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ORIGINAL ARTICLE

On the Diophantine equation $x^2 - 4p^m = \pm y^n$

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Abstract Let m and n be positive integers and p any odd prime. In this paper we consider the Diophantine equation $x^2 - 4p^m = \pm y^n$ in positive integers x and y where $(x, y) = 1$, and we show that under some not very restrictive conditions, this equation has only finitely many solutions (x, y, m, n), and we provide a small explicit upper bound for n which only depends on p.

1. Introduction

Let a, b, x, y, m, n be positive integers. Many special cases of the Diophantine equations

$$ax^2 + b^m = 4y^n, (x, y) = 1,$$

have been studied over the years. Ljunggren [4] studied this equation for $a = m = 1$ and $n = b = 3$ and proved that it only has two solutions. When $a = m = 1$, y

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any odd prime p and $b = 4p^r - 1$ for positive integer r, the mathematician Skinner [9] studied this equation and proved that it has only two solutions except when $b = 11$ or $b = 19$, in each case it has exactly three solutions. In [5,6] Maohua studied the equation $x^2 \pm D = 4p^n$, with some conditions and he found upper bound for the solution. In [1] the first author with Luca and Togbé studied the equation $x^2 + 5^a \cdot 13^b = y^n$ where a, $b \geqslant 0$ and they proved that it has only the following solutions

$$(x, y, a, b, n) = (70, 17, 0, 1, 3), (142, 29, 2, 2, 3), (4, 3, 1, 1, 4).$$

In this paper, we consider the Diophantine equations of the

$$x^2 - 4p^m = y^n, \tag{1}$$

and

$$x^2 + y^n = 4p^m, \tag{2}$$

where x, y, m, n are positive integers, p prime, n, $m \geqslant 3$, and we give the upper bound for n which only depends on p.

2. Auxiliary results

Lemma 1. [2] *Let p be a prime number. Let α_1 and α_2 be two algebraic numbers which are p-adic units. Denote by f the residue class degree of extension $Q_p(\alpha_1,\alpha_2)/Q_p$ and put $D = [Q(\alpha_1,\alpha_2 : Q]/f$. Let b_1 and b_2 be two positive integers and put*

$$\Lambda_u = \alpha_1^{b_1} - \alpha_2^{b_2}.$$

Denote by $A_1 > 1$ and $A_2 > 1$ two real numbers such that

$$\log A_i \geqslant \max\{h(\alpha_i), (\log p)/D\}, \quad i = 1, 2$$

and put

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

If α_1 and α_2 are multiplicatively independent, then we have the lower bound

$$v_p(\Lambda_u) \leqslant \frac{24p(p^f - 1)}{(p-1)(\log p)^4} D^4 \left(\max\left\{ \log b' + \log\log p + 0.4, \frac{10 \log p}{D}, 5 \right\} \right)^2 \times \log A_1 \log A_2.$$

Lemma 2. [3] *Let $\alpha_1 \geqslant 1$ and $\alpha_2 \geqslant 1$ be two real algebraic numbers. Let b_1 and b_2 two positive integers and put*

$$\Lambda_a = b_1 \log \alpha_1 - b_2 \log \alpha_2$$

Let $D = [Q(\alpha_1,\alpha_2):Q]$ and denote by $A_1 > 1$ and $A_2 > 1$ two real numbers satisfying

$$\log A_i \geqslant \max\{h(\alpha_i), 1/D\}, \quad i = 1,2$$

Finally, put

$$b' = \frac{b_1}{D\log A_2} + \frac{b2}{D\log A_1}.$$

If α_1 and α_2 are multiplicatively independent, then we have the lower bound

$$\log|\Lambda_a| \geqslant -32.31D^4(\max\{\log b' + 0.18, 0.5, 10/D\})^2 \log A_1 \log A_2.$$

Lemma 3. [8] *Let $d > 1$ be a squarfree integer, and let k be a positive odd integer, coprime to d. Denote by $\tau > 1$ the fundamental unit of the field $Q(\sqrt{d})$. If X, Y and Z are three positive integers satisfying*

$$X^2 - dY^2 = \pm k^Z,$$

then there exist positive integers a, b, t and v with $a \equiv b(mod2)$, a and b even if $d \not\equiv 1(mod4)$, such that

$$X + Y\sqrt{d} = \tau^{-t}\left(\frac{a + b\sqrt{d}}{2}\right)^v$$

Moreover, $0 < t \leqslant v$ and the integer Z/v divides the class number of the field $Q(\sqrt{d})$.

Lemma 4. [7] *Let p be an odd prime. Denote by h_p and R_p the class number and the regulator of the quadratic field $Q(\sqrt{d})$. Then we have the upper bounds.*

$$h_p \leqslant 0.5p^{1/2} \text{and} \quad 0.4812 < R_p \leqslant h_pR_p \leqslant p^{1/2}\log(4p).$$

3. Main result

We state our main result, depending only on the value of p in the following theorem.

Theorem. **Let** *$p \geqslant 11$ be a prime integer. The Diophantine Eqs. (1) and (2) such that n, $m \geqslant 3$ odd integers and $(x,y) = 1$ may have solutions only if $n \leqslant 4\times 10^6p^{3/2}(log4p)^2(R_p + 0.6)$.*

Proof. Since $(x,y) = 1$, therefore y and x are odd integers. If $y = 1$, then Eq. (1) is impossible modulo 4 and Eq. (2) becomes

$$x^2 - 1 = 4p^m,$$

which implies

$$x+1=2p^m,$$
$$x-1=2.$$

We get $p=2$ and $m=1$ which is not true, so $y>2$. Let (x,y,m,n) be a solution of (1) and (2) with m odd. Denote by $\tau>1$ the fundamental unit of the field $Q(\sqrt{d})$ and by h_p and $R_p=\log\tau$ its class number and regulator respectively. By Lemma 3, there exists an algebraic integer $\varepsilon=\frac{a+b\sqrt{p}}{2}$ in $Q(\sqrt{d})$, and positive integers t and v such that $0<t\leqslant v$ and

$$x+2p^{(m-1)/2}\sqrt{p}=\varepsilon^v\tau^{-t}\quad x-2p^{(m-1)/2}\sqrt{p}=\varepsilon^{-v}(\lambda\tau)^t \tag{3}$$

Where $\bar{\varepsilon}$ denotes the conjugate of ε over $Q(\sqrt{d})$ and $\lambda\in\{1,-1\}$ is the norm of τ.
Moreover,

$$v \text{ divides } n \text{ and } n \text{ divides } h_p v \tag{4}$$

From (3) we deduce the equation

$$4p^{(m-1)/2}\sqrt{p}=\varepsilon^v\tau^{-t}-\varepsilon^{-v}(\lambda\tau)^t. \tag{5}$$

If we put

$$\Lambda_u=\frac{4p^{(m-1)/2}\sqrt{p}}{\varepsilon^v\tau^{-t}}=\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^v-(\lambda\tau^2)^t, \tag{6}$$

then we have $v_p(\Lambda_u)=\frac{m}{2}$. In order to bound n, we apply Lemma 1 to (6) with the parameters

$$\alpha_1=\frac{\varepsilon}{\bar{\varepsilon}},\quad \alpha_2=\lambda\tau^2,\quad b_1=v,\quad b_2=t,\quad f=1,\quad d=2.$$

Here

$$h\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)=\log\varepsilon,\quad \text{and}\quad h(\lambda\tau^2)=R_p,$$

hence

$$\log A_1\geqslant \max\left\{\log\varepsilon,\log\frac{p}{2}\right\},\quad \text{and}\quad \log A_2\geqslant \max\left\{2R_p,\log\frac{p}{2}\right\}.$$

So we have

$$\log A_1=1.54\log\varepsilon,\quad \log A_2=\frac{R_p\log p}{0.96}$$

and we have

$$b'=\frac{0.48v}{R_p\log p}+\frac{t}{3.08\log\varepsilon}.$$

Hence $b'\log p<2v$, which implies

$$\log b'+\log\log p+0.4<\log 2+logv+0.4.$$

Assuming that α_1 and α_2 are multiplicatively independent, we get

$$m \leqslant 1232p(\log p)^{-3}R_p(\max\{\log v + 1.1, 5\log p\})^2 \times \log\varepsilon. \tag{7}$$

The case of Eq. (1). From (3) we have inferred that $\varepsilon^v\tau^{-t} \leqslant 4p^{m/2}$, hence

$$2v\log\varepsilon \leqslant 2t\log\tau + \log 16 + m\log p,$$

together with (7), it yields

$$2vm \leqslant 1232p(\log p)^{-3}R_p(m\log p + \log 16 + 2tR_p)(\max\{\log v + 1.1, 5 \times \log p\})^2. \tag{8}$$

From $(4p)^m > 4p^m \geqslant 2^n$ and (4) we deduce that

$$\frac{t}{m} \leqslant \frac{v}{m} \leqslant \frac{n}{m} \leqslant \frac{\log 4p}{\log 2},$$

hence, using (8) and $m \geqslant 3$, we get

$$v \leqslant 4093p(\log p)^{-3}\log 4pR_p(R_p + 0.6)(\max\{\log v + 1.1, 5\log p\})^2. \tag{9}$$

Assume first that $\max\{\log v + 1.1, 5\log p\} = 5\log p$, then we get from (4) and (9)

$$vh_p \leqslant 4093(\log p)^{-3}p\log 4ph_pR_p(R_p + 0.6) \times 25(\log p)^2.$$

From Lemma 4 we obtain

$$n \leqslant 102325P^{3/2}(\log 4p)^2(R_p + 0.6).$$

And if $\max\{\log v + 1.1, 5\log p\} = \log v + 1.1$, in order to get a better bound for n, we search an upper bound for v of the shape $v \leqslant \gamma p\log 4pR_p(R_p + 0.6)$, with a suitable constant γ.

Since $p \geqslant 11$, we see that γ must satisfy the inequality $\gamma \geqslant 37684(\log\gamma + 2.75)^2$. Thus we may choose $\gamma = 4 \times 10^6$ and from Lemma 4 we get

$$n \leqslant 4 \times 10^6 P^{3/2}(\log 4p)^2(R_p + 0.6).$$

The case of Eq. (2). Divides Eq. (5) by $\varepsilon^v\tau^{-t}$ we get

$$\Lambda_u = \frac{4p^{(m-1)/2}\sqrt{p}}{\varepsilon^v\tau^{-t}} = 1 - \left(\frac{\bar{\varepsilon}}{\varepsilon}\right)^v(\lambda\tau^2)^t. \tag{10}$$

If $\Lambda_u \geqslant \frac{1}{2}$, then $8p^{(m-1)/2}\sqrt{p} \geqslant \varepsilon^v\tau^{-t}$ which implies

$$2v\log\varepsilon - 2t\log\tau \leqslant \log 64 + m\log p.$$

If $\Lambda_u < \frac{1}{2}$, then $2\Lambda_u < 1$ and

$$|\log(1 - \Lambda_u)| \leqslant 2\Lambda_u. \tag{11}$$

From (10) we get

$$|v\log|\frac{\bar{\varepsilon}}{\varepsilon}| - t\log\tau^2| \leqslant |v\log\left(\frac{\bar{\varepsilon}}{\varepsilon}\right) - t\log(\lambda\tau^2)| \leqslant |\log(1 - \Lambda_u)| \tag{12}$$

We apply Lemma 2 with

$$\alpha_1 = \frac{\varepsilon}{\bar{\varepsilon}}, \quad \alpha_2 = \lambda\tau^2, \quad b_1 = v, \quad b_2 = t, \quad D = 2,$$

we get

$$\log A_1 = \log \varepsilon, \quad \log A_2 = R_p,$$

and we have

$$b' = \frac{t}{2\log\varepsilon} + \frac{v}{2R_p}.$$

Then $b' \leqslant 1.93v$, which implies $\log b' \leqslant \log 1.93 + \log v$. From (11) we have,

$$\log 2 + \log \Lambda_u \geqslant -517R_p \max\{\log v + 0.47, 5\}^2 \log\varepsilon. \tag{13}$$

Since $\frac{4p^{(m-1/2)}}{\varepsilon^v\tau^{-t}} = \Lambda_u$, therefore

$$\log \Lambda_u = \frac{m}{2}\log p + \log 4 - v\log\varepsilon + t\log\tau \tag{14}$$

From (13) and (14), we get

$$\begin{aligned} &\log 2 + \frac{m}{2}\log p + \log 4 - v\log\varepsilon + t\log\tau \\ &\quad \geqslant -5172R_p \max\{\log v + 0.47, 5\}^2 \log\varepsilon, v\log\varepsilon - t\log\tau \\ &\quad \leqslant \log 2 + \log 4 + \frac{m}{2}\log p + 517R_p \max\{\log v + 0.47, 5\}^2 \log\varepsilon, \end{aligned} \tag{15}$$

and using (7) we get

$$\begin{aligned} v\log\varepsilon - t\log\tau \leqslant\ & 3\log 2 + 616(\log p)^{-3} pR_p \max\{\log v + 1.1, 5\log p\}^2 \\ & \times \log\varepsilon + 517R_p \max\{\log v + 0.47, 5\}^2 \log\varepsilon. \end{aligned} \tag{16}$$

First assume $\varepsilon \leqslant \exp(2R_p)$, so

$$\frac{2}{\log y} \leqslant \frac{4R_p}{\log 3 \log\varepsilon}. \tag{17}$$

From (3) we get $\varepsilon^v\tau^{-t} > y^{n/2}$, which implies

$$v\log\varepsilon - t\log\tau > \frac{n}{2}\log y \tag{18}$$

Now (16) and (18) we get

$$\begin{aligned} n \leqslant\ & 5164.31pR_p^2(\log p)^{-3} \max\{\log n + 1.1, 5\log p\}^2 + 4334.33R_p^2 \max\{\log n \\ & + 0.475\}^2. \end{aligned}$$

As before, we search an upper bound for n of the shape $n \leqslant \gamma p^{3/2}(\log 4p)^2(R_p + 0.6)$. Using Lemma 4 and a few calculation, we show that it suffices that γ satisfies

$$\gamma \geqslant 5164.3 \max\{\log\gamma + 2.74\}^2 + 4334.33 \max\{\log\gamma + 3.37\}^2.$$

Thus, we can choose

$$\gamma = 4 \times 10^6 \tag{19}$$

which gives the bound

$$n \leqslant 4 \times 10^6 p^{3/2} (\log 3p)^2 (R_p + 0.6).$$

And if $\varepsilon > \exp(2R_p)$, then

$$v \log \varepsilon - tR_p \geqslant \frac{v}{2} \log \varepsilon \tag{20}$$

From (16) and (20) we get

$$v \leqslant 1.8062 + 1232(\log p)^{-3} pR_p \max\{\log v + 1.1, 5\log p\}^2 + 1034R_p \max\{\log v + 0.47, 5\}^2.$$

Hence, by (4)

$$n \leqslant 1.8062h_p + 1232(\log p)^{-3} p(h_pR_p) \max\{\log n + 1.1, 5\log p\}^2 + 1034(h_pR_p) \times \max\{\log n + 0.47, 5\}^2.$$

And it is easy to show that (19) also holds in this case. Hence the last statements of the theorem is proved. □

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