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On the principal frequency curve of the *p*-biharmonic operator

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Abstract A variational method on C^1 variety is adapted to show the existence of an increasing sequence of positive eigencurves of the *p*-biharmonic operator without any assumption on the regularity of the domain. A direct characterization of the principal curve (first one) is given.

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1. Introduction

Let Ω be any bounded domain in N-dimensional Euclidean space $\mathbb{R}^N$, $N \geqslant 1$; $1 < p < \infty$ and $\rho \in L^\infty(\Omega)$, $\rho \neq 0$, an indefinite weight function which can change its sign. We consider the following nonlinear eigenvalue problem:

$$E_p(\lambda) \quad \Delta_p^2 u - \lambda\rho(x)|u|^{p-2}u = \mu|u|^{p-2}u \quad \text{in } \Omega.$$

The solution u is required to belong to the Sobolev space $W_0^{2,p}(\Omega)$ and the real parameters $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$ play the role of eigenvalues. $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ is the operator of fourth order, the so called p-biharmonic operator, which has attracted growing interest, and figures in a variety of applications, where this operator is used to control the nonlinear artificial viscosity or diffusion of non-Newtonian fluids. The case $p = 2$ is reduced to the well-understood linear equation

$$\Delta_2^2 u + \lambda\rho(x)u + \mu u = 0,$$

which is the prototype of linear equations appearing in connection with Schrödinger's equation of fourth order. Here, $\Delta_2^2 = \Delta^2 = \Delta \cdot \Delta$ is the iterated Laplacian that appears often in the equations of Navier–Stokes as being a term of viscosity coefficient, and its reciprocal operator, denoted $(\Delta^2)^{-1}$, is the celebrated Green's operator, see Lions (1969).

Notice that the *biharmonic equation* is the partial differential equation (PDE) of fourth order which appears in quantum mechanics and in the theory of linear elasticity and PDEs modeling Stokes' flows. It is obtained when $p = 2$ and by setting

$$\Delta^2\psi = 0.$$

In *Cartesian coordinates* the biharmonic equation in 3-D is given by

$$\nabla^4\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2 \psi = 0.$$

That is,

$$\nabla^4\psi = \frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4} + \frac{\partial^4\psi}{\partial z^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + 2\frac{\partial^4\psi}{\partial x^2\partial z^2} + 2\frac{\partial^4\psi}{\partial y^2\partial z^2} = 0.$$

In *polar coordinates* (r, θ) it becomes

$$\nabla^4\psi = \psi_{rrrr} + \frac{2}{r^2}\psi_{rr\theta\theta} + \frac{1}{r^4}\psi_{\theta\theta\theta\theta} + \frac{2}{r}\psi_{rrr} - \frac{2}{r^3}\psi_{r\theta\theta} - \frac{1}{r^2}\psi_{rr} + \frac{4}{r^4}\psi_{\theta\theta} + \frac{1}{r^3}\psi_r = 0.$$

For a radial function $\psi \equiv \psi(r)$, the biharmonic equation becomes

$$\nabla^4\psi = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\left(r\frac{\partial\psi}{\partial r}\right)\right]\right\} = \psi_{rrrr} + \frac{2}{r}\psi_{rrr} - \frac{1}{r^2}\psi_{rr} + \frac{1}{r^3}\psi_r = 0.$$

Note that if $\psi(r) = \frac{1}{r}$ then we obtain, in $\mathbb{R}^3 \setminus \{0\}$, by calculation of derivatives that

$$\psi_r = -\frac{1}{r^2}$$

$$\psi_{rr} = \frac{2}{r^3}$$

$$\psi_{rrr} = -\frac{6}{r^4}$$

$$\psi_{rrrr} = \frac{24}{r^5}.$$

We deduce in this case that $\frac{1}{r}$ is a solution of the biharmonic equation.

In general, in N-dimensional space $\mathbb{R}^N \setminus \{0\},\ r = \left(\sum_{i=1}^{i=N} x_i^2\right)^{\frac{1}{2}}$and

$$\nabla^4\left(\frac{1}{r}\right) = \frac{3(N^2 - 8N + 15)}{r^5}.$$

Notice at the end that any harmonic function is biharmonic and the inverse fails.

The principal objective of this paper is to show that for any parameter $\lambda \in \mathbb{R}$, the problem $E_p(\lambda)$ has at least a nondecreasing sequence of positive eigenvalues $(\mu_k(\lambda))_{k\geqslant 1}$, object of Theorem 3.1, by using the Ljusternich–Schnirelmann theory on C^1 manifolds; see e.g. Szulkin (1988) for more details about this theory. We give a variational formulation (direct characterization) of $\mu_k(\lambda)$ involving a mini-max over sets of genus greater than k. It is important here to notice that our result is obtained without any assumptions at all on the regularity of the domain Ω.

We set

$$\mu_1(\lambda) = \inf\left\{\frac{\|\Delta v\|_p^p - \lambda\int_\Omega \rho(x)|v|^p dx}{\|v\|_p^p};\ v \in W_0^{2,p}(\Omega)\setminus\{0\}\right\}, \tag{1}$$

where $\|\cdot\|_p$ denotes the $L^p(\Omega)$-norm. It is not difficult to show that $u \to \|\Delta u\|_p$ defines a norm in $W_0^{2,p}(\Omega)$ and $W_0^{2,p}(\Omega)$ equipped with it is a uniformly convex Banach space for $1 < p < +\infty$. The norm $\|\Delta\cdot\|_p$ is uniformly equivalent on $W_0^{2,p}(\Omega)$ with the usual norm of $W_0^{2,p}(\Omega)$ (El Khalil et al., 2002).

We understand by the principal eigencurve (or the first frequency) of the p-biharmonic operator, the *graph* of the function $\mu_1 : \lambda \to \mu_1(\lambda)$ from $\mathbb{R}$ into $\mathbb{R}$, defined by (1).

Equations of the type

$$Au - \lambda Bu = \mu Cu, \tag{2}$$

where A and B are linear operators and C is the identity, are studied extensively by several authors. Preliminary work has been done by Richardson (1912) and by other authors on the first eigencurve of the Sturn–Liouville equation, i.e., the set of (λ, μ) satisfying (2) with $u \not\equiv 0$. Binding and Huang (1995) studied the case: $Au = -\Delta_p u,\ (p - Laplacian)\ Bu = \rho(x)|u|^{p-2}u$ and $Cu = |u|^{p-2}u$, on a bounded regular domain, generalizing the results of the linear case ($p = 2$) obtained by Hess and Kato (1980) and of Kato (1982).

For $\lambda = 0$ or $\mu = 0$, we cite the work of De Thélin (1986) for Ω is the unit ball and the works of Anane (1987) and Azorero and Alonso (1987) for a regular domain. We cite also the work of Ôtani and Theshima (1988) for a domain which is an interval in one-dimensional space ($N = 1$). In the case of a domain not necessarily regular, we indicate the paper of Lindqvist (1990) and the work of El Khalil et al. (2004). Here, in this paper, we study the case where

$$A = \Delta_p^2, \quad Bu = \rho(x)|u|^{p-2}u \quad \text{and } Cu = |u|^{p-2}u.$$

The rest of the paper is organized as follows. In Section 2, we establish some definitions and prove certain basic lemmas. In Section 3, we use a variational technique to prove the existence of a sequence of the eigencurves of the p-biharmonic operator with unbounded indefinite weight, and without any assumption on the regularity of the domain.

2. Preliminary notes

Throughout this paper, all solutions are weak ones, i.e., $u \in W_0^{2,p}(\Omega)$ is a solution of (1), if for all $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi dx = \int_\Omega (\lambda\rho(x) + \mu)|u|^{p-2}u\varphi dx. \tag{3}$$

If $u \in W_0^{2,p}(\Omega) \setminus \{0\}$, then u is called an eigenfunction of the p-biharmonic operator (or of E_λ) associated to the eigenvalue (λ, μ).

Definition 2.1. Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle , \rangle$. We shall deal with mappings T acting from X into X^*. The strong convergence in X (and in X^*) is denoted by $\to$ and the weak convergence by $\hookrightarrow$. T is said to belong to the class (S^+), if for any sequence $\{u_n\}$ in X converging weakly to $u \in X$ and $\limsup_{n\to+\infty}\langle Tu_n, u_n - u\rangle \leqslant 0$, it follows that u_n converges strongly to u in X. We write $T \in (S^+)$.

We consider the following functionals on $W_0^{2,p}(\Omega)$, defined by

$$\Phi_\lambda(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx - \frac{\lambda}{p}\int_\Omega \rho(x)|u|^p dx = \frac{1}{p}\int_\Omega |\Delta u|^p dx + \Phi(u)$$

and

$$\Psi(u) = \frac{1}{p}\|u\|_p^p.$$

We set

$$\mathcal{M} = \left\{u \in W_0^{1,p}(\Omega),\ p\Psi(u) = 1\right\}. \tag{4}$$

Lemma 2.1. *We have the following statements,*

(i) Φ_λ, Φ *et* Ψ *are even, and of class* C^1 *on* $W_0^{2,p}(\Omega)$.
(ii) $\mathcal{M}$ *is a closed* C^1*-manifold.*

Proof

(i) It is clear that Φ_λ, Φ and Ψ are even, and of class C^1 on $W_0^{2,p}(\Omega)$. $\Psi'(u) = |u|^{p-2}u,\ \Phi'(u) = \rho|u|^{p-2}u$ on Ω. And $\Phi'_\lambda(u) = \Delta_p^2(u) + \Phi'(u)$ on Ω.
(ii) $\mathcal{M} = \Psi^{-1}\left\{\frac{1}{p}\right\}$. Thus $\mathcal{M}$ is closed. Its derivative operator Ψ' satisfies $\Psi\prime(u) \neq 0\ \forall u \in \mathcal{M}$ (i.e., $\Psi'(u)$ is onto $\forall u \in \mathcal{M}$).

So ψ is a submersion, then $\mathcal{M}$ is a C^1-manifold. □

Remark 2.1. The functional

$$\begin{aligned} J:\ & W_0^{2,p}(\Omega)\ \rightarrow\ W^{-2,p'}(\Omega) \\ u\ & \rightarrow\ \|\Delta u\|_p^{2-p}\Delta_p^2 u \quad \text{if } u \neq 0 \\ u\ & \rightarrow\ 0 \quad \text{if } u = 0 \end{aligned}$$

is the duality map on $(W_0^{2,p}(\Omega),\ \|\Delta\ \cdot\ \|_p)$ associated with the Gauge function $\phi(t) = |t|^{p-2}t$.

The following lemma is the key to establish the existence result.

Lemma 2.2. *For any* $\lambda \in \mathbb{R}$*, we have*

(a) $\Psi'\ :\ W_0^{2,p}(\Omega)\ \rightarrow\ W^{-2,p'}(\Omega)$ *is completely continuous;*
(b) $\Phi'\ :\ W_0^{2,p}(\Omega)\ \rightarrow\ W^{-2,p'}(\Omega)$ *is completely continuous;*
(c) Φ_λ *is bounded from below on* $\mathcal{M}$*;*
(d) *the functional* Φ_λ *satisfies the Palais–Smale condition on* $\mathcal{M}$*, i.e., for* $\{u_n\}_n \subset \mathcal{M}$*, if* $\{\Phi_\lambda(u_n)\}_n$ *is bounded and*

$$\left(\Phi_\lambda|_{\mathcal{M}}\right)'(u_n)\ \rightarrow\ 0, \tag{5}$$

then $\{u_n\}_n$ *has a convergent subsequence in* $W_0^{2,p}(\Omega)$.

Proof. $\|\cdot\|_*$ is the dual norm of $W^{-2,p'}(\Omega)$ associated with $\|\Delta\cdot\|_p$.

(a) It is evident that Ψ' is completely continuous; it suffices to use the standard Sobolev embedding: $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$ (compactly), see Adams (1975) or Gilbarg and Trudinger (1983).
(b) Let $\{u_n\} \subset W_0^{2,p}(\Omega),\ u_n\ \rightharpoonup\ u$ (weakly) in $W_0^{2,p}(\Omega)$. By the Sobolev embedding, we deduce that $\{u_n\}$ converges strongly to u in $L^p(\Omega)$, and there exists $g \in L^p(\Omega)$ such that

$$|u_n| \leqslant g \quad \text{a.e. in } \Omega.$$

Thus

$$|u_n|^{p-1} \leqslant g^{p-1} \quad \text{a.e. in } \Omega.$$

Since $g^{p-1} \in L^{p'}(\Omega)$, we deduce that

$$|\rho(x)|u_n|^{p-2}u_n| \leqslant \|\rho\|_\infty |u_n|^{p-1} \leqslant \|\rho\|_\infty g^{p-1} \text{ a.e. in } \Omega.$$

It follows from the Dominated Convergence Theorem that

$$\rho(x)|u_n|^{p-2}u_n \;\rightarrow\; \rho(x)|u|^{p-2}u \quad \text{in } L^{p'}(\Omega).$$

i.e.,

$$\phi'(u_n) \;\rightarrow\; \phi'(u) \quad \text{in } L^{p'}(\Omega).$$

Recall that the following embeddings

$$W_0^{2,p}(\Omega) \;\rightarrow\; L^p(\Omega) \quad \text{and } L^{p'}(\Omega) \;\rightarrow\; W^{-2,p'}(\Omega)$$

are compact. Thus

$$\phi'(u_n) \;\rightarrow\; \phi'(u) \quad \text{in } W^{-2,p'}(\Omega).$$

(c) Let $u \in \mathcal{M}$. Then

$$\lambda \int_\Omega \rho(x)|u|^p dx \leqslant |\lambda| \|\rho\|_\infty.$$

This implies that

$$\phi_\lambda(u) \;\geqslant\; \frac{1}{p}\|\Delta u\|_p^p - \frac{1}{p}|\lambda|\|\rho\|_\infty. \tag{6}$$

Then

$$\Phi_\lambda(u) \;\geqslant\; \frac{1}{p}(\lambda_1 - |\lambda|\|\rho\|_\infty) > -\infty,$$

where $\lambda_1 = \mu_1(0)$ is the first eigenvalue of the p-biharmonic operator (El Khalil et al., 2002).

(D) Let $\{u_n\}_n \subset \mathcal{M}$ be such that $\{\Phi_\lambda(u_n)\}_n$ is bounded and

$$\left(\Phi_\lambda|_{\mathcal{M}}\right)'(u_n) \;\rightarrow\; 0.$$

$\{\Phi_\lambda(u_n)\}_n$ being bounded, so from (6), $\|\Delta u_n\|_p$ is bounded in $W_0^{2,p}(\Omega)$. Without loss of generality, we can assume that u_n converges weakly in $W_0^{2,p}(\Omega)$ to some function $u \in W_0^{2,p}(\Omega)$ and $\|\Delta u_n\|_p \rightarrow l$.For the rest, we distinguish two cases:

- Suppose that $l = 0$. In this case, we conclude that $\{u_n\}_n$ converges strongly to 0 in $W_0^{2,p}(\Omega)$.
- Suppose that $l \neq 0$. Then we argue as follows.

From (5), $\left(\Phi_\lambda|_{\mathcal{M}}\right)'(u_n) \;\rightarrow\; 0$. i.e.,

$$\epsilon_n := \phi'_\lambda(u_n) - g_n\psi'(u_n) \;\rightarrow\; 0 \quad \text{as } n \;\rightarrow\; \infty, \tag{7}$$

with

$$g_n = \frac{\langle \phi'_\lambda(u_n), u_n\rangle}{\langle \psi'(u_n, u_n)\rangle}.$$

The idea is to prove that

$$\limsup_{n\to+\infty} \langle \Delta_p^2(u_n), u_n - u\rangle \leqslant 0.$$

Indeed, note that

$$\langle \Delta_p^2(u_n), u_n - u\rangle = \|\Delta u_n\|_p^p - \langle \Delta_p^2(u_n), u\rangle.$$

Applying ϵ_n of (7) to u, we deduce that the quantity

$$\theta_n := \langle \Delta_p^2(u_n), u\rangle + \langle \phi'(u_n), u\rangle - g_n\langle \psi'(u_n), u\rangle \ \to\ 0, \quad \text{as } n \ \to\ \infty.$$

Thus

$$\langle \Delta_p^2(u_n), u_n - u\rangle = \|\Delta u_n\|_p^p - \theta_n + \langle \phi'(u_n), u\rangle - \Big(\|\Delta u_n\|_p^p + \langle \phi'(u_n), u_n\rangle\Big) \cdot \langle \psi'(u_n), u\rangle.$$

Hence

$$\langle \Delta_p^2(u_n), u_n - u\rangle = \|\Delta u_n\|_p^p(1 - \langle \psi'(u_n), u\rangle) - \theta_n + \langle \phi'(u_n), u\rangle - \langle \phi'(u_n), u_n\rangle \cdot \langle \psi'(u_n), u\rangle.$$

On the other hand, from Lemma 2.2(b), Φ' is completely continuous. Therefore

$$\Phi'(u_n) \ \to\ \Phi'(u), \quad \langle \Phi'(u_n), \quad u_n\rangle \ \to\ \langle \Phi'(u), u\rangle \text{ and } \langle \Phi'(u_n), u\rangle \ \to\ \langle \Phi'(u), u\rangle.$$

From Lemma 2.2(a), Ψ' is also completely continuous. So

$$\langle \Psi'(u_n), u_n\rangle \ \to\ \langle \Psi'(u), u\rangle \quad \text{in } W^{-2,p'}(\Omega)$$

and $p\Psi(u) = \langle \Psi'(u), u\rangle = 1$, because $p\Psi(u_n) = 1,\ \forall n \in \mathbb{N}$. Thus

$$1 - \langle \Psi'(u_n), u\rangle = \langle \Psi'(u), u\rangle - \langle \Psi'(u_n), u\rangle.$$

Hence

$$|1 - \langle \Psi'(u_n), u\rangle| \leqslant \|\Psi'(u_n) - \Psi'(u)\|_* \|\Delta u\|_p^p = \|\Psi'(u_n) - \Psi'(u)\|_* l^p,$$

where $\|\cdot\|_*$ is the dual norm associated to the norm $\|\Delta \cdot\|_p$. This implies that

$$1 - \langle \psi'(u_n), u\rangle \ \to\ 0 \quad \text{as } n \ \to\ \infty.$$

Combining with the above inequalities, we conclude that

$$\limsup_{n\to+\infty} \langle \Delta_p^2(u_n), u_n - u\rangle \leqslant \frac{1}{l^p} \limsup_{n\to+\infty}(1 - \langle \psi'(u_n), u\rangle) + \limsup_{n\to\infty}(\langle \phi'(u_n), u\rangle - \langle \phi'(u_n), u_n\rangle \cdot \langle \psi'(u_n), u\rangle).$$

Then

$$\limsup_{n\to+\infty}\langle\Delta_p^2(u_n),u_n-u\rangle\leqslant\langle\phi'(u),u\rangle(1-\langle\psi'(u),u\rangle).$$

On the other hand, we have $\langle\Psi'(u),u\rangle=1$, because

$$\Psi'(u_n),u_n>=p\Psi(u_n)=1$$

for any integer n. This implies that

$$\limsup_{n\to+\infty}\langle\Delta_p^2(u_n),u_n-u\rangle\leqslant 0. \tag{8}$$

We can write $\Delta_p^2u_n=\|\Delta u_n\|_p^{p-2}J(u_n)$, since $\|\Delta u_n\|_p\neq 0\ \forall n$ large enough, where J is the duality mapping defined in Remark 2.1. Therefore

$$\limsup_{n\to+\infty}\langle\Delta_p^2u_n,u_n-u\rangle=l^{p-2}\limsup_{n\to+\infty}\langle Ju_n,u_n-u\rangle$$

According to (8), we conclude that

$$\limsup_{n\to+\infty}\langle Ju_n,u_n-u\rangle\leqslant 0.$$

J being a duality mapping, thus it satisfies the condition S^+ given in Trojanski (1971), so that, $u_n\ \to\ u$ strongly in $W_0^{2,p}(\Omega)$.

This completes the proof of the lemma. □

3. Main results

Set

$$\Gamma_k=\{K\subset\mathcal{M}:\ K\text{ is symmetric, compact and }\gamma(K)\geqslant k\},$$

where $\gamma(K)=i$ is the genus of K, i.e., the smallest integer i such that there exists an odd continuous map from K to $\mathbb{R}^i-\{0\}$.

Now, by the Ljusternick–Schnirelmann theory, see e.g. Szulkin (1988), we have our main result.

Theorem 3.1. *For any $\lambda\in\mathbb{R}$and for any $k\in\mathbb{N}^*$,*

$$\mu_k(\lambda):=\inf_{A\in\Gamma_k}\max_{u\in A}p\Phi_\lambda(u)$$

is a critical value of Φ_λ restricted on $\mathcal{M}$. More precisely, there exist $u_k(\lambda)\in\mathcal{M},\mu_k(\lambda)\in\mathbb{R}$ such that

$$p\Phi_\lambda(u_k(\lambda))=\mu_k(\lambda)$$

and $u_k(\lambda)$ is an eigenfunction of E_λ associated to the eigenvalue $(\lambda,\mu_k(\lambda))$.

Proof. From the Ljusternick–Schnirelmann theory on C^1-manifolds of Szulkin (1988), We need only to prove that for any $k\in\mathbb{N}^*,\ \Gamma_{\Bbbk}\neq\emptyset$.

Indeed, since $W_0^{2,p}(\Omega)$ is separable, there exist $(e_i)_{i\geqslant 1}$ linearly dense in $W_0^{2,p}(\Omega)$ such that $supp\ e_i\cap\ supp e_j=\emptyset$ if $i\neq j$. We can assume that $e_i\in\mathcal{U}$ $\left(\text{if not, we take } e_i':=\frac{e_i}{\|e_i\|_p}\right)$.

Let $k\in\mathbb{N}^*$, denote $F_k=span\{e_1,e_2,\ldots,e_k\}$. F_k is a vector subspace and $dimF_k=k$.

If $v\in F_k$, then there exist $\alpha_1,\ldots\alpha_k$ in $\mathbb{R}$ such that $v=\sum_{i=1}^k\alpha_i e_i$. Thus $\Psi(v)=\sum_{i=1}^k|\alpha_i|^p\Psi(e_i)=\frac{1}{p}\sum_{i=1}^k|\alpha_i|^p$. It follows that the map $v\mapsto(p\Psi(v))^{\frac{1}{p}}:=\|v\|$ defines a norm on F_k. Consequently, there is a constant $c>0$ such that

$$c\|\Delta u\|_p\leqslant\|v\|\leqslant\frac{1}{c}\|\Delta u\|_p. \tag{9}$$

This implies that the set

$$V=F_k\cap\left\{v\in W_0^{2,p}(\Omega):\Psi(v)\leqslant\frac{1}{p}\right\}$$

is bounded, because

$$V\subset B\left(0,\frac{1}{c}\right),$$

where

$$B\left(0,\frac{1}{c}\right)=\left\{u\in W_0^{2,p}(\Omega),\quad\text{such that }\|\Delta u\|_p\leqslant\frac{1}{c}\right\}.$$

Thus V is a symmetric bounded neighborhood of $0\in F_k$.

By Szulkin (1988, Proposition 2.3(f)), we deduce that $\gamma(F_k\cap\mathcal{M})=k$, because $F_k\cap\mathcal{M}$ is compact (it is exactly the boundary of V) and $\Gamma_k\neq\emptyset$. This completes the proof of the theorem. □

Corollary 3.1

(i) $\mu_1(\lambda)$ *given by the above theorem is exactly the one defined by the characterization (1).*

(ii) $0<\mu_1(\lambda)\leqslant\mu_2(\lambda)\leqslant\ldots\leqslant\mu_n(\lambda)\ \to\ +\infty$.

Proof

(i) For $u\in\mathcal{M}$, we put $K_1=\{u,-u\}$. Then $\gamma(K_1)=1$. Since ϕ_λ is even, we obtain

$$p\Phi_\lambda(u)=\max_{K_1}p\Phi_\lambda\geqslant\inf_{K\in\Gamma_1}\max_K p\Phi_\lambda.$$

Then

$$\inf_{u\in\mathcal{M}} p\Phi_\lambda(u) \geqslant \inf_{K\in\Gamma_1} \max_K p\Phi_\lambda = \mu_1(\lambda).$$

On the other hand, $\forall K \in \Gamma_1,\ \forall u \in K$,

$$\sup_K p\Phi_\lambda \geqslant p\Phi_\lambda(u) \geqslant \inf_{u\in\mathcal{M}} p\Phi_\lambda(u).$$

So

$$\inf_{K\in\Gamma_1} \max_K p\phi_\lambda = \mu_1(\lambda) \geqslant \inf_{u\in\mathcal{M}} p\phi_\lambda(u).$$

Then

$$\mu_1(\lambda) = \inf\left\{\frac{\|\Delta v\|_p^p - \lambda\int_\Omega m(x)|v|^p\,dx}{\|v\|_p^p};\ v \in W_0^{2,p}(\Omega) \text{ and } \|v\|_p \neq 0\right\}.$$

(ii) From the definition of $\mu_i(\lambda),\ i \in \mathbb{N}^*$, we have $\mu_i(\lambda) \geqslant \mu_j(\lambda)$, for any $i \geqslant j$.

Now let $(e_n, e_j^*)_{n,j}$ be a bi-orthogonal system such that $e_n \in W_0^{2,p}(\Omega)$, $e_j^* \in W^{-2,p'}(\Omega)$. The e_n's are linearly dense in $W_0^{2,p}(\Omega)$; and the e_j^*'s are total for $W^{-2,p'}(\Omega)$. For $k \in \mathbb{N}^*$ set

$$F_k = span\{e_1, \ldots, e_k\}$$

and

$$F_k^\perp = span\{e_{k+1}, e_{k+2}, \ldots\}.$$

By Szulkin (1988, Proposition 2.3(g)), we have for any $A \in \Gamma_k,\ A \cap F_{k-1}^\perp \neq \emptyset$.

$$t_k := \inf_{A\in\Gamma_k} \sup_{u\in A\cap F_{k-1}^\perp} p\Phi_\lambda(u) \ \to\ +\infty.$$

Indeed, if not, for k large enough, there exists $u_k \in F_{k-1}^\perp$ with $\|u_k\|_p = 1$ such that

$$t_k \leqslant p\Phi_\lambda(u_k) \leqslant M$$

for some $M > 0$ independent of k. Thus from (5)

$$\|\Delta u_k\|_p \leqslant (M + |\lambda|\|\rho\|_\infty)^{\frac{1}{p}}.$$

This implies that $(u_k)_k$ is bounded in $W_0^{2,p}(\Omega)$. For a subsequence of $\{u_k\}$, if necessary, we can assume that $\{u_k\}$ converges weakly in $W_0^{2,p}(\Omega)$ and strongly in $L^p(\Omega)$.

By our choice of $F_{k-1}^\perp$, we have $u_k \hookrightarrow 0$ in $W_0^{2,p}(\Omega)$, and then by compactness, $u_k \ \to\ 0$ in $L^p(\Omega)$. This contradicts the fact that $\|u_k\|_p = 1\forall k$.

On the other hand, for any positive integer k and any $A \subset \Gamma_k$ we have $A \cap F_{k-1}^\perp \subset$ and

$$\max_{u\in A} p\Phi_\lambda(u) \leqslant \sup_{u\in A\cap F_{k-1}^\perp} p\Phi_\lambda(u).$$

Then

$$\mu_k(\lambda) = \inf_{A\in\Gamma_k} \max_{u\in A} p\Phi_\lambda(u) \leqslant \inf_{A\in\Gamma_k} \sup_{u\in A\cap F_{k-1}^{\perp}} p\Phi_\lambda(u) = t_k.$$

Consequently,

$$\mu_k(\lambda) \to +\infty,$$

as $k \to +\infty$. This completes the proof. □

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