ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 23 (2017) 133–140

On subspace-diskcyclicity

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Received 5 October 2015; received in revised form 1 June 2016; accepted 2 June 2016
Available online 14 June 2016

Abstract. In this paper, we define and study subspace-diskcyclic operators. We show that subspace-diskcyclicity does not imply diskcyclicity. We establish a subspace-diskcyclic criterion and use it to find a subspace-diskcyclic operator that is not subspace-hypercyclic for any subspaces. Also, we show that the inverses of invertible subspace-diskcyclic operators do not need to be subspace-diskcyclic for any subspaces. Finally, we prove that every finite-dimensional Banach space over the complex field supports a subspace-diskcyclic operator.

2010 Mathematics Subject Classification: primary 47A16; secondary 47A99

Keywords: Diskcyclic operators; Dynamics of linear operators in Banach spaces

1. INTRODUCTION

A bounded linear operator T on a separable Banach space $\mathcal{X}$ is hypercyclic if there is a vector $x \in \mathcal{X}$ such that its orbit $Orb(T,x) = \{T^n x : n \geq 0\}$ is dense in $\mathcal{X}$; such a vector x is called hypercyclic for T. The first example of a hypercyclic operator on a Banach space was constructed by Rolewicz in 1969 [11]. He showed that if B is the backward shift on $\ell^p(\mathbb{N})$ then λB is hypercyclic if and only if $|\lambda| > 1$.

The study of the scaled orbit and disk orbit is motivated by the Rolewicz example [11]. In 1974, Hilden and Wallen [7] defined the notion of supercyclicity. An operator T is called supercyclic if there is a vector x such that its scaled orbit $\mathbb{C}Orb(T,x)$ is dense in $\mathcal{X}$. The notion of a diskcyclic operator was introduced by Zeana [13]. An operator T is called diskcyclic if there is a vector $x \in \mathcal{X}$ such that its disk orbit $\mathbb{D}Orb(T,x)$ is dense in $\mathcal{X}$;

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2016.06.001

such a vector x is called diskcyclic for T. For more information about these operators, the reader may refer to [2,4,5].

In 2011, Madore and Martínez-Avendaño [9] considered the density of the orbit in a non-trivial subspace instead of the whole space, this phenomenon is called the subspace-hypercyclicity. An operator is called $\mathcal{M}$-hypercyclic or subspace-hypercyclic for a subspace $\mathcal{M}$ of $\mathcal{X}$ if there exists a vector such that the intersection of its orbit and $\mathcal{M}$ is dense in $\mathcal{M}$. They proved that subspace-hypercyclicity is an infinite dimensional phenomenon. Also, they asked whether the inverse of an invertible subspace-hypercyclic operator is again subspace-hypercyclic. This problem is still open. For more information on subspace-hypercyclicity, one may refer to [1,8,10].

In 2012, Xian-Feng et al. [12] defined the subspace-supercyclic operator as follows: An operator is called $\mathcal{M}$-supercyclic or subspace-supercyclic for a subspace $\mathcal{M}$ of $\mathcal{X}$ if there exists a vector such that the intersection of its scaled orbit and $\mathcal{M}$ is dense in $\mathcal{M}$.

Since both subspace-hypercyclicity and subspace-supercyclicity were studied, it is natural to define and study subspace-diskcyclicity. In the second section of this paper, we introduce the concept of subspace-diskcyclicity and subspace-disk transitivity. We show that not every subspace-diskcyclic operator is diskcyclic. We give the relation between different kinds of subspace-cyclicity. In particular, we give a set of sufficient conditions for an operator to be subspace-diskcyclic. We use these conditions to give an example of a subspace-diskcyclic operator which is not subspace-hypercyclic. Also, we give an example of a subspace-supercyclic operator that is not subspace-diskcyclic. Moreover, we give a simple example to show that the inverses of subspace-diskcyclic operators do not need to be subspace-diskcyclic which answers the corresponding question in [12, Question 1] for subspace-diskcyclicity. As a consequence, we show that every finite dimensional Banach space supports subspace-diskcyclic operators, which is not true for subspace-hypercyclicity.

2. Main results

In this paper, all Banach spaces $\mathcal{X}$ are infinite dimensional (unless stated otherwise) and separable over the field $\mathbb{C}$ of complex numbers. All subspaces of $\mathcal{X}$ are assumed to be non-trivial linear subspaces and topologically closed, and all relatively open sets are assumed to be non-empty. We will denote the closed unit disk by $\mathbb{D}$, the open unit disk by $\mathbb{U}$ and the set of all bounded linear operators on $\mathcal{X}$ by $\mathcal{B}(\mathcal{X})$.

Definition 2.1. Let $T \in \mathcal{B}(\mathcal{X})$, and let $\mathcal{M}$ be a subspace of $\mathcal{X}$. Then T is called a subspace-diskcyclic operator for $\mathcal{M}$ (or $\mathcal{M}$-diskcyclic, for short) if there exists a vector x such that $\mathbb{D}Orb(T,x) \cap \mathcal{M}$ is dense in $\mathcal{M}$; such a vector x is called a subspace-diskcyclic (or $\mathcal{M}$-diskcyclic, for short) vector for T.

Let $\mathbb{D}C(T,\mathcal{M})$ be the set of all $\mathcal{M}$-diskcyclic vectors for T, that is

$$\mathbb{D}C(T,\mathcal{M}) = \{x \in \mathcal{X} : \mathbb{D}Orb(T,x) \cap \mathcal{M} \text{ is dense in } \mathcal{M}\}.$$

Let $\mathbb{D}C(\mathcal{M},\mathcal{X})$ be the set of all $\mathcal{M}$-diskcyclic operators on $\mathcal{X}$, that is

$$\mathbb{D}C(\mathcal{M},\mathcal{X}) = \{T \in \mathcal{B}(\mathcal{X}) : \mathbb{D}Orb(T,x) \cap \mathcal{M} \text{ is dense in } \mathcal{M} \text{ for some } x \in \mathcal{X}\}.$$

The next example shows that subspace-diskcyclicity does not imply diskcyclicity.

Example 2.2. Suppose that T is a diskcyclic operator on $\mathcal{X}$, and x is a diskcyclic vector for T. Suppose that $\mathcal{N} = \mathcal{X} \oplus \{0\}$, and I is the identity operator on $\mathbb{C}^2$. Then, the operator $S = T \oplus I \in \mathcal{B}(\mathcal{X} \oplus \mathbb{C}^2)$ is not diskcyclic on $\mathcal{X} \oplus \mathcal{X}$; otherwise, we get I is a diskcyclic operator on $\mathbb{C}^2$ (see [4, Proposition 2.2]) which contradicts [4, Proposition 2.1]. However, it is clear that S is $\mathcal{N}$-diskcyclic, and $(x, 0)$ is an $\mathcal{N}$-diskcyclic vector for S.

From Example 2.2 above, it is clear that [4, Proposition 2.2] cannot be extended to subspace-diskcyclic operators, since I cannot be subspace-diskcyclic for any non-trivial subspace.

Definition 2.3. Let $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{M}$ be a subspace of $\mathcal{X}$. Then T is called subspace-disk transitive for $\mathcal{M}$ (or $\mathcal{M}$-disk transitive, for short) if for any two relatively open sets U and V in $\mathcal{M}$, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{U}^c$ such that $T^{-n}(\alpha U) \cap V$ contains a relatively open subset G of $\mathcal{M}$.

The next lemma gives some equivalent assertions to subspace-disk transitivity, which will be the tool to prove several facts in this paper.

Lemma 2.4. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{M}$ be a subspace of $\mathcal{X}$. Then the following assertions are equivalent:*

1. *T is $\mathcal{M}$-disk transitive,*
2. *For any two relatively open sets U and V in $\mathcal{M}$, there exist $\alpha \in \mathbb{U}^c$ and $n \in \mathbb{N}$ such that $T^{-n}(\alpha U) \cap V$ is non-empty and $T^n(\mathcal{M}) \subset \mathcal{M}$.*
3. *For any two relatively open sets U and V in $\mathcal{M}$, there exist $\alpha \in \mathbb{U}^c$ and $n \in \mathbb{N}$ such that $T^{-n}(\alpha U) \cap V$ is non-empty and relatively open in $\mathcal{M}$.*

Proof. (1) ⇒ (2): Let U and V be two relatively open sets in $\mathcal{M}$. By condition (1), there exist $\alpha \in \mathbb{U}^c$, $n \in \mathbb{N}$ and a relatively open set G in $\mathcal{M}$ such that $G \subset T^{-n}(\alpha U) \cap V$. It follows that

$$T^{-n}(\alpha U) \cap V \text{ is non-empty.} \tag{1}$$

Since $G \subset T^{-n}(\alpha U)$ it follows that $\frac{1}{\alpha}T^n G \subset U \subset \mathcal{M}$. Let $x \in \mathcal{M}$ and $x_0 \in G$. Then, there exists $r \in \mathbb{N}$ such that $(x_0 + rx) \in G$. Then, we get

$$\frac{1}{\alpha}T^n x_0 + \frac{1}{\alpha}T^n rx = \frac{1}{\alpha}T^n(x_0 + rx) \in \frac{1}{\alpha}T^n G \subset \mathcal{M}.$$

Since $x_0 \in G$ then $\frac{1}{\alpha}T^n x_0 \in \frac{1}{\alpha}T^n G \subset \mathcal{M}$, it follows that $\frac{r}{\alpha}T^n x \in \mathcal{M}$ and so

$$T^n x \in \mathcal{M}. \tag{2}$$

The proof follows from (1) and (2).
(2) ⇒ (3): Since $T^n|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$, then $T^{-n}(\alpha U)$ is relatively open in $\mathcal{M}$ for any relatively open set U in $\mathcal{M}$. It follows that $T^{-n}(\alpha U) \cap V$ is also relatively open in $\mathcal{M}$.
(3) ⇒ (1) is trivial. □

The next theorem shows that every subspace-disk transitive operator is subspace-diskcyclic for the same subspace. First, we need the following lemma.

We will suppose that $\{B_k : k \in \mathbb{N}\}$ is a countable open basis for the relative topology of a subspace $\mathcal{M}$.

Lemma 2.5. *Let T be an $\mathcal{M}$-diskcyclic operator. Then*

$$\mathbb{D}C(T,\mathcal{M}) = \bigcap_{k\in\mathbb{N}} \Big(\bigcup_{\substack{\alpha\in\mathbb{U}^c\\ n\in\mathbb{N}}} T^{-n}(\alpha B_k) \Big).$$

Proof. We have $x \in \mathbb{D}C(T,\mathcal{M})$ if and only if $\{\alpha T^n x : n \in \mathbb{N}, \alpha \in \mathbb{D}\setminus\{0\}\} \cap \mathcal{M}$ is dense in $\mathcal{M}$ if and only if for each $k > 0$, there are $\alpha \in \mathbb{D}\setminus\{0\}$ and $n \in \mathbb{N}$ such that $\alpha T^n x \in B_k$ if and only if $x \in \bigcap_{k\in\mathbb{N}} \big(\bigcup_{\substack{\alpha\in\mathbb{U}^c\\ n\in\mathbb{N}}} T^{-n}(\alpha B_k) \big)$. □

Theorem 2.6. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{M}$ be a subspace of $\mathcal{X}$. Suppose that T is $\mathcal{M}$-disk transitive, then $\bigcap_k \big(\bigcup_{\substack{\alpha\in\mathbb{U}^c\\ n\in\mathbb{N}}} T^{-n}(\alpha B_k) \big)$ is dense in $\mathcal{M}$.*

Proof. Since T is $\mathcal{M}$-disk transitive, then by Lemma 2.4, for each $i, j \in \mathbb{N}$, there exist $n_{i,j} \in \mathbb{N}$ and $\alpha_{i,j} \in \mathbb{U}^c$ such that

$$T^{-n_{i,j}}(\alpha_{i,j} B_i) \cap B_j$$

is non-empty relatively open in $\mathcal{M}$. Suppose that

$$A_i = \bigcup_{j=1}^{\infty} \big(T^{-n_{i,j}}(\alpha_{i,j} B_i) \cap B_j \big)$$

for all $i \in \mathbb{N}$. Then A_i is non-empty and relatively open in $\mathcal{M}$ since it is a countable union of relatively open sets in $\mathcal{M}$. Furthermore, each A_i is dense in $\mathcal{M}$ since it intersects each B_j. By the Baire category theorem, we get

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \big(T^{-n_{i,j}}(\alpha_{i,j} B_i) \cap B_j \big)$$

is a dense set in $\mathcal{M}$. Clearly,

$$\bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} T^{-n_{i,j}}(\alpha_{i,j} B_i) \cap B_j \subset \bigcap_{i} \bigcup_{\substack{\alpha\in\mathbb{U}^c\\ n\in\mathbb{N}}} T^{-n}(\alpha B_i) \cap \mathcal{M}.$$

It follows that $\bigcap_{i\in\mathbb{N}} \bigcup_{\substack{\alpha\in\mathbb{U}^c\\ n\in\mathbb{N}}} T^{-n}(\alpha B_i) \cap \mathcal{M}$ is dense in $\mathcal{M}$. The proof is completed. □

Corollary 2.7. *If T is an $\mathcal{M}$-disk transitive operator, then T is $\mathcal{M}$-diskcyclic.*

Proof. The proof follows from Lemma 2.5 and Theorem 2.6. □

It is clear from Definition 2.1, that every $\mathcal{M}$-hypercyclic operator is $\mathcal{M}$-diskcyclic which in turn is $\mathcal{M}$-supercyclic. On the other hand, the following two examples show that the reversed directions are not true in general. First we need the following lemmas.

Lemma 2.8. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{M}$ be a subspace of $\mathcal{X}$. Suppose that $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence of positive integers, $\{\lambda_{n_k}\}_{k\in\mathbb{N}} \subset \mathbb{D}\setminus\{0\}$ and $D_1, D_2 \in \mathcal{M}$ are two dense sets in $\mathcal{M}$ such that*

(a) *For every* $y \in D_2$*, there is a sequence* $\{x_k\}_{k\in\mathbb{N}}$ *in* $\mathcal{M}$ *such that* $\left\|\lambda_{n_k}^{-1}x_k\right\| \to 0$ *and* $T^{n_k}x_k \to y$ *as* $k \to \infty$*,*

(b) $\|\lambda_{n_k}T^{n_k}x\| \to 0$ *for all* $x \in D_1$ *as* $k \to \infty$*,*

(c) $T^{n_k}\mathcal{M} \subseteq \mathcal{M}$ *for all* $k \in \mathbb{N}$*.*

Then T *is* $\mathcal{M}$*-diskcyclic.*

Proof. Let U_1 and U_2 be two relatively open sets in $\mathcal{M}$. Then we can find $x \in D_1 \cap U_1$ and $y \in D_2 \cap U_2$. By hypothesis, there exist a sufficiently small positive ϵ and a large positive integer J, such that

$$\|\lambda_{n_J}T^{n_J}x\| < \frac{\epsilon}{2}, \qquad \left\|\lambda_{n_J}^{-1}x_J\right\| < \epsilon \quad \text{and} \quad \|T^{n_J}x_J - y\| < \frac{\epsilon}{2}.$$

Let $z = x+\lambda_{n_J}^{-1}x_J$, then $z \in \mathcal{M}$. Since $\|z - x\| = \left\|\lambda_{n_J}^{-1}x_J\right\| < \epsilon$, then $z \in U_1$. By condition (c), $\lambda_{n_J}T^{n_J}z \in \mathcal{M}$. Now, since $\lambda_{n_J}T^{n_J}z = \lambda_{n_J}T^{n_J}x + T^{n_J}x_J$, then

$$\|\lambda_{n_J}T^{n_J}z - y\| = \|\lambda_{n_J}T^{n_J}x + T^{n_J}x_J - y\| \leq \|\lambda_{n_J}T^{n_J}x\| + \|T^{n_J}x_J - y\| < \epsilon.$$

Thus $\lambda_{n_J}T^{n_J}z \in U_2$. By Lemma 2.4 and Corollary 2.7, T is $\mathcal{M}$-diskcyclic. □

Lemma 2.9 ($\mathcal{M}$*-Diskcyclic Criterion*)**.** *Let* $T \in \mathcal{B}(\mathcal{X})$ *and* $\mathcal{M}$ *be a subspace of* $\mathcal{X}$*. Suppose that* $\{n_k\}_{k\in\mathbb{N}}$ *is an increasing sequence of positive integers and* $D_1, D_2 \in \mathcal{M}$ *are two dense sets in* $\mathcal{M}$ *such that*

(a) *For every* $y \in D_2$*, there is a sequence* $\{x_k\}_{k\in\mathbb{N}}$ *in* $\mathcal{M}$ *such that* $\|x_k\| \to 0$ *and* $T^{n_k}x_k \to y$ *as* $k \to \infty$*,*

(b) $\|T^{n_k}x\|\,\|x_k\| \to 0$ *for all* $x \in D_1$ *as* $k \to \infty$*,*

(c) $T^{n_k}\mathcal{M} \subseteq \mathcal{M}$ *for all* $k \in \mathbb{N}$*.*

Then T *is* $\mathcal{M}$*-diskcyclic.*

Proof. We verify the hypotheses of Lemma 2.8. Let $\{\epsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers decreasing to 0, and let $\{z_n\}_{n\in\mathbb{N}} \subset D_1$ and $\{y_n\}_{n\in\mathbb{N}} \subset D_2$ be two countable dense subsets in $\mathcal{M}$. By hypothesis, for each $j = 1, \ldots, k$, there exists a sequence $\left\{x_k^{(j)}\right\}_{k\in\mathbb{N}}$ in $\mathcal{M}$ such that $\left\|x_k^{(j)}\right\| < \epsilon_k$, $T^{n_k}x_k^{(j)} \to y_j$ and

$$\|T^{n_k}z_i\|\left\|x_k^{(j)}\right\| < \epsilon_k^2 \tag{3}$$

for all $i = 1, \ldots, k$. Suppose that for each $k \geq 1$,

$$\lambda_{n_k} = \frac{1}{\epsilon_k}\max_{1\leq j\leq k}\left\{\left\|x_k^{(j)}\right\|\right\}.$$

It follows that $\lambda_{n_k} \in \mathbb{D}\backslash\{0\}$ for all $k \geq 1$, and

$$\frac{1}{\lambda_{n_k}}\left\|x_k^{(j)}\right\| \leq \frac{1}{\lambda_{n_k}}\max_{1\leq j\leq k}\left\{\left\|x_k^{(j)}\right\|\right\} = \epsilon_k \quad \text{for all } j \leq k. \tag{4}$$

By (3), we get

$$\lambda_{n_k} \|T^{n_k} z_i\| = \frac{1}{\epsilon_k} \max_{1\le j\le k} \left\{ \left\| x_k^{(j)} \right\| \right\} \|T^{n_k} z_i\| < \epsilon_k \quad \text{for all } i \le k. \tag{5}$$

As $k \to \infty$, the proof follows from (4) and (5) and from density of $\{z_n\}$ and $\{y_n\}$ in D_1 and D_2, respectively. □

The next example shows that $\mathcal{M}$-diskcyclicity does not imply $\mathcal{M}$-hypercyclicity. First, we need the following lemma.

Lemma 2.10. *Let T be an invertible bilateral weighted shift on $\ell^p(\mathbb{Z})$ and $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers. Suppose that $\mathcal{M}$ is a subspace of $\ell^p(\mathbb{Z})$ with the canonical basis $\{e_{m_i}\}$ such that $T^{n_k}\mathcal{M} \subseteq \mathcal{M}$. If there exists an $i \in \mathbb{N}$ such that $T^{n_k} e_{m_i} \to 0$, then $T^{n_k} e_{m_r} \to 0$ for all $r \in \mathbb{N}$.*

Proof. Since $T^{n_k}\mathcal{M} \subseteq \mathcal{M}$, the proof is similar to the proof of [6, Lemma 3.1]. □

Example 2.11. Let $F : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ be a bilateral weighted forward shift operator, defined by $F(e_n) = w_n e_{n+1}$ for all $n \in \mathbb{Z}$, where

$$w_n = \begin{cases} 3 & \text{if } n \ge 0, \\ 4 & \text{if } n < 0. \end{cases}$$

Let $\mathcal{M}$ be the subspace of $\ell^p(\mathbb{Z})$ defined as follows:

$$\mathcal{M} = \left\{ \{a_n\}_{n=-\infty}^{\infty} \in \ell^p(\mathbb{Z}) : a_{2n} = 0 \right\},$$

then F is an $\mathcal{M}$-diskcyclic operator, not $\mathcal{M}$-hypercyclic.

Proof. We will apply the $\mathcal{M}$-diskcyclic criterion to give the proof. Let D be a dense subset of $\mathcal{M}$, consisting of all sequences with finite support. Let $n_k = 2k$ for all $k \in \mathbb{N}$. It is clear that the set $C = \{e_m : m \in O\}$ is the canonical basis for $\mathcal{M}$, where O is the set of all odd integer numbers. Let $x, y \in D$, then $x = \sum_{i\in O} x_i e_i$ and $y = \sum_{i\in O} y_i e_i$, where $x_i, y_i \in \mathbb{C}$ for all $i \in O$.

Let B be a bilateral weighted backward shift on $\ell^p(\mathbb{Z})$ defined by $Be_n = z_n e_{n-1}, n \in \mathbb{Z}$, where

$$z_n = \begin{cases} \dfrac{1}{3} & \text{if } n > 0, \\ \dfrac{1}{4} & \text{if } n \le 0. \end{cases}$$

Suppose that $x_k = B^{2k} y$ for all $k \in \mathbb{N}$. Since $|w_n| \ge 4$ and $|z_n| \ge 1/4$ for all $n \in \mathbb{Z}$, then F and B are invertible with $F^{-1} = B$. Since B and F are linear and invertible, then it is sufficient by the triangle inequality, Lemma 2.10 and [3, Lemma 3.4] to assume that $x = y = e_1$. Since

$$B^{2k} e_1 = \left(\prod_{j=0}^{1-2k} z_j \right) e_{1-2k},$$

then $\|B^{2k}e_1\| = \frac{1}{4^{2k}} \to 0$ as $k \to \infty$. Hence

$$\|x_k\| \to 0 \quad \text{as } k \to \infty. \tag{6}$$

It is easy to show that for a large enough k,

$$F^{2k}x_k = y. \tag{7}$$

It follows from (6) and (7) that the condition (a) in Lemma 2.9 holds.

Moreover, we have

$$\|F^{2k}e_1\| \, \|B^{2k}e_1\| = \left\| \prod_{j=1}^{2k} w_j \right\| \left\| \prod_{j=0}^{1-2k} z_j \right\| = \left(\frac{3}{4}\right)^{2k} \to 0,$$

as $k \to \infty$. Hence the condition (b) in Lemma 2.9 holds. It can be easily deduced from the definition of $\mathcal{M}$ that for each $x \in \mathcal{M}$ and each $k \in \mathbb{N}$, the sequence $F^{2k}x$ will have a zero entry on all even positions, that is

$$F^{2k}x \in \mathcal{M}.$$

It follows that the condition (c) in Lemma 2.9 holds. Thus F is an $\mathcal{M}$-diskcyclic operator.

Note that the operator F is clearly not $\mathcal{M}$-hypercyclic since

$$\|F^{n_k}e_i\| = \left\| \prod_{j=i}^{i+n_k-1} w_j \right\| \to \infty$$

for any increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers, and any $i \in \mathbb{Z}$, that is, its orbit cannot be dense in any subspace. □

The next simple example shows that $\mathcal{M}$-supercyclicity does not imply $\mathcal{M}$-diskcyclicity.

Example 2.12. Let I be the identity operator on the space $\mathbb{C}^k$ for some $k \geq 2$, and let $\mathcal{M}$ be a one dimensional subspace of $\mathbb{C}^k$. Then it is clear that $\mathbb{C}Orb(I, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$ for some non-zero vector $x \in \mathbb{C}^k$, that is, I is $\mathcal{M}$-supercyclic. However, $\mathbb{D}Orb(I, x) \cap \mathcal{M}$ cannot be dense in $\mathcal{M}$ for any $x \in \mathbb{C}^k$, that is, I is not $\mathcal{M}$-diskcyclic.

The following example gives several useful consequences, some of them answering the corresponding questions in [12, Question 3.3], [9, Question 1] and [10, Question 1] for subspace-diskcyclicity.

Example 2.13. Let $T = kx \in \mathcal{B}(\mathbb{C}^n)$, $k \in \mathbb{D}^c, n \geq 2$. Let $\mathcal{M} = \{y : y = (a, 0, 0, \ldots, 0), y \in \mathbb{C}^n\}$ be a one dimensional subspace of $\mathbb{C}^n$. Then

1. T and T^* are $\mathcal{M}$-diskcyclic operators,
2. T^{-1} is not a subspace-diskcyclic operator for any subspace,
3. There is some vector $x \in \mathbb{C}^n$ such that $\mathbb{D}Orb(T^{-1}, x)$ is somewhere dense in $\mathcal{M}$, but not everywhere dense in $\mathcal{M}$.

Proof. For (1), let $x = (1, 0, 0, \dots, 0)$, then

$$\mathbb{D}Orb(T, x) \cap \mathcal{M} = \{(\alpha k^n, 0, 0, \dots, 0) : \alpha \in \mathbb{D}, n \geq 0\}.$$

Let $z = (b, 0, 0, \dots, 0) \in \mathcal{M}$, and let us choose an $m \in \mathbb{N}$ such that $|k^m| \geq |b|$. Then it is clear that $z = \left(k^m \left(\frac{b}{k^m}\right), 0, 0, \dots, 0\right) \in \mathbb{D}Orb(T, x) \cap \mathcal{M}$. It follows that T is an $\mathcal{M}$-diskcyclic operator. By the same way, we can show that $T^* = \bar{k}x$ is $\mathcal{M}$-diskcyclic.

For (2), since $T^{-1}x = \frac{1}{k}x$ then $\mathbb{D}Orb(T^{-1}, x)$ is bounded for all $x \in \mathbb{C}^n$, and hence T^{-1} cannot be dense in any proper subspace of $\mathbb{C}^n$. Thus, T^{-1} is not $\mathcal{M}$-diskcyclic.

For (3), let $x = (1, 0, 0, \dots, 0)$, then $\left(\overline{\mathbb{D}Orb(T^{-1}, x) \cap \mathcal{M}}\right)^{\circ} = \{(y, 0, 0, \dots, 0) : y \in \mathbb{C}, |y| < 1\} \neq \phi$. Therefore, $\mathbb{D}Orb(T^{-1}, x)$ is somewhere dense in $\mathcal{M}$. However, by part (2), $\mathbb{D}Orb(T^{-1}, x)$ is not everywhere dense in $\mathcal{M}$. □

It follows from the above example that compact and hyponormal subspace-diskcyclic operators exist on $\mathbb{C}$.

Since every two n-dimensional Banach spaces over the complex scalar field are isomorphic, then from Example 2.13 above, one may easily conclude the following proposition.

Proposition 2.14. *There are subspace-diskcyclic operators on every finite dimensional Banach space over the scalar field* $\mathbb{C}$.

Acknowledgments

The authors are very grateful to the referee and editor for the valuable suggestions and comments to improve our paper.

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