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On the bi-harmonic maps with potential

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Abstract. In this note we characterize the harmonic maps and biharmonic maps with potential, and we prove that every biharmonic map with potential on a complete manifold satisfying some conditions is a harmonic map with potential.

Keywords: Harmonic maps with potential; Biharmonic maps with potential; Complete manifold; H-energy

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1. INTRODUCTION

The concept of harmonic maps with potential, was initially suggested by Ratto in [14] and recently developed by several authors : V. Branding [2], Y. Chu [5], A. Fardoun et al. [11], R. Jiang [12] and others.

In this paper we establish the second variation of the H-energy functional (Theorem 1), we introduce the notion of biharmonic maps with potential and we characterize the biharmonic maps with potential (Theorem 3). Also we prove that every biharmonic map with potential

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on a complete manifold satisfying some conditions is a harmonic map with potential (Theorem 5).

2. Harmonic maps with potential

Consider a smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds, and let H be a smooth function on N. For any compact domain D of M the H-energy functional of φ is defined by

$$E_H(\varphi; D) = \int_D \big[e(\varphi) - H(\varphi)\big] v_g, \tag{2.1}$$

where $e(\varphi)$ is the energy density of φ defined by

$$e(\varphi) = \sum_i \frac{1}{2} h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{2.2}$$

v_g is the volume element and $\{e_i\}$ is an orthonormal frame on (M, g).

Definition 1 (*[14]*). A map is called harmonic with potential H if it is a critical point of the H-energy functional over any compact subset D of M.

Let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then

$$\frac{d}{dt} E_H(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_H(\varphi), v) v_g, \tag{2.3}$$

where $v = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_H(\varphi) = \tau(\varphi) + (\operatorname{grad}^N H) \circ \varphi, \tag{2.4}$$

and $\tau(\varphi)$ is the tension field of φ given by

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi = \sum_i \Big(\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i)\Big) \tag{2.5}$$

(see [1]).

Corollary 1 (*[10,14]*). *A smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is harmonic with potential H if and only if*

$$\tau_H(\varphi) = 0.$$

Remark 1. Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. If the potential H is constant, then φ is harmonic with potential H if and only if φ is harmonic map.
One can refer to [3,4,7–9] for background on harmonic maps and generalized harmonic maps.

2.1. The second variation of the H-energy functional

Theorem 1. *Let $\varphi : (M, g) \to (N, h)$ be a harmonic map with potential H between Riemannian manifolds and $\{\varphi_{t,s}\}_{t,s\in(-\epsilon,\epsilon)}$ be a two-parameter variation with compact support* *in D. Set*

$$v=\frac{\partial\varphi_{t,s}}{\partial t}\Big|_{t=s=0},\quad w=\frac{\partial\varphi_{t,s}}{\partial s}\Big|_{t=s=0}. \tag{2.6}$$

Under the notation above we have the following

$$\frac{\partial^2}{\partial t\partial s}E_H(\varphi_{t,s};D)\Big|_{t=s=0}=\int_D h(J_H^{\varphi}(v),w)v_g, \tag{2.7}$$

where $J_H^{\varphi}(v)\in\Gamma(\varphi^{-1}TN)$ *is given by*

$$\begin{aligned}J_H^{\varphi}(v)&=-\operatorname{trace}R^N(v,d\varphi)d\varphi-\operatorname{trace}(\nabla^{\varphi})^2v-\left(\nabla_v^N\operatorname{grad}^N H\right)\circ\varphi\\&=-\sum_i\Big[R^N(v,d\varphi(e_i))d\varphi(e_i)-\nabla_{e_i}^{\varphi}\nabla_{e_i}^{\varphi}v+\nabla_{\nabla_{e_i}^M e_i}^{\varphi}v\Big]\\&\quad-\left(\nabla_v^N\operatorname{grad}^N H\right)\circ\varphi.\end{aligned} \tag{2.8}$$

Here R^N is the curvature tensor of (N,h) defined by

$$R^N(U,V)W=\nabla_U^N\nabla_V^N W-\nabla_V^N\nabla_U^N W-\nabla_{[U,V]}^N W,$$

for all $U,V,W\in\Gamma(TN)$, ∇^M (resp. ∇^N) is the Levi-Civita connection of (M,g) (resp. of (N,h)), ∇^{φ} denotes the pull-back connection on $\varphi^{-1}TN$.

Proof of Theorem 1. Define $\phi:M\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\to N$ by

$$\phi(x,t,s)=\varphi_{t,s}(x),\quad (x,t,s)\in M\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon), \tag{2.9}$$

let ∇^{ϕ} denote the pull-back connection on $\phi^{-1}TN$. Note that, for any vector field X on M considered as a vector field on $M\times(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$, we have

$$[\partial_t,X]=0,\quad [\partial_s,X]=0,\quad [\partial_t,\partial_s]=0. \tag{2.10}$$

Then, by (2.1) we obtain

$$\frac{\partial^2}{\partial t\partial s}E_H(\varphi_{t,s};D)\Big|_{t=s=0}=\int_D\frac{\partial^2}{\partial t\partial s}\big[e(\varphi_{t,s})-H(\varphi_{t,s})\big]\Big|_{t=s=0}v_g. \tag{2.11}$$

By calculation, from the divergence theorem we get

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}\int_D e(\varphi_{t,s})\Big|_{(t,s)=(0,0)}&=\sum_i\int_D h\big(R^N(v,d\varphi(e_i))w,d\varphi(e_i)\big)\\&\quad-\sum_i\int_D h\Big(\nabla_{\partial_t}^{\phi}d\phi(\partial_s)\Big|_{(t,s)=(0,0)},\tau(\varphi)\Big)\\&\quad-\sum_i\int_D h\big(w,\nabla_{e_i}^{\varphi}\nabla_{e_i}^{\varphi}v\big).\end{aligned} \tag{2.12}$$

$$\begin{aligned}-\frac{\partial^2}{\partial t\partial s}H(\varphi_{t,s})\Big|_{(t,s)=(0,0)}&=-h\Big(\nabla_{\partial_t}^{\phi}d\phi(\partial_s)\Big|_{(t,s)=(0,0)},(\operatorname{grad}^N H)\circ\varphi\Big)\\&\quad-h\big(w,(\nabla_v^{\varphi}\operatorname{grad}^N H)\circ\varphi\big).\end{aligned} \tag{2.13}$$

From formulas (2.11)–(2.13) and the harmonicity with potential H of φ, Theorem 1 follows. □

3. Biharmonic maps with potential

Consider a smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds and let H be a smooth function on N. A natural generalization of harmonic maps with potential is given by integrating the square of the norm of $\tau_H(\varphi)$. More precisely, the H-bienergy functional of φ is defined by

$$E_{2,H}(\varphi; D) = \frac{1}{2}\int_D |\tau_H(\varphi)|^2 \, v_g. \tag{3.1}$$

Definition 2. A map is called biharmonic with potential H, if it is a critical point of the H-bienergy functional over any compact subset D of M.

3.1. First variation of the H-bienergy functional

Theorem 2. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, H a smooth function on N, D a compact subset of M and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation with compact support in D. Then*

$$\frac{d}{dt}E_{2,H}(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_{2,H}(\varphi), v)\, v_g, \tag{3.2}$$

where $\tau_{2,H}(\varphi) \in \Gamma(\varphi^{-1}TN)$ is given by

$$\begin{aligned}\tau_{2,H}(\varphi) &= -\operatorname{trace} R^N(\tau_H(\varphi), d\varphi)d\varphi - \operatorname{trace}(\nabla^{\varphi})^2\tau_H(\varphi)\\ &\quad -\big(\nabla^N_{\tau_H(\varphi)} \operatorname{grad}^N H\big)\circ\varphi\\ &= -\sum_i\Big[R^N(\tau_H(\varphi), d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau_H(\varphi) + \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau_H(\varphi)\Big]\\ &\quad -\big(\nabla^N_{\tau_H(\varphi)} \operatorname{grad}^N H\big)\circ\varphi.\end{aligned} \tag{3.3}$$

Proof. Define $\phi : M \times (-\epsilon, \epsilon) \to N$ by $\phi(x, t) = \varphi_t(x)$. First note that

$$\frac{d}{dt}E_{2,H}(\varphi_t; D) = \int_D h(\nabla^{\phi}_{\partial_t}\tau_H(\varphi_t), \tau_H(\varphi_t))\, v_g. \tag{3.4}$$

Calculating in a normal frame at $x \in M$ we have

$$\nabla^{\phi}_{\partial_t}\tau_H(\varphi_t) = \sum_i \nabla^{\phi}_{\partial_t}\nabla^{\phi}_{e_i}d\varphi_t(e_i) + \nabla^{\phi}_{\partial_t}(\operatorname{grad}^N H)\circ\varphi_t, \tag{3.5}$$

by the definition of the curvature tensor of (N, h) we have

$$\nabla^{\phi}_{\partial_t}\nabla^{\phi}_{e_i}d\varphi_t(e_i) = R^N(d\phi(\partial_t), d\varphi_t(e_i))d\varphi_t(e_i) + \nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_t}d\varphi_t(e_i). \tag{3.6}$$

By the compatibility of ∇^{ϕ} with h we have

$$\begin{aligned}h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_t}d\varphi_t(e_i), \tau_H(\varphi_t)) &= e_i\big(h(\nabla^{\phi}_{\partial_t}d\varphi_t(e_i), \tau_H(\varphi_t))\big)\\ &\quad - h(\nabla^{\phi}_{\partial_t}d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_H(\varphi_t)).\end{aligned} \tag{3.7}$$

The second term of (3.7) is

$$\begin{aligned}-h(\nabla^{\phi}_{\partial_t}d\varphi_t(e_i), \nabla^{\phi}_{e_i}\tau_H(\varphi_t)) &= -e_i\big(h(d\phi(\partial_t), \nabla^{\phi}_{e_i}\tau_H(\varphi_t))\big)\\ &\quad + h(d\phi(\partial_t), \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\tau_H(\varphi_t)).\end{aligned} \tag{3.8}$$

From the definition of ∇^{ϕ} and the symmetry of the hessian tensor (i.e. $Hess(H)(X, Y) = h(\nabla_X \operatorname{grad}(H), Y) = Hess(H)(Y, X)$ (see [6,13])), we have

$$\begin{aligned} h\big(\nabla^{\phi}_{\partial_t}(\operatorname{grad}^N H)\circ\varphi_t, \tau_H(\varphi_t)\big) &= h\big((\nabla^N_{d\phi(\partial_t)} \operatorname{grad}^N H)\circ\varphi_t, \tau_H(\varphi_t)\big) \\ &= h\big((\nabla^N_{\tau_H(\varphi_t)} \operatorname{grad}^N H)\circ\varphi_t, d\phi(\partial_t)\big). \end{aligned} \tag{3.9}$$

By (3.4)–(3.9), $v = d\phi(\partial_t)$ when $t = 0$, and the divergence theorem (see [1]), Theorem 2 follows. □

From Theorem 2, we deduce:

Theorem 3. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and H be a smooth function on N. Then φ is biharmonic with potential H if and only if*

$$\begin{aligned} \tau_{2,H}(\varphi) = &-\operatorname{trace} R^N(\tau_H(\varphi), d\varphi)d\varphi - \operatorname{trace}(\nabla^{\varphi})^2\tau_H(\varphi) \\ &- \big(\nabla^N_{\tau_H(\varphi)} \operatorname{grad}^N H\big)\circ\varphi = 0. \end{aligned} \tag{3.10}$$

From Theorem 3, we have

Corollary 2. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and H be a smooth function on N. If φ is harmonic with potential H, then φ is biharmonic with potential H.*

Definition 3. Let H be a smooth function on N. A map $\varphi : (M, g) \to (N, h)$ is called a proper biharmonic map with potential H if and only if φ is a biharmonic map with potential H which is not a harmonic map with potential H.

Theorem 4. *Let $\varphi : (M, g) \to \mathbb{R}^n$ be a harmonic map and H be a non constant function on $\mathbb{R}^n$. Then φ is proper biharmonic with potential H if and only if*

$$\nabla_{\operatorname{grad} H} \operatorname{grad} H + \sum_i \nabla_{d\varphi(e_i)}(\nabla_{d\varphi(e_i)} \operatorname{grad} H) = 0$$

where $(e_1, \ldots, e_m)$ is a local orthonormal frame on M.

Proof. The proof follows directly from Theorem 3 and formula (2.4). □

Remark 2. Using Theorem 4 we can construct many examples of proper biharmonic map with potential.

Example 1. $Id_{\mathbb{R}^n}$ is a proper biharmonic map with potential H if and only if H is a solution of the following equation

$$\sum_i \Big(\frac{\partial^2 H}{\partial x_1^2} + \frac{1}{2}\Big(\frac{\partial H}{\partial x_i}\Big)^2\Big) = const.$$

Theorem 5. *Let (M, g) be a complete Riemannian manifold with infinite volume, (N, h) a Riemannian manifold with non-positive sectional curvature, and H a smooth function on N with* Hess $H \leq 0$. *Then, every biharmonic map with potential H from (M, g) to (N, h),* *satisfying*

$$\int_M |\tau_H(\varphi)|^2 \, v_g < \infty \tag{3.11}$$

is harmonic with potential H.

Proof. Assume that $\varphi : (M, g) \to (N, h)$ is a biharmonic map with potential H, fix a point $x \in M$ and let $\{e_1, \ldots, e_m\}$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i} e_j = 0$, at x for all $i, j = 1, \ldots, m$. By formula (3.10) we have

$$\begin{aligned}\tau_{2,H}(\varphi) &= -\sum_i R^N(\tau_H(\varphi), d\varphi(e_i))d\varphi(e_i) - \sum_i \nabla^\varphi_{e_i} \nabla^\varphi_{e_i} \tau_H(\varphi)\\ &\quad - \big(\nabla^N_{\tau_H(\varphi)} \operatorname{grad}^N H\big) \circ \varphi\\ &= 0,\end{aligned}$$

and then

$$\begin{aligned}-\sum_i h\big(\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} \tau_H(\varphi), \tau_H(\varphi)\big) &= \sum_i h\big(R^N(\tau_H(\varphi), d\varphi(e_i))d\varphi(e_i), \tau_H(\varphi)\big)\\ &\quad + h\big(\nabla^N_{\tau_H(\varphi)} \operatorname{grad}^N H, \tau_H(\varphi)\big) \circ \varphi.\end{aligned}$$

Since the sectional curvature of N is non-positive and Hess $H \leq 0$, we conclude that

$$-\sum_i h\big(\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} \tau_H(\varphi), \tau_H(\varphi)\big) \leq 0. \tag{3.12}$$

Let ρ be a smooth function with compact support on M. By (3.12) we have

$$-\sum_i h\big(\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} \tau_H(\varphi), \rho^2 \, \tau_H(\varphi)\big) \leq 0,$$

which is equivalent to

$$-\sum_i e_i\big[h\big(\nabla^\varphi_{e_i} \tau_H(\varphi), \rho^2 \, \tau_H(\varphi)\big)\big] + \sum_i h\big(\nabla^\varphi_{e_i} \tau_H(\varphi), \nabla^\varphi_{e_i} \rho^2 \, \tau_H(\varphi)\big) \leq 0. \tag{3.13}$$

Formula (3.13) is equivalent to

$$\begin{aligned}&-\sum_i e_i\big[h\big(\nabla^\varphi_{e_i} \tau_H(\varphi), \rho^2 \, \tau_H(\varphi)\big)\big] + \sum_i \rho^2 \, |\nabla^\varphi_{e_i} \tau_H(\varphi)|^2 \, v_g\\ &\quad + \sum_i 2 \, \rho \, e_i(\rho) \, h\big(\nabla^\varphi_{e_i} \tau_H(\varphi), \tau_H(\varphi)\big) \, v_g\\ &\leq 0.\end{aligned} \tag{3.14}$$

If we denote by $\omega(X) = h\big(\nabla^\varphi_X \tau_H(\varphi), \rho^2 \, \tau_H(\varphi)\big)$, then the inequality (3.14) becomes

$$-\, div^M \omega + \sum_i \rho^2 \, |\nabla^\varphi_{e_i} \tau_H(\varphi)|^2 \, v_g + \sum_i 2 \, \rho \, e_i(\rho) \, h\big(\nabla^\varphi_{e_i} \tau_H(\varphi), \tau_H(\varphi)\big) \, v_g \leq 0. \tag{3.15}$$

Using the divergence theorem (see [1,6]), we have $\int_M div^M \omega = 0$ and by integrating the formula (3.15) we obtain

$$\sum_i \int_M \rho^2 \, |\nabla^\varphi_{e_i} \tau_H(\varphi)|^2 \, v_g + \sum_i \int_M 2 \, \rho \, e_i(\rho) \, h\big(\nabla^\varphi_{e_i} \tau_H(\varphi), \tau_H(\varphi)\big) \, v_g \leq 0. \tag{3.16}$$

By the Young inequality we have

$$-2\,\rho\, e_i(\rho)\, h\big(\nabla^{\varphi}_{e_i}\tau_H(\varphi), \tau_H(\varphi)\big) \leq \epsilon\, \rho^2\, |\nabla^{\varphi}_{e_i}\tau_H(\varphi)|^2 + \frac{1}{\epsilon}\, e_i(\rho)^2\, |\tau_H(\varphi)|^2. \tag{3.17}$$

By (3.16) and (3.17) we have

$$\sum_i \int_M \rho^2 |\nabla^{\varphi}_{e_i}\tau_H(\varphi)|^2\, v_g \leq \epsilon \sum_i \int_M \rho^2\, |\nabla^{\varphi}_{e_i}\tau_H(\varphi)|^2\, v_g + \frac{1}{\epsilon}\sum_i \int_M e_i(\rho)^2\, |\tau_H(\varphi)|^2\, v_g. \tag{3.18}$$

If we put $\epsilon = \frac{1}{2}$, by (3.18) we have

$$\frac{1}{2}\sum_i \int_M \rho^2 |\nabla^{\varphi}_{e_i}\tau_H(\varphi)|^2\, v_g \leq 2 \sum_i \int_M e_i(\rho)^2 |\tau_H(\varphi)|^2\, v_g. \tag{3.19}$$

Choose the smooth cut-off function $\rho = \rho_R$, i.e.

$$\begin{cases} \rho \leq 1, & \text{on } M; \\ \rho = 1, & \text{on the ball } B(x,R); \\ \rho = 0, & \text{on } M \setminus B(x,2R); \\ |\mathrm{grad}^M \rho| \leq \dfrac{2}{R}, & \text{on } M. \end{cases}$$

Replacing $\rho = \rho_R$ in (3.19) we obtain

$$\frac{1}{2}\sum_i \int_M |\nabla^{\varphi}_{e_i}\tau_H(\varphi)|^2\, v_g \leq \frac{8}{R^2}\int_M |\tau_H(\varphi)|^2\, v_g. \tag{3.20}$$

Since $\int_M |\tau_H(\varphi)|^2\, v_g < \infty$, when $R \to \infty$, we have

$$\frac{1}{2}\sum_i \int_M |\nabla^{\varphi}_{e_i}\tau_H(\varphi)|^2\, v_g = 0.$$

In this way $\nabla^{\varphi}_{e_i}\tau_H(\varphi) = 0$ and $e_i\big(|\tau_H(\varphi)|^2\big) = h(\nabla^{\varphi}_{e_i}\tau_H(\varphi), \tau_H(\varphi)) = 0$ for all $i = 1, \ldots, m$, i.e. the function $|\tau_H(\varphi)|^2$ is constant on M. Finally, since the volume of M is infinite (i.e. $V(M) = \int_M v_g = +\infty$), from the formula (3.11) we conclude that $|\tau_H(\varphi)|^2 = constant = 0$, i.e. φ is harmonic with potential H. $\square$

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