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On stochastic solutions of nonlocal random functional integral equations

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Abstract. In this paper, we use Schauder's fixed point to establish the existence of at least one solution for a functional nonlocal stochastic differential equation under sufficient conditions in the space of all square integrable stochastic processes with a finite second moment. We state and prove the conditions which guarantee the uniqueness of the solution. We solve a nonlinear example analytically and obtain the initial condition which makes the solution passes through a random position with a given normal distribution at a specified time. Also, the Milstein scheme to this example is studied.

Keywords: Schauder's fixed point; Existence; Uniqueness; Nonlocal conditions; Stochastic differential equation; Anderson–Darling; Milstein

Mathematics Subject Classification: 60H10; 65C30

1. Introduction

Differential and integral equations are significant tools in applied sciences such as viscoelasticity, bio-chemistry, electrical engineering, electro-magnetics, finance, and many other fields, for more details, see [12,14,15]. When random fluctuations have great effects on the parameters and evolution in the mathematical model which describes a certain

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phenomenon, a stochastic differential equation should be the starting point for deriving the suitable model. Recently, nonlocal stochastic models were introduced by many authors to describe the evolution of the studied phenomena, see [3,5,6,10]. This paper is a contribution in that field. We study the existence and uniqueness of the following nonlocal functional stochastic differential equation.

$$dx(t) = f(t, x(t), Ax(t))dt + g(t, x(t), B(t)x(t))dW(t) \tag{1}$$

$$x(0) = x_0 + h(t_1, t_2, \ldots, t_p, x(\cdot)) \tag{2}$$

where $t \in [0, T]$, $T < \infty$, $0 \leq t_1 < t_2 \cdots < t_p \leq T$, $p \in \mathbb{N}$, and the nonlocal condition $h(t_1, t_2, \ldots, t_p, x(\cdot))$ is used in the sense that in the place of $\cdot$ we can substitute only elements of the set $\{t_1, t_2, \ldots, t_p\}$, see [3]. The problem will be studied in the space $C := C(J, L_2(\Omega, \mathcal{A}, P))$, where $\mathcal{A} := \mathcal{A}_T$, $J := [0, T]$, which contains all mean square continuous stochastic processes defined from J into $L_2(\Omega, \mathcal{A}, P)$ and adapted to the filtration $\{\mathcal{A}_t\}_{t \in J}$, see the next section for more details. The process $W(t)$ is a real martingale with respect to the filtration $\{\mathcal{A}_t\}_{t \in J}$ and the random variable $x_0 \in C$ is independent of $\{W(t) : t \in J\}$. The operator A is a closed linear operator defined on C taking values in C, and $\{B(t) : t \in J\}$ is a family of linear bounded operators defined on C with values in C, see [5]. The functions $f(t)$, and $g(t)$ are $\mathcal{A}_t$-measurable scalar functions satisfying certain conditions will be defined later. M.A. Abdou, et al., in [2], studied a nonlocal random integral equation of Volterra type using the theory of admissibility of integral operator. We extend some results developed in [2] and study a functional nonlocal stochastic integral equation of Volterra–Itô – Doob type using the fixed point technique. The subsequent sections are organized as follows. In Section 2, the necessary definitions and theorems are introduced. We state and prove the main results in Section 3. An illustrative example is studied in Section 4 and a conclusion is shown in Section 5.

2. Preliminaries

In this section, we shall present the mathematical background which will be essential for the main results in the next section, for more details, see [1,4,7–9,11,13].

Let $(\Omega, \mathcal{A}, P)$, where $\mathcal{A} := \mathcal{A}_T$, be a complete probability space with a filtration $\{\mathcal{A}_t\}_{t \in J}$, where Ω is a nonempty set known as a sample space, $\mathcal{A}$ is a σ-algebra of events of Ω occurring during the time interval J, P is a complete probability measure and $\{\mathcal{A}_t\}_{t \in J}$ is an increasing family of sub-σ-algebras $\mathcal{A}_t \subset \mathcal{A}$ satisfying the usual conditions. Let $L_2 := L_2(\Omega, \mathcal{A}, P)$ be the space of all square integrable real stochastic processes $\{x(t) : t \in J\}$ with a finite second moment (*i.e.* $E\{|x(t)|^2\} < \infty$), for all $t \in J$, and equipped with the norm $\|x(t)\|_{L_2} = \left\{E(|x(t)|^2)\right\}^{\frac{1}{2}} = \left\{\int_\Omega |x(t)|^2 dP\right\}^{\frac{1}{2}}$ for each $t \in J$, where E is the expectation over the random variables $x(t)$. Let $C := C(J, L_2(\Omega, \mathcal{A}, P))$ be the space of all continuous stochastic processes defined from J into $L_2(\Omega, \mathcal{A}, P)$ which are adapted to the filtration $\{\mathcal{A}_t\}_{t \in J}$ and equipped with the norm $\|x\|_C = \sup_{t \in J} \left\{\|x(t)\|_{L_2}\right\}$

Definition 2.1. *(Real Stochastic Process)* Let $x(t, \omega)$ be a real valued function defined on $J \times \Omega$. Then, $x(t, \omega)$ is called a real stochastic process if the following two conditions are satisfied:

(i) For every fixed $t \in J$, the function $x(t, \cdot)$ is a random variable defined on Ω.
(ii) For every fixed $\omega \in \Omega$, the function $x(\cdot, \omega)$ is a measurable function defined on J and called the trajectory of the stochastic process.

In what follows, we will not mention ω explicitly and use $x(t)$ instead of $x(t, \omega)$. Also, we use $h(x(\cdot))$ to mean $h(t_1, t_2, \ldots, t_p, x(\cdot))$ which is interpreted in the introduction, and $\eta := [T^2 + F(T) - F(0)]$.

Definition 2.2. *(Real Martingale)* Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in J}, P)$ be a filtered probability space. Then, a real stochastic process $\{x(t) : t \in J\}$ is said to be a real martingale with respect to the filtration $\{\mathcal{A}_t\}_{t\in J}$ if the following three conditions are satisfied:

(i) For all $t \in J$, $x(t)$ is measurable with respect to $\mathcal{A}_t$.
(ii) For all $t \in J$, $E(|x(t)|) < \infty$.
(iii) For all $t_1, t_2 \in J$, $t_1 < t_2$, $E(x(t_2)|\mathcal{A}_{t_1}) = x(t_1)$ almost surely.

Theorem 2.1. *(Closed Graph Theorem) Let X and Y be two Banach spaces. Let K be a linear operator from X into Y. Then the operator K is continuous if and only if it is closed.*

Theorem 2.2. *(Schauder's Fixed Point Theorem) Let S be a closed bounded convex subset in a Banach space. Let the operator K be a completely continuous operator on S. Let $KS \subset S$. Then the operator K has at least one fixed point in S.*

Theorem 2.3. *(Itô's Theorem) Let $x(t)$ be an Itô process. Let $g(t, x(t))$ be a real valued function defined on $J \times \mathbb{R}$ with continuous partial derivatives g_t, g_x, g_{xx}. Then the stochastic process $Y_t = g(t, x(t))$ is also an Itô process with a differential given by $dY_t = g_t\, dt + g_x\, dx + \frac{1}{2} g_{xx}\, (dx)^2$, where dx is the differential of $x(t)$.*

3. Main results

Suppose the following stochastic functional differential equation:

$$dx(t) = f(t, x(t), Ax(t))dt + g(t, x(t), B(t)x(t))dW(t)$$
$$x(0) = x_0 + h(x(\cdot))$$

which is equivalent to the following stochastic functional integral equation:

$$x(t) = x_0 + h(x(\cdot)) + \int_0^t f(\tau, x(\tau), Ax(\tau))d\tau + \int_0^t g(\tau, x(\tau), B(\tau)x(\tau))dW(\tau) \tag{3}$$

where the first integral is a mean square Riemann integral and the second is an Itô – Doob integral. The function $W(t)$ is a real martingale and adapted to the filtration $\{\mathcal{A}_t\}_{t\in J}$. The random variable $x_0 \in C$ is independent of $W(t)$ for $t \geq 0$. The operator A is a closed linear and defined on C with values in C. The operators $\{B(t) : t \in J\}$ are linear bounded defined on C into C, see [5]. The measurable real random functions f and g are defined on $J \times C \times C$ with values in the space L_2. The random function $h(x(\cdot))$ is called the stochastic perturbing term and it is a given function defined on C with values in the space C. The functions f, g, and h will be specified in the conditions below. By a solution of the considered stochastic differential equation (1), and (2), we mean a stochastic process $x(t)$ adapted to the filtration $\{\mathcal{A}_t\}_{t\in J}$, continuous in mean-square which satisfies the stochastic integral equation (3) almost surely. Now, we shall assume the following conditions:

H_1: There is a real continuous monotone nondecreasing mapping $F(t)$ defined on J, such that $s < t$ implies holding the following equality almost surely: $E\{|W(t) - W(s)|^2\} = E\{|W(t) - W(s)|^2 \setminus \mathcal{A}_s\} = F(t) - F(s)$

H_2: The functions $f(t, x, y)$, and $g(t, x, y)$ are mean square continuous in (x, y) for each $t \in J$.

H_3: f and g have the following restriction on growth

$$|f(t, x, y)| \leq \alpha\sqrt{(1 + |x|^2 + |y|^2)}$$

$$|g(t, x, y)| \leq \alpha\sqrt{(1 + |x|^2 + |y|^2)}$$

for all $t \in J$, $x \in C$, $y \in C$, where the constant $\alpha > 0$.

Now, let us define the following integral operator K by:

$$\begin{aligned} Kx(t) := x_0 + h(x(\cdot)) + \int_0^t f(\tau, x(\tau), Ax(\tau))d\tau \\ + \int_0^t g(\tau, x(\tau), B(\tau)x(\tau))dW(\tau), \qquad t \in [0, T] \end{aligned} \tag{4}$$

Lemma 3.1. *The operator K defined by Eq. (4) maps the space $C\,(J, L_2(\Omega, \mathcal{A}, P))$ into itself and it is continuous for each $t \in J$.*

Proof. Clearly, the function Kx is square integrable with respect to the probability measure, has a finite second moment, and adapted to $\{\mathcal{A}_t\}_{t\in J}$ for each $t \in J$. Let $x \in C$. Let $t_1 \in J$ and $t_2 \in J$ with $t_2 > t_1$ such that $(t_2 - t_1) < \delta$.

$$\begin{aligned} |Kx(t_2) - Kx(t_1)|^2 \leq 2\left|\int_{t_1}^{t_2} f(\tau, x(\tau), Ax(\tau))\, d\tau\right|^2 \\ + 2\left|\int_{t_1}^{t_2} g(\tau, x(\tau), B(\tau)x(\tau))\, dW(\tau)\right|^2 \end{aligned}$$

Taking expectation of both sides, applying the Cauchy–Schwarz inequality, and the condition H_1 yields:

$$\begin{aligned} E|Kx(t_2) - Kx(t_1)|^2 \leq 2(t_2 - t_1)\int_{t_1}^{t_2} E|f(\tau, x(\tau), Ax(\tau))|^2 d\tau \\ + 2\int_{t_1}^{t_2} E|g(\tau, x(\tau), B(\tau)x(\tau))|^2 dF(\tau) \end{aligned}$$

From the closed graph theorem it follows that the operator A is bounded. So, there is a real constant $\beta \geq 0$ such that $\|Ax\|_C \leq \beta\|x\|_C$. Also, $B(t)$ is a family of bounded operators. Therefore, there is a real constant , $\gamma(t) \geq 0$, depends on t, such that $\|B(t)x\|_C \leq \gamma(t)\|x\|_C \leq \gamma_1\|x\|_C$, where $\gamma_1 := \max_{t\in J}\{\gamma(t)\}$. Let $\zeta := \max\{\beta, \gamma_1\}$.
Applying the growth condition yields

$$E|Kx(t_2) - Kx(t_1)|^2 \leq 2\alpha^2\left[(t_2 - t_1)^2 - (F(t_2) - F(t_1))\right]\left[1 + (1 + \zeta^2)\|x\|_C^2\right]$$

but the function $F(t)$ is continuous, so $t_2 \longrightarrow t_1$ implies $[(F(t_2) - F(t_1))] \longrightarrow 0$. Therefore, $E|Kx(t_2) - Kx(t_1)|^2 \longrightarrow 0$ when $t_2 \longrightarrow t_1$. Consequently, the function Kx is continuous in mean-square on J. Hence, the integral operator K maps the space $C(J, L_2(\Omega, \mathcal{A}, P))$ into itself. Let $x_n \longrightarrow x$ in $C(J, L_2(\Omega, \mathcal{A}, P))$ almost surely with $\{x_n\}_{n=1}^{\infty} \subseteq C(J, L_2(\Omega, \mathcal{A}, P))$. Clearly, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded in $C(J, L_2(\Omega, \mathcal{A}, P))$. Now, from the conditions H_2 and H_3 the continuity of h in x, and applying the Lebesgue dominated convergence theorem yields

$$\begin{aligned} E|Kx_n(t) - Kx(t)|^2 &\leq 3E|h(x_n(\cdot)) - h(x(\cdot))|^2 \\ &+ 3T\int_0^t E|f(\tau, x_n(\tau), Ax_n(\tau)) - f(\tau, x(\tau), Ax(\tau))|^2 d\tau \\ &+ 3\int_0^t E|g(\tau, x_n(\tau), B(\tau)x_n(\tau)) - g(\tau, x(\tau), B(\tau)x(\tau))|^2 dF(\tau) \\ &\longrightarrow 0 \ \ as \ \ n \longrightarrow \infty \end{aligned}$$

Therefore, $Kx_n \longrightarrow Kx$ in $C(J, L_2(\Omega, \mathcal{A}, P))$ as $x_n \longrightarrow x$ in $C(J, L_2(\Omega, \mathcal{A}, P))$ and hence, the operator K is continuous. ■

Theorem 3.1. *Suppose the conditions H_1, H_2 , and H_3 are satisfied. Let $S := \{x \in C : \|x\|_C \leq r\}$. Let the operator $K : S \longrightarrow C$. Then the stochastic integral equation (3) has at least one solution in S, provided that $r := 2\sqrt{\frac{\|x_0\|_C^2 + \|h\|_C^2 + \eta\alpha^2}{1 - 4\eta\alpha^2(1+\zeta^2)}}$, and $4\eta\alpha^2(1 + \zeta^2) < 1$.*

Proof. Clearly, the set S is bounded, closed and convex nonempty subset of the space C. Using Lemma 3.1, the operator $K : S \longrightarrow C$ is continuous. Let $\{Kx_n\}_{n=1}^{\infty}$ be a sequence of continuous functions in the set KS. Let $t_1 \in J$ and $t_2 \in J$ with $t_2 > t_1$ such that $(t_2 - t_1) < \delta$. Using an argument similar to the one used in Lemma 3.1 yields

$$E|Kx_n(t_2) - Kx_n(t_1)|^2 \leq 2\alpha^2 \left[(t_2 - t_1)^2 - (F(t_2) - F(t_1))\right]\left[1 + (1 + \zeta^2)r^2\right]$$

Consequently, $E|Kx_n(t_2) - Kx_n(t_1)|^2 \longrightarrow 0$ when $t_2 \longrightarrow t_1$ for all $n \in \mathbb{N}$. Therefore, the sequence $\{Kx_n\}_{n=1}^{\infty}$ is equicontinuous in mean-square. Also, it is uniformly bounded in mean-square because for every $n \in \mathbb{N}$ we have:

$$E|Kx_n(t)|^2 \leq 4\|x_0\|_C^2 + 4\|h\|_C^2 + 4\eta\alpha^2\left[1 + (1 + \zeta^2)r^2\right]$$

Using the Arzela-Ascoli theorem, there exists a convergent subsequence $\{Kx_{n_k}\}_{n_k=1}^{\infty}$ in $\{Kx_n\}_{n=1}^{\infty}$ which converges uniformly in KS and hence the set KS is compact. Consequently, the operator K is completely continuous. Also, $KS \subset S$ because

$$E|Kx(t)|^2 \leq 4\|x_0\|_C^2 + 4\|h\|_C^2 + 4\eta\alpha^2[1 + (1 + \zeta^2)r^2]$$

Substituting $r = 2\sqrt{\frac{\|x_0\|_C^2 + \|h\|_C^2 + \eta\alpha^2}{1 - 4\eta\alpha^2(1+\zeta^2)}}$ and simplifying yields $\|Kx\|_C \leq r$ for each $x \in S$. Using the Schauder fixed point theorem, the operator K has at least one fixed point in S. ■

Theorem 3.2. *Suppose the conditions H_1 and H_3 are satisfied. Let $S := \{x \in C : \|x\|_C \leq r\}$, where $r := 2\sqrt{\frac{\|x_0\|_C^2 + \|h\|_C^2 + \eta\alpha^2}{1 - 4\eta\alpha^2(1+\zeta^2)}}$, and $4\eta\alpha^2(1 + \zeta^2) < 1$. Let the operator $K : S \longrightarrow C$. Let the functions f, g, and h satisfy the following conditions:*

H_4: *There exists a constant* $\alpha > 0$ *such that:*

$$|f(t, x_2, y_2) - f(t, x_1, y_1)| \leq \alpha\sqrt{(|x_2 - x_1|^2 + |y_2 - y_1|^2)}$$

$$|g(t, x_2, y_2) - g(t, x_1, y_1)| \leq \alpha\sqrt{(|x_2 - x_1|^2 + |y_2 - y_1|^2)}$$

$$|h(t, x_2) - h(t, x_1)| \leq \alpha\ |x_2 - x_1|$$

for all $t \in J$, x_1, *and* $x_2 \in C$, y_1, *and* $y_2 \in C$.

Then Eq. (3) *has a unique solution in S, provided that:*

$$0 \leq \alpha\sqrt{3\left[1 + \eta(1 + \zeta^2)\right]} < 1$$

Proof. Clearly, the set S is a closed subspace of $C\ (J, L_2(\Omega, \mathcal{A}, P))$. By the same argument as the one used in Theorem 3.1 we can prove that, $KS \subset S$. So, it remains to prove the integral operator K, defined by Eq. (4), is a contraction for each $t \in J$. Let x and y belong to the set S. Applying the Cauchy–Schwarz inequality, the Lipschitz condition, and taking expectation of both sides yields

$$E|Kx - Ky|^2 \leq 3\alpha^2\left[1 + \eta(1 + \zeta^2)\right] E|x - y|^2$$

Taking supremum of both sides over $t \in J$ yields

$$\|Kx - Ky\|_C \leq \alpha\sqrt{3\left[1 + \eta(1 + \zeta^2)\right]}\ \|x - y\|_C$$

Consequently, the integral operator K is a contraction. Therefore, using the Banach fixed point theorem, the stochastic integral equation (3) has a unique solution in S. ■

4. Illustrative example

Example 4.1. Consider the following stochastic differential equation

$$dx = \left(\frac{3}{2}x^5 - x^3\right)\ dt - x^3\ dW(t)$$

$$x(0.25) = z, \qquad t \in [0, 1]$$

where the stochastic process $W(t)$ is a Brownian motion, and z is a random variable has a normal distribution with mean 0.03 and a standard deviation 0.01 with 0.05 level of significance. The closed form solution for this nonlinear problem is given by:

$$x(t) = x(0)\left[1 + 2x^2(0)(t + W(t))\right]^{\frac{-1}{2}}$$

It is easy to check this solution using the Itô theorem. We claim that in order to have $x(0.25)$ normally distributed with mean 0.03 and standard deviation 0.01, the initial random variable $x(0)$ should have the same distribution as $x(0.25)$ with 0.05 level of significance. To test our claim, we have generated 30 samples for $x(0.25)$ when $x(0)$ has the claimed distribution. Each sample has size 150. We tested each sample for normality with the Anderson–Darling test. The data in each tested sample showed p-value supports our null hypothesis as shown in Table 1, and Fig. 1.

Table 1
The P-values, Mean, and Standard deviation for the tested 30 samples.

P-value	Mean	StDev.	P-value	Mean	StDev.	P-value	Mean	StDev.
0.960	0.0294	0.0105	0.354	0.0289	0.0104	0.657	0.0298	0.0109
0.196	0.0302	0.0095	0.261	0.0303	0.0086	0.730	0.0302	0.0087
0.659	0.0301	0.0102	0.417	0.0285	0.0099	0.244	0.0297	0.0112
0.576	0.0297	0.0089	0.900	0.0304	0.0101	0.451	0.0289	0.0092
0.569	0.0298	0.0088	0.324	0.0288	0.0094	0.932	0.0298	0.0099
0.814	0.0308	0.0091	0.276	0.0296	0.0103	0.844	0.0309	0.0087
0.494	0.0303	0.0108	0.925	0.0289	0.0098	0.212	0.0290	0.0105
0.892	0.0302	0.0103	0.358	0.0307	0.0096	0.404	0.0306	0.0099
0.674	0.0303	0.0100	0.246	0.0298	0.0107	0.328	0.0303	0.0106
0.865	0.0311	0.0112	0.945	0.0301	0.0101	0.426	0.292	0.0092

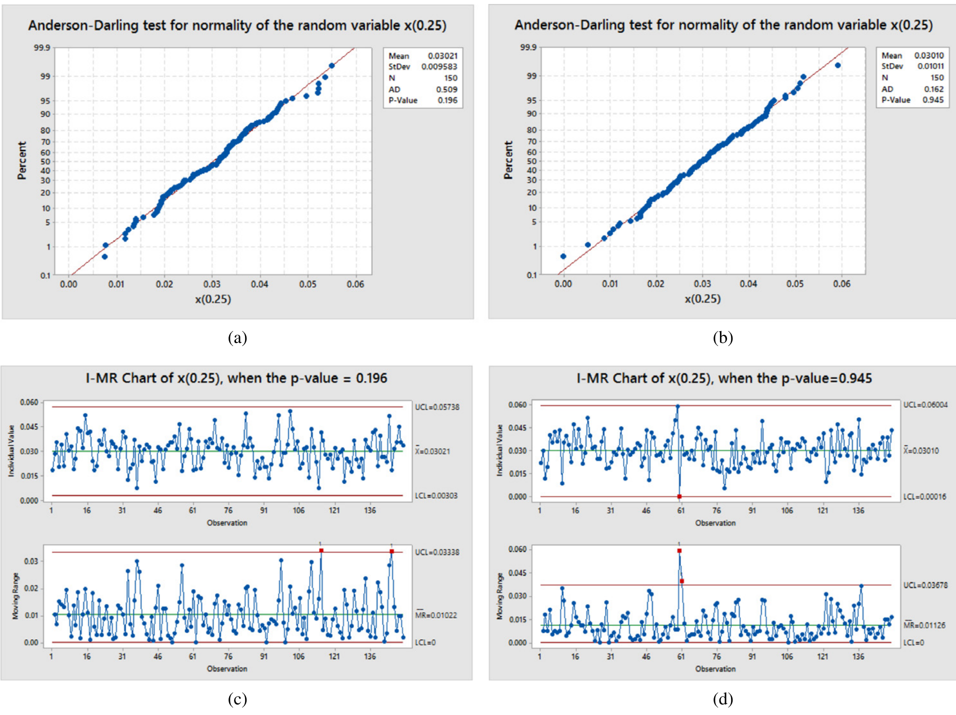

Fig. 1. The Probability plot of the random variable $x(0.25)$ and its stability using the individual and moving range (I-MR for short) chart.

For some more details about the solution picture, we apply the following Milstein scheme

$$x_{i+1} = x_i + \left[\frac{3}{2}x_i^5 - x_i^3\right] dt_i - x_i^3 dW_i + \frac{3}{2}x_i^5 \left[(dW_i)^2 - dt_i\right]$$

where $i = 0, 1, 2, 3, \ldots.., n < \infty$, the set $\{t_0, t_1, \ldots, t_n\}$ makes a partition on [0, 1], x_i is the estimated value of $x(t_i)$. the initial condition x_0 is selected from normal distribution with

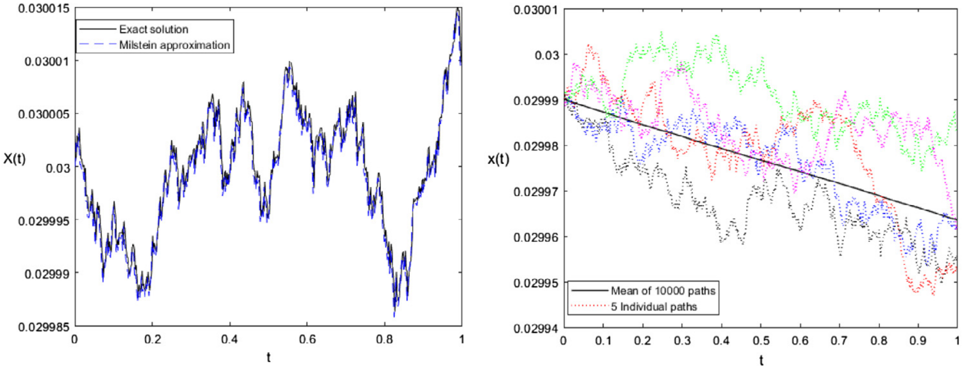

Fig. 2. The graph on the left shows the actual path and Milstein approximate path while the right shows the expected path of the solution $x(t)$ with five individual paths when the state $= 0$, $n = 2^9$, and the initial random variable has normal distribution with mean $= 0.03$, and standard deviation $= 0.01$ with 0.05 level of significance.

mean 0.03 and standard deviation 0.01. $dt_i = (t_{i+1} - t_i)$, $dW_i = (W(t_{i+1}) - W(t_i))$ is a random variable has normal distribution with mean zero and variance dt_i, see Fig. 2.

5. Conclusion

In the present work, we have studied a nonlocal stochastic functional integral equation. We used the Schauder fixed point to establish the existence of at least one mean square continuous solution. As an application, of our results, we studied a nonlinear example in some more detail. We obtained its closed form solution. The distribution to be achieved by the initial condition in order to make the solution have a certain normal distribution at a specified time is investigated. We recognized, in this example, that in order to have the solution at time t = 0.25 normally distributed, the initial random variable should have the same distribution with 0.05 level of significance. We used the Anderson–Darling test to check our claim and concluded that it is true. The stability of the solution distribution at time 0.25 was investigated via the individual and moving range chart. Finally, for more details about the solution picture, we pictured the average of 10 000 individual paths and the approximated path using Milstein scheme.

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