mathbb{Z}$, $u_j\#\chi_n$ belongs to $\mathcal{E}(\mathbb{R})$ which is a regularization of u_j and $u_j\#\chi_n\to u_j$ in $\mathcal{D}'(\mathbb{R})$ as $n\to\infty$. Let $h_{j,n}:=u_j\#\chi_n$. Then for $K:=$support χ and all $j\in\mathbb{Z}$, it follows from the hypothesis (i) and (ii) that

$$\begin{aligned} &P(-iT)h_{j,n} = u_{j+1}\#\chi_n = h_{j+1,n}, \\ &\|h_{j,n}\|_\infty \leqslant \widetilde{M_j}, \end{aligned} \tag{4.17}$$

where $\widetilde{M_j} := M_j \sum_{n \leqslant N} \sup_{x \in K} |T^n \chi_n(x)|$ is a positive constant. It then follows from (4.17) and Theorem 4.2 that $u_n = u_{n,+} + u_{n,-}$ where $P(-iT)u_{n,+} = u_{n,+}$ and $P(-iT)u_{n,-} = -u_{n,-}$.

If 1 (or (-1)) is not in the range of P then $u_{n,+} = 0$ (or $u_{n,-} = 0$). Letting $n \to \infty$, we obtain the result. □

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