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On the equivalence of two curvature conditions for Lorentzian hypersurfaces

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Abstract. Let $n \geq 3$. We show that semi-symmetry and Ricci-semisymmetry conditions are equivalent for any n-dimensional Lorentzian hypersurface in a Lorentzian space form with nonzero curvature. We also show that these curvature conditions are equivalent for any n-dimensional Lorentzian isoparametric hypersurface in Minkowski space $\mathbb{R}_1^{n+1}$, and we construct an example of a Ricci-semisymmetric 5-dimensional Lorentzian hypersurface in $\mathbb{R}_1^6$ which is not semi-symmetric.

Keywords: Lorentzian hypersurfaces in space forms; Semi-symmetry and Ricci-semisymmetry conditions

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1. Introduction

Recall that if T is a tensor field of type (r, s) on a pseudo-Riemannian manifold (M, g), and if ∇ and $\mathcal{R}$ denote the Levi-Civita connection and the curvature tensor of M, respectively, then for any vector fields X and Y on M, we define the action of $\mathcal{R}(X, Y)$ on T as follows

$$\mathcal{R}(X, Y) \cdot T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T.$$

We also define the tensor $\mathcal{R} \cdot T$ of type $(r, s+2)$ as follows

$$(\mathcal{R} \cdot T)(X_1, \ldots, X_s, X, Y) = (\mathcal{R}(X, Y) \cdot T)(X_1, \ldots, X_s).$$

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A pseudo-Riemannian manifold (M, g) is said to be semi-symmetric if

$$\mathcal{R} \cdot \mathcal{R} = 0 \tag{1}$$

holds on M. It is said to be Ricci-semisymmetric if

$$\mathcal{R} \cdot S = 0 \tag{2}$$

holds on M.

It is well known that the class of semi-symmetric manifolds includes the class of locally symmetric manifolds ($\nabla\mathcal{R} = 0$) as a proper subset and that the class of Ricci-semisymmetric manifolds includes the class of locally Ricci-symmetric manifolds (that is, $\nabla S = 0$) as a proper subset. It is clear that every semi-symmetric manifold is Ricci-semisymmetric. However, the converse is not true in general. It turns out that the conditions (1) and (2) are equivalent on any 3-dimensional pseudo-Riemannian manifold. For $n \geq 3$, P.J. Ryan proved in [8] that (1) and (2) are equivalent for any hypersurface in a Riemannian space form with nonzero curvature, and also for any hypersurface in Euclidean space with non-negative or constant scalar curvature. Further, in [7] it was proved that (1) and (2) are equivalent for complete hypersurfaces of Euclidean space. In [1], an example was given of a Ricci-semisymmetric hypersurface in Euclidean space which is not semi-symmetric. Recall that for a semi-Riemannian manifold (M, g), if the signature of the metric g is $(-, +, \ldots, +)$ and $\dim M \geq 2$, then (M, g) is called a Lorentzian manifold.

In this paper, we classify the shape operators of Ricci-semisymmetric Lorentzian hypersurfaces in Lorentzian space forms. We consider the equivalence of semi-symmetry and Ricci-semisymmetry conditions for Lorentzian hypersurfaces in Lorentzian space forms, and we give an example of a 5-dimensional Ricci-semisymmetric Lorentzian hypersurface of Minkowski space $\mathbb{R}^6_1$ which is not semi-symmetric.

2. Preliminaries

Recall that a Lorentzian vector space $(V, \langle , \rangle)$ is a vector space V of dimension $n > 1$ that is endowed with a scalar product $\langle , \rangle$ of index one. An endomorphism A of $(V, \langle , \rangle)$ is said to be self-adjoint if it satisfies $\langle AX, Y\rangle = \langle X, AY\rangle$ for all $X, Y \in V$. We know that a self-adjoint endomorphism in a Lorentzian vector space need not be diagonalizable. Self-adjoint endomorphisms are classified according to the following well known result (see [6]).

Lemma 1. *Let $(V, \langle , \rangle)$ be an n-dimensional Lorentzian vector space, and let A be a self-adjoint endomorphism of $(V, \langle , \rangle)$. Then, A has one of the following forms:*

(i) $A = diag(\lambda_1, \ldots, \lambda_n)$,

(ii) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \oplus diag(\lambda_3, \ldots, \lambda_n)$, *with* $b \neq 0$,

(iii) $A = \begin{bmatrix} \lambda & 0 \\ \epsilon & \lambda \end{bmatrix} \oplus diag(\lambda_3, \ldots, \lambda_n)$, $\epsilon = \pm 1$,

(iv) $A = \begin{bmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix} \oplus diag(\lambda_4, \ldots, \lambda_n)$,

where in cases (i) *and* (ii) *, A is represented relative to an orthonormal basis $\{e_1, \ldots, e_n\}$ with nonzero products $-\langle e_1, e_1\rangle = \langle e_i, e_i\rangle = 1$, with $2 \leq i \leq n$. In cases* (iii) *and* (iv)*, A is represented relative to a pseudo-orthonormal basis $\{e_1, \ldots, e_n\}$ with nonzero products $-\langle e_1, e_2\rangle = \langle e_i, e_i\rangle = 1$, with $3 \leq i \leq n$.*

Let $\overline{M}_1^{n+1}(\overline{c})$ be an $(n+1)$-dimensional Lorentzian space form, that is, a complete simply connected and connected $(n+1)$-dimensional Lorentzian manifold of constant curvature $\overline{c}$. If $f : M_1^n \to \overline{M}_1^{n+1}(\overline{c})$ is an isometric immersion from an n-dimensional Lorentzian manifold (M_1^n, g) into $\overline{M}_1^{n+1}(\overline{c})$, then we say that M_1^n is a Lorentzian hypersurface of $\overline{M}_1^{n+1}(\overline{c})$.

Let M_1^n be a Lorentzian hypersurface of $\overline{M}_1^{n+1}(\overline{c})$, and let ξ be a local spacelike unit normal field on $\overline{M}_1^n$. For any vector fields X and Y tangent to $\overline{M}_1^n$, the Gauss formula and the Weingarten formula are

$$\overline{\nabla}_X Y = f_*(\nabla_X Y) + h(X,Y)\xi,$$
$$\overline{\nabla}_X \xi = -f_*(AX),$$

where $\overline{\nabla}$ and ∇ denote, respectively, the Levi-Civita connections on $\overline{M}_1^{n+1}(\overline{c})$ and M_1^n, and h is the second fundamental form, and A is defined by $g(AX, Y) = h(X,Y)$. In fact, A is nothing but the shape operator of M_1^n derived from ξ. Since h is symmetric, A_x is self-adjoint on $T_x(M_1^n)$, for all x. At each point x, the type number of M_1^n at x is defined to be the rank of A_x, and it is denoted by $k(x)$.

Let M_1^n be a Lorentzian hypersurface of $\overline{M}_1^{n+1}(\overline{c})$. If the shape operator A is diagonalizable, M_1^n is said to be isoparametric if A has constant eigenvalues. If A is not diagonalizable, M_1^{n+1} is said to be isoparametric if the minimal polynomial of A is constant.

3. Ricci-semisymmetric Lorentzian hypersurfaces

Let (M_1^n, g) be a Lorentzian hypersurface in a Lorentzian space form $\overline{M}_1^{n+1}(\overline{c})$. The Ricci tensor field of M_1^n can be written as

$$S(X,Y) = \overline{c}(n-1)g(X,Y) + mg(AX,Y) - g\left(A^2X, Y\right)$$

where $m = trace\,(A)$ (see [3] or [7]). Let $\overline{S}$ denote the tensor field satisfying

$$S(X,Y) = g\left(\overline{S}X, Y\right).$$

It is clear that $\mathcal{R}\cdot S = 0$ if and only if $\mathcal{R}\cdot\overline{S} = 0$, and it is easy to prove the following fact.

Lemma 2. *$\mathcal{R}\cdot\overline{S} = 0$ if and only if $\mathcal{R}(X,Y)$ commutes with $\overline{S}$ for all X, Y.*

Proposition 3. *Let $n \geq 3$ and $\overline{c} \neq 0$, and let M_1^n be a Ricci-semisymmetric Lorentzian hypersurface in $\overline{M}_1^{n+1}(\overline{c})$. Then the shape operator cannot admit complex eigenvalues.*

Proof. Assume the contrary that the shape operator A has a complex eigenvalue at some point $x \in M_1^n$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $T_x M_1^n$ relative to which A takes the second form of Lemma 1. Then,

$$\begin{aligned}\overline{S}\mathcal{R}(e_1,e_2)e_1 &= (a^2+b^2+\overline{c})(\overline{c}(n-1)+ma-a^2+b^2)e_2\\ &\quad -b(a^2+b^2+\overline{c})(2a-m)e_1\\ \mathcal{R}(e_1,e_2)\overline{S}e_1 &= (\overline{c}(n-1)+ma-a^2+b^2)(a^2+b^2+\overline{c})e_2\\ &\quad -b(a^2+b^2+\overline{c})(2a-m)e_1.\end{aligned}$$

By Lemma 2, we have $\overline{S}\mathcal{R}(e_1,e_2)e_1 = \mathcal{R}(e_1,e_2)\overline{S}e_1$, which is equivalent to

$$(a^2+b^2+\overline{c})(2a-m)=0. \tag{3}$$

Similarly, for $i \geq 3$, we have

$$\begin{aligned} \overline{S}\mathcal{R}(e_1,e_i)e_1 &= (a\lambda_i+\overline{c})(\overline{c}(n-1)+m\lambda_i-\lambda_i^2)e_i \\ \mathcal{R}(e_1,e_i)\overline{S}e_1 &= (a\lambda_i+\overline{c})(\overline{c}(n-1)+m\lambda_i-\lambda_i^2)e_i+b^2(2a-m)\lambda_i e_i. \end{aligned}$$

Since $\overline{S}\mathcal{R}(e_1,e_i)e_1 = \mathcal{R}(e_1,e_i)\overline{S}e_1$, then

$$(a\lambda_i+\overline{c})\left[(a-\lambda_i)(m-a-\lambda_i)+b^2\right]+b^2(2a-m)\lambda_i=0. \tag{4}$$

Also, for $i \geq 3$, we have

$$\begin{aligned} \overline{S}\mathcal{R}(e_1,e_i)e_2 &= b\lambda_i(\overline{c}(n-1)+m\lambda_i-\lambda_i^2)e_i \\ \mathcal{R}(e_1,e_i)\overline{S}e_2 &= b\lambda_i(\overline{c}(n-1)+ma-a^2+b^2)e_i-b(2a-m)(a\lambda_i+\overline{c})e_i. \end{aligned}$$

Since $\overline{S}\mathcal{R}(e_1,e_i)e_2 = \mathcal{R}(e_1,e_i)\overline{S}e_2$, then

$$\left[(a-\lambda_i)(m-a-\lambda_i)+b^2\right]\lambda_i=(2a-m)(a\lambda_i+\overline{c}). \tag{5}$$

By multiplying (4) by λ_i and substituting (5) into the resulting equation, we get

$$(2a-m)\left[(a\lambda_i+\overline{c})^2+b^2\lambda_i^2\right]=0.$$

Since $b \neq 0$ and $\overline{c} \neq 0$, the last equation implies that $m=2a$.

Now, by substituting this into (4) and (5), we deduce that $a\lambda_i+\overline{c}=0$ and $\lambda_i=0$, respectively. This implies that $\overline{c}=0$, a contradiction. Thus, A cannot admit complex eigenvalues. ∎

In the case $\overline{c}=0$, we return to the proof of the last proposition. From Eq. (3) we get that $m=2a$, and by substituting this into (5) we deduce that $\lambda_i=0$ for any i. In this case, it turns out that the hypersurface is semi-symmetric. In this case, we have the following proposition and example (see [2]).

Proposition 4. *Let M_1^n be a semi-symmetric Lorentzian hypersurface in $\mathbb{R}_1^{n+1}$, $n \geq 3$, and let $x \in M_1^n$. If the shape operator A_x admits a complex eigenvalue $a+ib$ with $b \neq 0$, then*

$$A_x=\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \oplus 0_{n-2}.$$

In particular, $k(x) \leq 2$.

Examples of semi-symmetric Lorentzian hypersurfaces in Minkowski space whose shape operators have a complex eigenvalue do exist. Here is an example.

Example 5. Let M_1^2 be the surface defined by the parametrization $X:\mathbb{R}\times(0,+\infty)\to\mathbb{R}_1^3$ given by $X(s,t)=(\cosh s\sinh t, \sinh s\sinh t, s)$. It is easy to see that M_1^2 is a Lorentzian surface in $\mathbb{R}_1^3$ with induced metric $g = \cosh^2 t \left(ds^2 - dt^2\right)$ and normal vector field

$$N_{\pm} = \frac{\pm 1}{\cosh t} \left(\sinh s, \cosh s, -\sinh t\right).$$

The shape operator associated to N_+ is

$$A = \begin{pmatrix} 0 & \dfrac{1}{\cosh^2 t} \\ -\dfrac{1}{\cosh^2 t} & 0 \end{pmatrix}.$$

Now, in order to obtain examples of semi-symmetric Lorentzian hypersurfaces of dimensions $n \geq 3$ in $\mathbb{R}_1^{n+1}$ whose shape operators have a complex eigenvalue, it suffices to consider cylinders over the above Lorentzian surface, that is, products of the form $M_1^2 \times E^{n-2}$.

The following result describes the shape operators of Ricci-semisymmetric Lorentzian hypersurfaces.

Theorem 6. *Let $n \geq 3$, and let M_1^n be a Ricci-semisymmetric Lorentzian hypersurface in $\overline{M}_1^{n+1}(\overline{c})$. Then,*

(i) *If $\overline{c} = 0$, then A_x takes one of the following forms*

$$\begin{aligned} A_x &= \lambda I_p \oplus \mu I_{n-p}, \quad \textit{where either } \lambda\mu = 0 \textit{ or } \lambda = \mu, \\ A_x &= \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \oplus 0_{n-2}, \\ A_x &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \oplus 0_{n-2}, \quad \textit{with } b \neq 0, \\ A_x &= \begin{bmatrix} \lambda & 0 \\ \epsilon & \lambda \end{bmatrix} \oplus diag\left(\mu, 0, \ldots, 0\right), \quad \textit{with } \lambda\mu = 0, \\ A_x &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus 0_{n-3} \end{aligned}$$

where the first three forms are relative to orthonormal bases, and the last two forms are relative to pseudo-orthonormal bases.

(ii) *If $\overline{c} \neq 0$, then A_x takes one of the following forms*

$$\begin{aligned} A_x &= \lambda I_n, \quad \textit{with } \lambda \in \mathbb{R}, \\ A_x &= \lambda I_p \oplus \mu I_{n-p}, \quad \textit{with } \lambda\mu \neq 0 \textit{ and } \lambda \neq \mu. \textit{ In this case, we have either} \\ &\quad \lambda\mu + \overline{c} = 0 \textit{ or } 1 \leq p \leq n-1, \\ A_x &= \begin{bmatrix} \lambda & 0 \\ 0 & 0_{n-2} \end{bmatrix}, \quad \textit{with } \lambda \neq 0, \\ A_x &= \begin{bmatrix} \lambda & 0 \\ \epsilon & \lambda \end{bmatrix} \oplus \lambda I_{n-2}, \quad \textit{with } \lambda = 0 \textit{ or } \overline{c} = -\lambda^2, \end{aligned}$$

where the first three forms are relative to orthonormal bases, and the last two forms are relative to pseudo-orthonormal bases.

Proof. By Propositions 3 and 4, we know that if the shape operator A_x admits a complex eigenvalue then $\overline{c} = 0$ and

$$A_x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \oplus 0_{n-2}, \quad \text{with } b \neq 0.$$

Thus, by Lemma 1, there are three cases to be considered.

Case 1. A_x is diagonalizable. In this case, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x M_1^n$ such that $Ae_i = \lambda_i e_i$, $1 \leq i \leq n$. We easily verify that the Ricci-semisymmetry condition (2) is equivalent to

$$(\lambda_i - \lambda_j)(\lambda_i \lambda_j + \overline{c})(m - \lambda_i - \lambda_j) = 0, \quad i \neq j.$$

In this case, we see that the diagonal forms of A_x described in the statement of Theorem 6 easily follow from the above equation (compare [7], Theorem 4.5 and its proof).

Case 2. The shape operator A_x has the form (iii) of Lemma 1. Let $\{e_1, \ldots, e_n\}$ be a pseudo-orthonormal basis of $T_x M_1^n$ such that A_x has such a form. We compute

$$\overline{S}\mathcal{R}(e_1, e_2)e_1 = (\lambda^2 + \overline{c})\left[(\overline{c}(n-1) + m\lambda - \lambda^2)e_1 + \epsilon(m - 2\lambda)e_2\right],$$

and

$$\mathcal{R}(e_1, e_2)\overline{S}e_1 = \left(\overline{c}(n-1) + m\lambda - \lambda^2\right)(\lambda^2 + \overline{c})e_1 + \epsilon(m - 2\lambda)(\lambda^2 + \overline{c})e_2.$$

By Lemma 2, we have $\overline{S}\mathcal{R}(e_1, e_2)e_1 = \mathcal{R}(e_1, e_2)\overline{S}e_1$, which is equivalent to

$$(m - 2\lambda)\left(\lambda^2 + \overline{c}\right) = 0. \tag{6}$$

Similarly, for $i \neq 1, 2$, we have

$$\overline{S}\mathcal{R}(e_1, e_i)e_1 = -\epsilon\lambda_i\left(\overline{c}(n-1) + m\lambda_i - \lambda_i^2\right)e_i,$$

and

$$\mathcal{R}(e_1, e_i)\overline{S}e_1 = -\epsilon\left[\lambda_i\left(\overline{c}(n-1) + m\lambda - \lambda^2\right) + (m - 2\lambda)(\lambda\lambda_i + \overline{c})\right]e_i,$$

from which we get

$$\lambda_i(\lambda_i - \lambda)(m - \lambda_i - \lambda) = (m - 2\lambda)(\lambda\lambda_i + \overline{c}). \tag{7}$$

Also, for $i \neq 1, 2$, we have

$$\overline{S}\mathcal{R}(e_1, e_i)e_2 = -(\lambda\lambda_i + \overline{c})\left(\overline{c}(n-1) + m\lambda_i - \lambda_i^2\right)e_i,$$

and

$$\mathcal{R}(e_1, e_i)\overline{S}e_2 = -\left(\overline{c}(n-1) + m\lambda - \lambda^2\right)(\lambda\lambda_i + \overline{c})e_i,$$

from which we get

$$(\lambda\lambda_i + \overline{c})(\lambda_i - \lambda)(m - \lambda_i - \lambda) = 0. \tag{8}$$

By multiplying (7) by $(\lambda\lambda_i + \bar{c})$ and using (8), we deduce that

$$(m - 2\lambda)(\lambda\lambda_i + \bar{c}) = 0. \tag{9}$$

Also, by subtracting (9) from (6), we get

$$\lambda(m - 2\lambda)(\lambda - \lambda_i) = 0. \tag{10}$$

Finally, for $i \neq j$, we have

$$\overline{S}\mathcal{R}(e_i, e_j)e_j = (\lambda_i\lambda_j + \bar{c})\left(\bar{c}(n-1) + m\lambda_i - \lambda_i^2\right)e_i,$$

and

$$\mathcal{R}(e_i, e_j)\overline{S}e_j = (\lambda_i\lambda_j + \bar{c})\left(\bar{c}(n-1) + m\lambda_j - \lambda_j^2\right)e_i.$$

Since $\overline{S}\mathcal{R}(e_i, e_j)e_j = \mathcal{R}(e_i, e_j)\overline{S}e_j$, we get

$$(\lambda_i\lambda_j + \bar{c})(\lambda_i - \lambda_j)(m - \lambda_i - \lambda_j) = 0. \tag{11}$$

If $m \neq 2\lambda$, we deduce from (6) and (9) that $\bar{c} = -\lambda^2 = -\lambda\lambda_i$, that is $\lambda(\lambda - \lambda_i) = 0$. If $\lambda \neq 0$, then $\lambda = \lambda_i$ for all i. This covers the last form for A_x of case (ii) with the assumption that $\bar{c} = -\lambda^2$. If $\lambda = 0$, then $\bar{c} = 0$, and we get from (6) that

$$\lambda_i(m - \lambda_i) = 0. \tag{12}$$

Since $m = \sum_{i=3}^{n}\lambda_i \neq 0$, it follows that there exists some i_0 such that $\lambda_{i_0} \neq 0$. Now, (12) implies that $m = \lambda_{i_0}$, from which we deduce that all other λ_i are equal to zero. This covers the fourth form for A_x of case (i), with the assumption that $\lambda = 0$ and $\mu \neq 0$.

If $m = 2\lambda$, then (7), (8), and (11) become

$$\lambda_i(\lambda - \lambda i) = 0 \tag{13}$$

$$(\lambda\lambda_i + \bar{c})(\lambda_i - \lambda) = 0 \tag{14}$$

$$(\lambda_i\lambda_j + \bar{c})(\lambda_i - \lambda_j)(m - \lambda_i - \lambda_j) = 0. \tag{15}$$

We notice that, by (13), each nonzero λ_i must be equal to λ. Since $m = 2\lambda$, this implies $\lambda_i = 0$ for all i. By substituting this into (14), we get $\bar{c}\lambda = 0$.

If $\bar{c} = 0$, then we obtain the fourth form for A_x of case (i), with the assumption that $\mu = 0$. If $\bar{c} \neq 0$, then we obtain the last form for A_x of case (ii) with the assumption that $\lambda = 0$.

Case 3. The shape operator has the form (iv) of Lemma 1. Let $\{e_1, \ldots, e_n\}$ be a pseudo-orthonormal basis of $T_xM_1^n$ such that A_x has such a form. We compute

$$\begin{aligned}\overline{S}\mathcal{R}(e_1, e_2)e_2 &= -(\lambda^2 + \bar{c})(\bar{c}(n-1) + m\lambda - \lambda^2)e_2 + \left[(\lambda^2 + \bar{c}) - \lambda(m - 2\lambda)\right]e_1 \\ &\quad - \left[(\lambda^2 + \bar{c})(m - 2\lambda) + \lambda\left(\bar{c}(n-1) + m\lambda - \lambda^2\right)\right]e_3\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}(e_1, e_2)\overline{S}e_2 &= -\left(\bar{c}(n-1) + m\lambda - \lambda^2\right)(\lambda^2 + \bar{c})e_2 \\ &\quad - \lambda\left(\bar{c}(n-1) + m\lambda - \lambda^2\right)e_3 - \left[(\lambda^2 + \bar{c}) - \lambda(m - 2\lambda)\right]e_1.\end{aligned}$$

By Lemma 2, we have $\overline{S}\mathcal{R}(e_1, e_2)e_2 = \mathcal{R}(e_1, e_2)\overline{S}e_2$, which is equivalent to

$$(\lambda^2 + \overline{c})(m - 2\lambda) = 0, \tag{16}$$

$$3\lambda^2 - m\lambda + \overline{c} = 0. \tag{17}$$

Similarly, for $i \geq 4$, we have

$$\overline{S}\mathcal{R}(e_1, e_i)e_2 = -(\lambda\lambda_i + \overline{c})(\overline{c}(n-1) + m\lambda_i - \lambda_i^2)e_i$$

and

$$\mathcal{R}(e_1, e_i)\overline{S}e_2 = -(\lambda\lambda_i + \overline{c})\left(\overline{c}(n-1) + m\lambda + \lambda^2\right)e_i,$$

from which, we get

$$(\lambda\lambda_i + \overline{c})(\lambda - \lambda_i)(m - \lambda - \lambda_i) = 0. \tag{18}$$

Also, we have

$$\begin{aligned}\overline{S}\mathcal{R}(e_2, e_3)e_2 = {} & \lambda(\overline{c}(n-1) + m\lambda - \lambda^2)e_2 - [\lambda - (m - 2\lambda)]\, e_1 \\ & + \left[\lambda(m - 2\lambda) + \left(\overline{c}(n-1) + m\lambda - \lambda^2\right)\right] e_3\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}(e_2, e_3)\overline{S}e_2 = {} & \left[\left(\overline{c}(n-1) + m\lambda - \lambda^2\right) + (m - 2\lambda)\left(\lambda^2 + \overline{c}\right)\right] e_2 \\ & + (3\lambda - m)\, u + (\overline{c}(n-1) + m\lambda + \overline{c})\, e_3\end{aligned}$$

from which, we get $m = 3\lambda$. By Substituting this into (17), we get $\overline{c} = 0$. It follows then from (16) that $\lambda = 0$.

Now, for $i \geq 4$, we compute

$$\overline{S}\mathcal{R}(e_2, e_i)e_i = 0$$

and

$$\mathcal{R}(e_2, e_i)\overline{S}e_i = -\lambda_i^3 e_3,$$

from which, we get $\lambda_i = 0$. This covers the last form for A_x of case (i). And the proof of Theorem 6 is then complete. ∎

4. On the equivalence of semi-symmetry and Ricci-semisymmetry conditions

In this section we shall prove that the Ricci-semisymmetry and semi-symmetry conditions are equivalent on Lorentzian hypersurfaces in Lorentzian space forms with nonzero curvature, and we construct an example of a Ricci-semisymmetric Lorentzian hypersurface in $\mathbb{R}^6_1$ which is not semi-symmetric.

Proposition 7. *Let $n \geq 3$, and let M^n_1 be a Lorentzian hypersurface in a space form $\overline{M}^{n+1}_1(\overline{c})$, with $\overline{c} \neq 0$. Then $\mathcal{R} \cdot S = 0$ if and only if $\mathcal{R} \cdot \mathcal{R} = 0$.*

Proof. If the shape operator is diagonalizable, then we easily verify that the conditions $\mathcal{R} \cdot S = 0$ and $\mathcal{R} \cdot \mathcal{R} = 0$ are equivalent (compare [8], Proposition 7).

If the shape operator is nondiagonalizable, then we see from the Proof of Theorem 4.5 in [2] that $\mathcal{R} \cdot \mathcal{R} = 0$ if and only if the following equations are satisfied for $i \neq j$

$$\begin{aligned} \left(\lambda^2 + \overline{c}\right) \lambda_i &= 0 \\ \left(\lambda \lambda_i + \overline{c}\right) \lambda \left(\lambda - \lambda_i\right) &= 0 \\ \lambda_j \left(2\lambda\lambda_i + \overline{c} - \lambda_i^2\right) &= 0. \end{aligned}$$

But we have seen in the proof of Theorem 6 that, in this case, we have either $\lambda^2 + \overline{c} = 0$ and $\lambda_i = \lambda^2$ or $\lambda_i = 0$ for all i. Thus, the above equations are satisfied. ■

Proposition 8. *Let $n \geq 3$, and let M_1^n be a Lorentzian isoparametric hypersurface of the Minkowski space $\mathbb{R}_1^{n+1}$. Then $\mathcal{R} \cdot S = 0$ if and only if $\mathcal{R} \cdot \mathcal{R} = 0$.*

Proof. If the shape operator A_x is diagonalizable, then M_1^n has at most one nonzero eigenvalue (see [4], Corollary 2.7). In this case, it is clear that $\mathcal{R} \cdot S = 0$ and $\mathcal{R} \cdot \mathcal{R} = 0$ are equivalent. If the shape operator A_x is nondiagonalizable then, by Theorem 6, the shape operator A_x has one of the following forms

$$A_x = \begin{bmatrix} \lambda & 0 \\ \epsilon & \lambda \end{bmatrix} \oplus diag\left(\mu, 0, \ldots, 0\right), \text{ with } \lambda\mu = 0, \text{ or } A_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus 0_{n-3}.$$

If A_x has the first form then, as in Theorem 4.5 in [2], we get $\mathcal{R} \cdot \mathcal{R} = 0$ if and only if the following equations are satisfied for $3 \leq i, j, k \leq n$

$$\begin{aligned} \lambda\lambda_i &= 0 \\ \lambda_i \lambda \left(\lambda - \lambda_i\right) &= 0 \\ \lambda_j (2\lambda\lambda_i - \lambda_i^2) &= 0, \ i \neq j \\ \lambda_i \lambda_j \lambda_k \left(\lambda_i - \lambda_j\right) &= 0, \ i, j \text{ and } k \text{ are distinct.} \end{aligned}$$

Since $\lambda\mu = 0$, the above equations are satisfied.

If A_x has the second form then, as in the proof of Theorem 4.5 in [2], $\mathcal{R} \cdot \mathcal{R} = 0$ if and only if $\overline{c} = \lambda = \lambda_i = 0$ for $3 \leq i \leq n$, which is clearly satisfied. ■

The following example, which is inspired from [5], shows that the conditions $\mathcal{R} \cdot \mathcal{R} = 0$ and $\mathcal{R} \cdot S = 0$ are not equivalent for general Lorentzian hypersurfaces.

Example 9. We consider

$$S^2 = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \right\}.$$

and

$$S_1^2 = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1 \right\}.$$

Inside the product $M^4 = S_1^2 \times S^2$, we consider the cone

$$C^5 = \left\{ (tp, tq) \in \mathbb{R}^6 : (p, q) \in M^4, t > 0 \right\}.$$

It is clear that we can parametrize C^5 by

$$\begin{aligned} & f(t, u, v, \phi, \psi) \\ & \quad = (t \cosh u \cos v, t \cosh u \sin v, t \sinh u, t \sin \phi \cos \psi, t \sin \phi \sin \psi, t \cos \phi), \end{aligned}$$

and it is easy to check that the induced metric ds^2 on C^5 is

$$ds^2 = 2dt^2 - t^2 du^2 + t^2 \cosh^2 u dv^2 + t^2 d\phi^2 + t^2 \sin^2 \phi d\psi^2.$$

Since $t > 0$, then C^5 is Lorentzian hypersurface of the Minkowski space $\mathbb{R}_1^6$. Note that $\xi = \frac{1}{\sqrt{2}} (-p, q)$ is a unit normal on C^5.

Let x and y be parameters in S_1^2 and S^2, respectively. Therefore, we have

$$\begin{aligned} \partial_x &= (tp_x, 0) \\ \partial_y &= (0, tq_y) \\ \partial_t &= (0, 0). \end{aligned}$$

Now, we compute

$$\begin{aligned} D_{\partial x} \xi &= -\frac{1}{\sqrt{2}} (p_x, 0) \\ D_{\partial y} \xi &= \frac{1}{\sqrt{2}} (0, q_y) \\ D_{\partial t} \xi &= (0, 0). \end{aligned}$$

By Weingarten formula

$$\begin{aligned} A_x(\partial_x) &= \frac{1}{\sqrt{2}} (p_x, 0) = \frac{1}{\sqrt{2}t} \partial_x \\ A_x(\partial_y) &= -\frac{1}{\sqrt{2}} (0, q_y) = -\frac{1}{\sqrt{2}t} \partial_y \\ A_x(\partial_t) &= 0. \end{aligned}$$

It follows that $A_x = diag(0, \frac{1}{\sqrt{2}t}, \frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t})$, and it is clear now that the eigenvalues of A_x satisfy the condition $\mathcal{R} \cdot S = 0$ but not the condition $\mathcal{R} \cdot \mathcal{R} = 0$.

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