ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 22 (2016) 132–137

On the existence of positive solutions for an ecological model with indefinite weight

SALEH SHAKERI[a,*], GHASEM A. AFROUZI[b], ARMIN HADJIAN[c]

[a] Department of Mathematics, Islamic Azad University-Ayatollah Amoli Branch, Amol, Iran
[b] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[c] Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran

Received 10 September 2014; received in revised form 1 December 2014; accepted 9 December 2014
Available online 12 February 2015

Abstract. This study concerns the existence of positive solutions for the following nonlinear boundary value problem:

$$\begin{cases} -\Delta u = am(x)u - bu^2 - c\dfrac{u^p}{u^p+1} - K & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u, while a, b, c, p, K are positive constants with $p \geq 2$ and Ω is a bounded smooth domain of $\mathbb{R}^N$ with $\partial\Omega$ in C^2. The weight function m satisfies $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, also $\|m\|_\infty = l < \infty$. We prove the existence of positive solutions under certain conditions.

Keywords: Ecological systems; Indefinite weight; Grazing and constant yield harvesting; Sub–super solution method

2010 Mathematics Subject Classification: 35J65; 35J25

1. INTRODUCTION

In this note, we mainly consider the following reaction–diffusion equation:

$$\begin{cases} -\Delta u = am(x)u - bu^2 - c\dfrac{u^p}{u^p+1} - K & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: s.shakeri@iauamol.ac.ir (S. Shakeri), afrouzi@umz.ac.ir (G.A. Afrouzi), hadjian83@gmail.com (A. Hadjian).

Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2014.12.002

where $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u, while a, b, c, p, K are positive constants with $p \geq 2$ and Ω is a bounded smooth domain of $\mathbb{R}^N$ with $\partial\Omega$ in C^2. The weight function m satisfies $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, also $\|m\|_\infty = l < \infty$. We denote by λ_1 the first eigenvalue of

$$\begin{cases} -\Delta\phi = \lambda m(x)\phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with positive principal eigenfunction ϕ_1 satisfying $\|\phi_1\|_\infty = 1$ (see [4]).

Here u is the population density and $am(x)u - bu^2$ represents the logistics growth. This model describes grazing of a fixed number of grazers on a logistically growing species (see [6,8]). The herbivore density is assumed to be a constant which is a valid assumption for managed grazing systems and the rate of grazing is given by $\frac{cu^p}{1+u^p}$. At high levels of vegetation density this term saturates to c as the grazing population is a constant. This model has also been applied to describe the dynamics of fish populations (see [6,12]). The diffusive logistic equation with constant yield harvesting, in the absence of grazing was studied in [9]. Recently, in the case when $m(x) = 1$ problem (1.1) has been studied by R. Shivaji et al. (see [2]).

The purpose of this paper is to improve the result of [2] with weight m. We shall establish our abstract existence result via the method of sub–super solutions. The concepts of sub–super solution were introduced by Nagumo [7] in 1937 who proved, using also the shooting method, the existence of at least one solution for a class of nonlinear Sturm–Liouville problems. In fact, the premises of the sub–super solutions method can be traced back to Picard. He applied, in the early 1880s, the method of successive approximations to prove the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems. This is the starting point of the use of sub–super solutions in connection with monotone methods. Picard's techniques were applied later by Poincaré [10] in connection with problems arising in astrophysics. We refer the reader to [11].

Definition 1.1. We say that ψ (*resp.* z) in $C^2(\Omega) \cap C(\overline{\Omega})$ is a *subsolution* (*resp. super solution*) of (1.1), if ψ (*resp.* z) satisfies

$$\begin{cases} -\Delta\psi \leq am(x)\psi - b\psi^2 - c\dfrac{\psi^p}{\psi^p + 1} - K & \text{in } \Omega, \\ \psi \geq 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

$$\left(\textit{resp.} \begin{cases} -\Delta z \geq am(x)z - bz^2 - c\dfrac{z^p}{z^p + 1} - K & \text{in } \Omega, \\ z \geq 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega \end{cases} \right). \tag{1.4}$$

Then the following lemma holds (see [1]).

Lemma 1.2 (*See [1]*)**.** *If there exist sub–super solutions ψ and z respectively, such that $\psi \leq z$ on Ω, Then* (1.1) *has a positive solution u such that $\psi \leq u \leq z$ in Ω.*

Proposition 1.3. *If $a \leq \lambda_1$ then* (1.1) *has no positive solution.*

Proof. Suppose not, i.e., assume that there exists a positive solution u of (1.1), then u satisfies

$$\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \left[am(x)u - bu^2 - c\frac{u^p}{u^p+1} - K \right] u dx.$$

But

$$\int_{\Omega} |\nabla u|^2 dx \geq \lambda_1 \int_{\Omega} am(x)u^2 dx.$$

Thus, we have

$$\int_{\Omega} \left[am(x)u - bu^2 - c\frac{u^p}{u^p+1} - K \right] u dx \geq \lambda_1 \int_{\Omega} am(x)u^2 dx,$$

and hence

$$(a - \lambda_1) \int_{\Omega} m(x)u^2 dx \geq \int_{\Omega} \left[bu^2 + c\frac{u^p}{u^p+1} + K \right] u dx \geq 0.$$

Since $u > 0$, $m(x) \geq m_0 > 0$, this requires $a > \lambda_1$, which is a contradiction. Hence (1.1) has no positive solution. □

2. Existence of solution

In this section we prove the existence of solution for problem (1.1) by comparison method (see [5]). It is easy to see that any subsolution of

$$\begin{cases} -\Delta u = am_0 u - bu^2 - c\dfrac{u^p}{u^p+1} - K & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

is a subsolution of (1.1). Also any super solution of

$$\begin{cases} -\Delta u = alu - bu^2 - c\dfrac{u^p}{u^p+1} - K & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

is a super solution of (1.1), where l is as defined above.

We denote by λ_1' the first eigenvalue of

$$\begin{cases} -\Delta \phi = \lambda' \phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega, \end{cases} \tag{2.3}$$

with positive principal eigenfunction ϕ_1' satisfying $\|\phi_1'\|_\infty = 1$.

Theorem 2.1. *If $a > \frac{\lambda_1'}{m_0}$, $b > 0$ and $c > 0$, then there exists a $K_0(a, b, c, p, m_0) > 0$ such that for $K < K_0(a, b, c, p, m_0)$, (1.1) has a positive solution.*

Proof. We use the method of sub–super solutions. We recall the anti-maximum principle (see [3]) in the following form. Let λ' is as defined above, then there exist $\sigma = \sigma(\Omega) > 0$

and a solution $z_{\lambda'}$ (with $z_{\lambda'} > 0$ in Ω and $\frac{\partial z_{\lambda'}}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outer unit normal to Ω) of

$$\begin{cases} -\Delta z - \lambda' z = -1 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

for $\lambda' \in (\lambda_1', \lambda_1' + \sigma)$. Fix

$$\lambda_*' \in \left(\lambda_1', \min\{\lambda_1' + \sigma, m_0 a\}\right).$$

Let $z_{\lambda_*'} > 0$ be the solution of (2.4) when $\lambda' = \lambda_*'$ and $\alpha = \|z_{\lambda_*'}\|_\infty$. Define

$$\psi := \mu K z_{\lambda_*'},$$

where $\mu \geq 1$ is to be determined later. We will choose μ and $K > 0$ properly so that ψ is a subsolution. Then,

$$-\Delta\psi = -\Delta(\mu K z_{\lambda_*'}) = \lambda_*'\psi - \mu K.$$

Thus, ψ is a subsolution if $\lambda_*'\psi - \mu K \leq a m_0 \psi - b\psi^2 - c\frac{\psi^p}{\psi^p+1} - K$. That is if

$$(am_0 - \lambda_*')\psi - b\psi^2 - c\frac{\psi^p}{\psi^p + 1} + (\mu - 1)K \geq 0.$$

Consider

$$H(y) = (am_0 - \lambda_*')y - by^2 - c\frac{y^p}{y^p + 1} + (\mu - 1)K.$$

It can be written as

$$H(y) = h_1(y) + h_2(y),$$

where

$$h_1(y) = (am_0 - \lambda_*')y - by^2 - cy^p + (\mu - 1)K$$

and

$$h_2(y) = \frac{cy^{2p}}{y^p + 1}.$$

Obviously, $h_2(y) \geq 0$ for all $y \geq 0$. So if we can find K and μ such that $h_1(y) \geq 0$ for $0 \leq y \leq \mu K\alpha$, then ψ will be a subsolution. Now $h_1(0) = (\mu - 1)K$, $h_1''(y) = -2b - cp(p-1)y^{p-2} < 0$ and there exists a unique y_0 such that $h_1(y_0) = 0$. This means that ψ is a subsolution if $h_1(\mu K\alpha) \geq 0$, i.e. if

$$(am_0 - \lambda_*')\mu K\alpha - b(\mu K\alpha)^2 - c(\mu K\alpha)^p + (\mu - 1)K \geq 0.$$

Let

$$G(K) = (am_0 - \lambda_*')\mu\alpha - b(\mu\alpha)^2 K - c(\mu\alpha)^p K^{p-1} + (\mu - 1).$$

Notice that $G(0) = (am_0 - \lambda'_*)\mu\alpha + (\mu - 1) > 0$, since $\mu \geq 1$ and $am_0 > \lambda'_*$. Also we have $G'(K) = -b(\mu\alpha)^2 - c(p-1)(\mu\alpha)^p K^{p-2} < 0$. Hence, given μ and p, there exists a unique $K^* = K^*(a, b, c, \mu, p, m_0) > 0$ with $G(K^*) = 0$. Since

$$G(K) \leq (am_0 - \lambda'_*)\mu\alpha - b(\mu\alpha)^2 K + (\mu - 1)K = \widetilde{G}(K),$$

we see that

$$K^* \leq \frac{(am_0 - \lambda'_*)\mu\alpha + (\mu - 1)}{b\mu^2\alpha^2} := K_1(a, b, \mu, m_0).$$

Note that $K_1(a, b, \mu, m_0)$ is bounded for $\mu \in [1, \infty)$. Hence K^* is bounded for $\mu \in [1, \infty)$. Let

$$K_0(a, b, c, m_0, p) = \sup_{\mu \geq 1} K^*(a, b, c, \mu, m_0, p).$$

Now let $\widetilde{K} < K_0(a, b, c, m_0, p)$. By definition there will exist a $\widetilde{\mu} \geq 1$ such that

$$\widetilde{K} < K^*(a, b, c, \widetilde{\mu}, m_0, p) < K_0(a, b, c, m_0, p).$$

Choose $\psi = \widetilde{\mu}\widetilde{K}z$. With $\mu = \widetilde{\mu}$ we have $G(\widetilde{K}) \geq 0$ and hence

$$(am_0 - \lambda'_*)\widetilde{\mu}\widetilde{K}\alpha - b(\widetilde{\mu}\widetilde{K}\alpha)^2 - c(\widetilde{\mu}\widetilde{K}\alpha)^p + (\widetilde{\mu} - 1)\widetilde{K} \geq 0.$$

Hence ψ turns out to be a subsolution to (1.1).

We next construct the super solution z for (1.1) such that $z \geq \psi$. Let $z = \overline{M}e$, where $\overline{M} > 0$ is such that $alu - bu^2 - c\frac{u^p}{u^p+1} - K \leq \overline{M}$ for all $u \geq 0$ and e is the unique positive solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly,

$$-\Delta z = \overline{M} \geq alz - bz^2 - c\frac{z^p}{z^p + 1} - K.$$

Thus, z is a super solution of (2.2). Therefore, z is a super solution of (1.1).

Since by the Hopf maximum principle $\frac{\partial e}{\partial \nu} < 0$ on $\partial\Omega$ (where ν is the outer unit normal to Ω), we can choose $M \gg 1$ so that $z = \overline{M}e \geq \psi$. Hence by Lemma 1.2 the problem has a positive solution for all $K < K_0(a, b, c, m_0, p)$. The proof is complete. □

3. An extension to system (3.1)

In this section, we consider the extension of (1.1) to the following system:

$$\begin{cases} -\Delta u = a_1 m(x)u - b_1 u^2 - c_1\dfrac{v^p}{v^p + 1} - K_1 & \text{in } \Omega, \\ -\Delta v = a_2 m(x)v - b_2 v^2 - c_2\dfrac{u^p}{u^p + 1} - K_2 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Delta u = \mathrm{div}(\nabla u)$ is the Laplacian of u, while $a_1, a_2, b_1, b_2, c_1, c_2, p, K_1, K_2$ are positive constants with $p \geq 2$, and Ω is a bounded smooth domain of $\mathbb{R}^N$ with $\partial\Omega$ in C^2. The weight function m satisfies $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, also $\|m\|_\infty = l < \infty$. We prove the following result by finding sub–super solutions to reaction–diffusion system (3.1).

Theorem 3.1. *If* $\min\{a_1, a_2\} > \frac{\lambda_1'}{m_0}$, *then there exists a* $K_0^*(a_1, a_2, b_1, b_2, c_1, c_2, p, m_0) > 0$ *such that for* $\max\{K_1, K_2\} < K_0^*$, (3.1) *has a positive solution. Here* λ_1' *is the first eigenvalue of operator* $-\Delta$ *with Dirichlet boundary conditions.*

Proof. Choose $\lambda_*' \in (\lambda_1', \min\{\lambda_1' + \sigma, m_0\tilde{a}\})$ where $\tilde{a} = \min\{a_1, a_2\}$ and σ is as before from the anti-maximum principle in the previous section. Define

$$(\psi_1, \psi_2) := (\mu K_1 z_{\lambda_*'}, \mu K_2 z_{\lambda_*'}).$$

By the same argument as in the proof of Theorem 2.1 we can show that (ψ_1, ψ_2) is a subsolution of (3.1) for $\max\{K_1, K_2\} < K_0^*$. Also it is easy to check that constant function $(z_1, z_2) := (\overline{M}e, \overline{M}e)$ is a super solution of (3.1) for $\overline{M}$ large. Further M can be chosen large enough so that $(z_1, z_2) \geq (\psi_1, \psi_2)$ on Ω. Hence for $\max\{K_1, K_2\} < K_0^*$, (3.1) has a positive solution and the proof is complete. $\square$

Acknowledgments

The authors would like to thank the unknown referee for his/her valuable comments and suggestions. This research work has been supported by a research grant from Islamic Azad University-Ayatollah Amoli Branch (52393911019014).

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.
[2] R. Causey, S. Sasi, R. Shivaji, An ecological model with grazing and constant yield harvesting, Bull. Belg. Math. Soc. Simon Stevin 17 (2010) 833–839.
[3] Ph. Clément, L.A. Peletier, An anti-maximum principle for second-order elliptic operators, J. Differential Equations 34 (1979) 218–229.
[4] P. Drábek, J. Hernández, Existence and uniqueness of positive solutions for some quasilinear elliptic problems, Nonlinear Anal. 44 (2001) 189–204.
[5] P. Drábek, P. Kerjčí, P. Takáč, Nonlinear Differential Equations, in: CRC Research Notes Math., vol. 404, Chapman & Hall/CRC, Boca Raton, FL-London-New York-Washington, DC, 1999.
[6] R.M. May, Thresholds and breakpoints in ecosystems with a multiplicity of stable states, Nature 269 (1977) 471–477.
[7] M. Nagumo, Über die Differentialgleichung $y'' = f(x, y, y')$, Proc. Phys.-Math. Soc. Japan 19 (1937) 861–866.
[8] I. Noy-Meir, Stability of grazing systems: An application of predator–prey graphs, J. Ecol. 63 (1975) 459–481.
[9] S. Oruganti, J. Shi, R. Shivaji, Diffusive logistic equation with constant yield harvesting, I: steady states, Trans. Amer. Math. Soc. 354 (2002) 3601–3619.
[10] H. Poincaré, Les fonctions fuchsiennes et l'équation $\Delta u = e^u$, J. Math. Pures Appl. 4 (1898) 137–230.
[11] V. Rădulescu, Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods, in: Contemporary Mathematics and Its Applications, vol. 6, Hindawi Publishing Corporation, New York, 2008.
[12] J.H. Steele, E.W. Henderson, Modeling long-term fluctuations in fish stocks, Science 224 (1984) 985–987.