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On regularization and error estimates for non-homogeneous backward Cauchy problem

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Abstract In this paper we study the general non-homogeneous Backward Cauchy problem $\begin{cases} u_t + Au = f, & 0 < t < T, \\ u(T) = g, \end{cases}$ for positive selfadjoint unbounded operator A on the Hilbert space $\mathcal{H}$. The problem is known to be severely ill-posed. We give extensions of the quasi-boundary methods to the non-homogeneous case. We prove several sharp results on regularizations and error-estimates. Other results, including some explicit convergence rates are proved. Finally applications to the non-homogeneous backward heat equation with Bessel operator are given.

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1. Introduction

Motivated by many physical problems, e.g., inverse problems of heat equations, backward Cauchy problems have received much attention since 1960s (see [7,8,10,11,14,19]). Since this type of problems are not stable, an usual method to deal with such problems is to find corresponding well-posed problems such that their solutions can approximate the solutions of corresponding ill-posed problems. In this context, many approaches have been tried. The method of quasi-values of Ivanov [9], the method of regularization of Tikhonov [19], the method of quasi-reversibility of Lattes and Lions [10], the Gajewski and Zacharia's method based on eigenfunction expansion [6], the method of auxiliary boundary conditions [9,14], the quasi-boundary value method [2,5,18], the method of reduction to a Dirichlet problem called Carasso's method [3,4] and the C-regularized semigroups method [14].

In this paper we consider the following non-homogeneous backward Cauchy problem

$$\begin{cases} u_t + Au = f, & 0 < t < T, \\ u(T) = g, \end{cases} \qquad \text{(BCP)}$$

where A is an unbounded positive selfadjoint operator on the Hilbert space $\mathcal{H}$, admitting an orthonormal basis of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}^*}$ in $\mathcal{H}$, corresponding respectively to the positive unbounded increasing sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^*}$. The given data g is in $\mathcal{H}$ and the element f belongs to the space $L^2((0,T),\mathcal{H})$. The (BCP) problem is known to be severely ill-posed, solutions do not always exist, they depend on strong conditions of convergence. We shall see below that these conditions may not be satisfied. Even in this restricted class of conditions, the solutions obtained do not depend continuously upon the given data and the (BCP) problem remains highly intable.

We note that this type of problems has been considered by many authors, using different approaches. Such authors as Lattes and Lions [10], Miller [15] and Showalter [17] have approximated the homogeneous (BCP) by perturbing the operator A. In their work Clark and Oppenheimer [2], Denche and Bessila [5] followed what Showalter [18] did in a more general context, and approximate the homogeneous (BCP) by perturbing the final value condition; they approximate the homogeneous (BCP) problem respectively with

$$\begin{cases} u_t + Au = 0, & 0 < t < T, \\ u(T) + \alpha u(0) = g, \end{cases}$$

and

$$\begin{cases} u_t + Au = 0, & 0 < t < T, \\ u(T) - \alpha u'(0) = g. \end{cases}$$

This method is called quasi-boundary value method. A similar approach known as the method of auxiliary boundary conditions was given in [9,14].

Through many studies around the backward Cauchy problem, the general setting of the non-homogeneous case is scarcely considered. The general case where A is the infinitesimal generator of a holomorphic semigroup is studied in [1] where the approximate problem is given by

$$\begin{cases} u_t + h(A)u = f, & 0 < t < T, \\ u(T) = g, \end{cases}$$

for a suitable function h. The particular case when A is generated by a second order differential operator with periodic boundary conditions, was recently treated by Trong and Tuan in [20]. This paper is generalized to the non linear cases in [21]. The same non linear case was recently treated by a truncation method in [16].

In the present paper, we shall regularize the non-homogeneous backward Cauchy problem by the sequence of problems:

$$\begin{cases} u_t + Au = f_\alpha, & 0 < t < T, \\ u(T) = g_\alpha, \end{cases} \qquad \text{(ABCP)}$$

where

$$f_\alpha = \sum_{k \geqslant 1} \frac{e^{-\lambda_k T}}{\alpha\lambda_k^p + e^{-\lambda_k T}} f_k \varphi_k \quad \text{and} \quad g_\alpha = \sum_{k \geqslant 1} \frac{e^{-\lambda_k T}}{\alpha\lambda_k^p + e^{-\lambda_k T}} g_k \varphi_k, \quad 0 < \alpha < 1.$$

We construct a regularized solution and give error estimates. We obtain several other results, including some explicit convergence rates. Finally applications to the non-homogeneous backward heat equation with Bessel operator are given.

For the homogeneous case where $f = 0$ and $p = 1$, this method coincides with the modified quasi-boundary value method proposed in [5]. In the particular case of an operator A generated by a second order differential operator with periodic boundary conditions, and $p = 1$, we obtain the method used in [20].

2. The approximated problem

We approximate the above (BCP) problem by the following sequence of problems:

$$\begin{cases} u_t + Au = f_\alpha, & 0 < t < T, \\ \qquad u(T) = g_\alpha, \end{cases} \qquad (ABCP)$$

where we define for any $0 < \alpha < 1$:

$$f_\alpha = \sum_{k \geqslant 1} \frac{e^{-\lambda_k T}}{\alpha\lambda_k^p + e^{-\lambda_k T}} f_k \varphi_k \text{ and } \quad g_\alpha = \sum_{k \geqslant 1} \frac{e^{-\lambda_k T}}{\alpha\lambda_k^p + e^{-\lambda_k T}} g_k \varphi_k.$$

The families $\{f_k\}_{k\in\mathbb{N}^*}$ and $\{g_k\}_{k\in\mathbb{N}^*}$ are the sequences of generalized Fourier–Bessel coefficients of f and g in $\mathcal{H}$. The power $p \in \mathbb{R}_+$ is an arbitrary real number.

Definition 1. A solution of the (BCP) problem in the classical sense is a function $u : [0, T] \to \mathcal{H}$, such that $u(t) \in \mathcal{D}(A)$, for any $t \in (0, T), u \in C^1((0, T), \mathcal{H})$ and satisfies the equation and the final value condition.

Lemma 1. *For $g \in \mathcal{H}$ and $f \in L^2((0, T), \mathcal{H})$, the (BCP) problem admits a solution in the classical sense if and only f:*

$$\sum_{k \geqslant 1} \left(g_k e^{\lambda_k T} - \int_0^T e^{\lambda_k s} f_k(s) ds \right)^2 < +\infty. \tag{1}$$

Proof. Assume that the (BCP) problem admits a solution in the classical sense. This implies:

$$u(t) = \sum_{k \geqslant 1} u_k(t) \varphi_k, u(t) \in \mathcal{D}(A), \text{ for any } t \in (0, T) \text{ and } u \in C^1((0, T), \mathcal{H}).$$

From the equation and the final value condition in the (BCP) problem, we deduce that

$$u(t) = \sum_{k \geqslant 1} \left(g_k e^{\lambda_k (T-t)} - \int_t^T e^{\lambda_k (s-t)} f_k(s) ds \right) \varphi_k, \tag{2}$$

where f_k and g_k are the generalized Fourier–Bessel coefficients of f and g in $\mathcal{H}$. Since $u(t) \in \mathcal{H}$ for all $t \in [0,T]$, therefore:

$$\|u(t)\|^2 = \sum_{k \geqslant 1} \left(g_k e^{\lambda_k (T-t)} - \int_t^T e^{\lambda_k (s-t)} f_k(s) ds \right)^2 < +\infty,$$

in particular for $t = 0$, we get

$$\|u(0)\|^2 = \sum_{k \geqslant 1} \left(g_k e^{\lambda_k T} - \int_t^T e^{\lambda_k s} f_k(s) ds \right)^2 < +\infty.$$

Conversely assume that the condition (1) is satisfied. Then we define the function $u(t)$ by the expression in (2); using a Cauchy–Schwarz and elementary inequalities, we get

$$\|u(t)\|^2 \leqslant 2 \sum_{k \geqslant 1} \left(g_k e^{\lambda_k T} - \int_0^T e^{\lambda_k s} f_k(s) ds \right)^2 + 2T \|f\|^2_{L^2((0,T),\mathcal{H})}.$$

Similarly, we have for $Au(t)$

$$\|Au(t)\|^2 = \sum_{k \geqslant 1} \lambda_k^2 e^{-2\lambda_k t} \left(g_k e^{\lambda_k T} - \int_t^T e^{\lambda_k s} f_k(s) ds \right)^2,$$

whereupon

$$\|Au(t)\|^2 \leqslant 2 \sum_{k \geqslant 1} \left(g_k e^{\lambda_k T} - \int_0^T e^{\lambda_k s} f_k(s) ds \right)^2 + 2T \|f\|^2_{L^2((0,T),\mathcal{H})},$$

therefore, $u(t) \in \mathcal{D}(A)$, for any $t \in (0,T)$. In a similar manner, since we have:

$$u'(t) = \sum_{k\geqslant 1}\left[f_k(t) - \lambda_k e^{-\lambda_k t}(g_k e^{\lambda_k T} - \int_t^T e^{\lambda_k s} f_k(s)ds)\right]\varphi_k,$$

we verify that $u \in C^1((0,T),\mathcal{H})$. Finally, we check directly that $u(t)$verifies well the (BCP) problem and the proof is complete. □

3. Convergence rates and error-estimates

In this section, we establish several results related to stability and error-estimates.

Theorem 1. *If $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$, the approximate non-homogeneous backward problem (ABCP) admits a unique classical solution $u_\alpha(t)$, which depends continuously upon the data $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$. Moreover, we have for any $t \in [0,T]$, any $p \in \mathbb{R}_+^*$*

$$\|u_\alpha(t)\|_{\mathcal{H}} \leqslant \frac{1}{\alpha}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^p (\|g\|_{\mathcal{H}} + \sqrt{T}\|f\|_{L^2((0,T),\mathcal{H})}). \tag{3}$$

Proof. For the approximate problem (ABCP), if $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$, a unique solution exists and is given by

$$u_\alpha(t) = \sum_{k\geqslant 1}\frac{e^{-\lambda_k t}}{\alpha\lambda_k^p + e^{-\lambda_k T}}\left(g_k - \int_t^T e^{\lambda_k(s-T)} f_k(s)ds\right)\varphi_k; \tag{4}$$

we have

$$\|u_\alpha(t)\| \leqslant \left\|\sum_{k\geqslant 1}\frac{e^{-\lambda_k t}}{\alpha\lambda_k^p + e^{-\lambda_k T}} g_k\varphi_k\right\| + \left\|\sum_{k\geqslant 1}\frac{e^{-\lambda_k t}}{\alpha\lambda_k^p + e^{-\lambda_k T}}\int_t^T e^{\lambda_k(s-T)} f_k(s)ds\varphi_k\right\|. \tag{5}$$

If we consider the function

$$h(\lambda) = \frac{1}{\alpha\lambda^p + e^{-\lambda T}},$$

then $h(\lambda_0) = \sup_{\lambda\geqslant 0} h(\lambda)$ exists, the extremum λ_0 is the root of the equation

$$\log\frac{T}{p\alpha} = \lambda T\left(1 + \frac{p-1}{T}\frac{\log\lambda}{\lambda}\right).$$

For α small enough, we may take λ_0 approximately $\frac{1}{T}\log\frac{T}{p\alpha}$; in this case

$$h(\lambda_0) = \frac{1}{\alpha\lambda_0^p\left(1 + \frac{p}{T\lambda_0}\right)} \leqslant \frac{1}{\alpha}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^p,$$

therefore, substituting in the inequality (5) and using the Cauchy–Schwarz inequality, we obtain for any $p \in \mathbb{R}_+$:

$$\|u_\alpha(t)\| \leqslant \frac{1}{\alpha}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^p (\|g\|_{\mathcal{H}} + \sqrt{T}\|f\|_{L^2((0,T),\mathcal{H})}).$$

In particular we deduce that $u_\alpha(t) \in \mathcal{H}$, for any $t \in [0,T]$. In an analogous manner to the proof of Lemma 1, we show that $u_\alpha(t) \in \mathcal{D}(A)$, for any $t \in (0,T)$ and $u_\alpha \in C^1((0,T),\mathcal{H})$, this completes the proof. □

With respect to the given data g in $\mathcal{H}$, stability may be written precisely in the following corollary as:

Corollary 1. *If g_1 and g_2 are given data in $\mathcal{H}$ corresponding respectively to the solutions $u_{1\alpha}(t)$ and $u_{2\alpha}(t)$ then*

$$\|u_{1\alpha}(t) - u_{2\alpha}(t)\| \leqslant \frac{1}{\alpha}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^p \|g_1 - g_2\|, \textit{for any } p \in \mathbb{R}_+^*.$$

Proof. Using the corresponding solutions $u_\alpha(t)$ given by (4), and the supremum estimate in Theorem 1, we get

$$\|u_{1\alpha}(t) - u_{2\alpha}(t)\|^2 = \sum\nolimits_{k\geqslant 1} \frac{e^{-2\lambda_k t}}{(\alpha\lambda_k^p + e^{-\lambda_k T})^2}(g_{1k} - g_{2k})^2$$

$$\leqslant \frac{1}{\alpha^2}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^{2p} \|g_1 - g_2\|^2. \qquad □$$

Remark 1. In the backward Cauchy problem (BCP), according to Theorem 1 if we choose $f = 0$ we obtain extensions of the quasi-boundary method in [2,5] with better norm estimate and stability of order $\frac{1}{\alpha}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^p$, $p \in \mathbb{R}_+^*$. The papers [2,5] treat respectively $p = 0$ and $p = 1$. For the particular $p = 0$, Theorem 1 remains valid, this extends again [2] to the non-homogeneous case in which estimate (3) becomes:

$$\|u_\alpha(t)\| \leqslant \frac{1}{\alpha}(\|g\|_{\mathcal{H}} + \sqrt{T}\|f\|_{L^2((0,T),\mathcal{H})}).$$

In the next theorem, we show that, with respect to the variable $t \in [0,T]$ the previous uniform norm estimate may be improved.

Theorem 2. *Let $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$, then for any $p \in \mathbb{R}_+^*, 0 < \alpha < 1$ and $t \in [0,T]$, we have*

$$\|u_\alpha(t)\| \leqslant \frac{1}{\alpha^{1-\frac{t}{T}}}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^{p\left(1-\frac{t}{T}\right)}\left(\|g\|_{\mathcal{H}} + \sqrt{T-t}\|f\|_{L^2((0,T),\mathcal{H})}\right).$$

Proof. Using

$$\frac{e^{-2\lambda_k t}}{\left(\alpha\lambda_k^p + e^{-\lambda_k T}\right)^2} = \frac{e^{-2\lambda_k t}}{\left[\left(\alpha\lambda_k^p + e^{-\lambda_k T}\right)^{\frac{t}{T}}\left(\alpha\lambda_k^p + e^{-\lambda_k T}\right)^{1-\frac{t}{T}}\right]^2} \leqslant \frac{1}{\left(\alpha\lambda_k^p + e^{-\lambda_k T}\right)^{2\left(1-\frac{t}{T}\right)}},$$

and the supremum estimate as in Theorem 1, we get

$$\left\|\sum_{k\geqslant 1}\frac{e^{-\lambda_k t}}{\alpha\lambda_k^p + e^{-\lambda_k T}} g_k\varphi_k\right\| \leqslant \frac{1}{\alpha^{1-\frac{t}{T}}}\left(\frac{T}{\log\frac{T}{n\alpha}}\right)^{n\left(1-\frac{t}{T}\right)}\|g\|_{\mathcal{H}}.$$

In a similar manner we obtain

$$\left\|\sum_{k\geqslant 1}\frac{e^{-\lambda_k t}}{\alpha\lambda_k^p + e^{-\lambda_k T}}\int_t^T e^{\lambda_k(s-T)} f_k(s)ds\varphi_k\right\| \leqslant \frac{1}{\alpha^{\left(1-\frac{t}{T}\right)}}\left(\frac{T}{\log\frac{T}{p\alpha}}\right)^{p\left(1-\frac{t}{T}\right)} \times \sqrt{(T-t)}\|f\|_{L^2((0,T),\mathcal{H})},$$

going back to inequality (5) we get the desired result. □

Theorem 3. *For any $g \in \mathcal{H}$, the sequence $u_\alpha(T)$ converges to g in $\mathcal{H}$ as α tends to zero.*

Proof. For all $0 < \alpha < 1$ and $p \in \mathbb{R}_+^*$, from (4) we have

$$\|u_\alpha(T) - g\|^2 = \sum_{k\geqslant 1}\left[\frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\right]^2 g_k^2,$$

then, for $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $\sum_{k\geqslant N+1} g_k^2 \leqslant \frac{\epsilon}{2}$, hence

$$\|u_\alpha(T) - g\|^2 = \sum_{k=1}^N\left[\frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\right]^2 g_k^2 + \sum_{k\geqslant N+1}\left[\frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\right]^2 g_k^2,$$

this implies

$$\|u_\alpha(T) - g\|^2 \leqslant \alpha^2\sum_{k=1}^N \lambda_k^{2p} e^{2\lambda_k T} g_k^2 + \frac{\epsilon}{2},$$

therefore for the choice of $\alpha < \sqrt{\frac{\epsilon}{2}}\left(\sum_{k=1}^N \lambda_k^{2p} e^{2\lambda_k T} g_k^2\right)^{-\frac{1}{2}}$, we obtain

$$\|u_\alpha(T) - g\|^2 \leqslant \epsilon,$$

and the proof is complete. □

Theorem 4. *If there exists* $\epsilon \in (0,2)$ *such that the series* $\sum_{k\geqslant 1}\lambda_k^{\epsilon p}e^{\epsilon\lambda_k T}g_k^2$ *is convergent for* $p \in \mathbb{R}_+^*$ *, then* $\|u_\alpha(T) - g\|$*converges to zero with order* $\epsilon^{-1}\alpha^{\frac{\epsilon}{2}}$.

Proof. From (4) we have

$$\begin{aligned}\|u_\alpha(T) - g\|^2 &= \sum_{k\geqslant 1}\left[\frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\right]^2 g_k^2 \\ &= \alpha^{2-\beta}\sum_{k\geqslant 1}\lambda_k^{2p}\frac{\alpha^\beta}{(\alpha\lambda_k^p + e^{-\lambda_k T})^2}g_k^2, \quad \beta \in (0,2).\end{aligned}$$

If we take

$$h_k(\alpha,\beta) = \frac{\alpha^\beta}{(\alpha\lambda_k^p + e^{-\lambda_k T})^2},$$

then we have an extremum–maximum at:

$$\alpha_0 = \frac{\beta}{2-\beta}\frac{1}{\lambda_k^p}e^{-\lambda_k T}, \forall p \in \mathbb{R}_+^*.$$

Hence

$$\begin{aligned}\|u_\alpha(T) - g\|^2 &= \left(\frac{\beta}{2-\beta}\right)^\beta \alpha^{2-\beta}\sum_{k\geqslant 1}\frac{\lambda_k^{2p}}{\lambda_k^{\beta p}}\frac{e^{-\beta\lambda_k T}}{(\alpha_0\lambda_k^p + e^{-\lambda_k T})^2}g_k^2 \\ &= \left(\frac{\beta}{2-\beta}\right)^\beta \alpha^{2-\beta}\sum_{k\geqslant 1}\lambda_k^{(2-\beta)p}\frac{e^{(2-\beta)\lambda_k T}}{(\alpha_0\lambda_k^p + 1)^2}g_k^2 \\ &\leqslant \left(\frac{\beta}{2-\beta}\right)^\beta \alpha^{2-\beta}\sum_{k\geqslant 1}\lambda_k^{(2-\beta)p}e^{(2-\beta)\lambda_k T}g_k^2,\end{aligned}$$

if we take $\epsilon = (2-\beta)$, therefore,

$$\sum_{k\geqslant 1}\lambda_k^{\epsilon p}e^{\epsilon\lambda_k T}g_k^2 < +\infty \Rightarrow \|u_\alpha(T) - g\| \leqslant \frac{c}{\epsilon}\alpha^{\frac{\epsilon}{2}}. \qquad \square$$

In the next two Theorems, we would like to show that much weaker functional conditions on the given data g also give error-estimates on the convergence rate of the elements $u_\alpha(T)$ to g in the Hilbert space $\mathcal{H}$.

Theorem 5. *Let us assume that* $g \in \mathcal{D}(A^s), \forall s \in]0,1]$, *then there exists a constant* $c > 0$ *depending upon* $g \in \mathcal{H}$ *for which,*

$$\|u_\alpha(T) - g\| \leqslant \frac{c}{\log^s\left(\frac{T}{s\alpha}\right)}.$$

Proof. We write

$$\|u_\alpha(T) - g\|^2 = \sum_{k\geqslant 1}\left[\frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\right]^2 g_k^2 = \sum_{k\geqslant 1}\frac{\alpha^2\lambda_k^{2p-2s}}{(\alpha\lambda_k^p + e^{-\lambda_k T})^2}\lambda_k^{2s}g_k^2.$$

Then, the function

$$h(\lambda) = \frac{\lambda^{(p-s)}}{(\alpha\lambda^p + e^{-\lambda T})}, \tag{6}$$

admits an extremum–maximum λ_0 root of the equation

$$\log\frac{T}{s\alpha} = \lambda T + (p-1)\log\lambda + \log\left(1 + \frac{p-s}{\lambda T}\right),$$

for small enough α we may choose $\lambda_0 \approx \frac{1}{T}\log\frac{T}{s\alpha}$, in this case we get

$$\sup_{\lambda\geqslant 0} h(\lambda) = h(\lambda_0) = \frac{1}{\alpha}\frac{\lambda_0^{-s}}{1 + \frac{s}{\lambda_0 T + (p-s)}} \leqslant \frac{1}{\alpha}\lambda_0^{-s},$$

hence, we deduce that

$$\|u_\alpha(T) - g\| \leqslant \frac{c}{\log^s\left(\frac{T}{s\alpha}\right)},$$

where the constant c^2 is $\sum_{k\geqslant 1}\lambda_k^{2s}g_k^2 = \|A^s g\|^2$. □

Remark 2. If the parameter $s = 1$ we obtain estimates as in [5,20] with better stability, see Corollary 1. Theorems 4 and 5 show, how error-estimates depend upon the convergence rates of the element $g \in \mathcal{H}$. This convergence rate is expressed in how faster the series $\|g\|^2_{\mathcal{H}} = \sum_{k\geqslant 1} g_k^2$ is convergent. Faster convergence rates of order $\alpha^{\frac{\epsilon}{2}}$ need strong hyperbolic convergence rate on $g \in \mathcal{H}$ as $\sum_{k\geqslant 1}\lambda_k^{\epsilon}e^{\epsilon\lambda_k T}g_k^2 < +\infty$ see experimental application in [12,13]. Weaker parabolic conditions as $\sum_{k\geqslant 1}\lambda_k^{2s}g_k^2 < +\infty$ induce slower error-estimates. However these latter are important, they show the differences between the hyperbolic models as in [12,13] and the parabolic one as in [2,5,20]. And that the parabolic case admits a larger class of given data $g \in \mathcal{H}$.

In the next theorem, we prove that weaker logarithmic conditions on the data g also give error-estimates.

Theorem 6. *Let us assume that* $\sum_{k\geqslant 1}\log^{2s}\lambda_k g_k^2 < +\infty$, *for* $s > 0$, *then there exists a constant* $c > 0$ *depending upon the data* g *in* $\mathcal{D}(\log^s A)$ *for which,*

$$\|u_\alpha(T) - g\| \leqslant \frac{c}{\log^s\left(\frac{1}{T}\log\left(\frac{T}{s\alpha}\right)\right)}.$$

Proof. Write

$$\|u_\alpha(T) - g\|^2 = \sum_{k\geqslant 1}\frac{\alpha^2\lambda_k^{2p}}{(\alpha\lambda_k^p + e^{-\lambda_k T})^2}g_k^2 = \sum_{k\geqslant 1}\frac{\alpha^2\lambda_k^{2p}}{\log^{2s}\lambda_k(\alpha\lambda_k^p + e^{-\lambda_k T})^2}\log^{2s}\lambda_k g_k^2.$$

The function

$$h(\lambda) = \frac{\lambda^p}{\log^s \lambda(\alpha\lambda^p + e^{-\lambda T})}, \quad \lambda \geqslant e, s > 0,$$

verifies

$$h(\lambda) \leqslant h_1(\lambda), \text{ where } h_1(\lambda) = \frac{\lambda^p}{\alpha\lambda^p \log^s \lambda + e^{-\lambda T}};$$

$h_1(\lambda)$ admits an extremum–maximum at λ_0 root of the equation

$$\log \frac{T}{s\alpha} = \lambda T\left(1 + \frac{(p-1)}{T}\frac{\log \lambda}{\lambda} + \frac{(s-1)}{T}\frac{1}{\lambda}\log\log \lambda + \frac{1}{\lambda}\log\left(1 + \frac{p}{\lambda T}\right)\right).$$

For α small enough we may choose $\lambda_0 \approx \frac{1}{T}\log \frac{T}{s\alpha}$, in this case we get

$$\sup_{\lambda \geqslant e} h_1(\lambda) = h_1(\lambda_0) \leqslant \frac{1}{\alpha}\frac{1}{\log^s \lambda_0},$$

therefore

$$\|u_\alpha(T) - g\|^2 \leqslant \frac{1}{\log^{2s} \lambda_0}\sum_{k\geqslant 1} \log^{2s} \lambda_k g_k^2,$$

which implies the existence of $c^2 = \sum_{k\geqslant 1} \log^{2s} \lambda_k g_k^2$ such that

$$\|u_\alpha(T) - g\| \leqslant \frac{c}{\left(\log\left(\frac{1}{T}\log\left(\frac{T}{s\alpha}\right)\right)\right)^s},$$

and the proof is complete. □

4. Extensions to the uniform case

After studying in the previous section the close relationship between different conditions on $g \in \mathcal{H}$, stability and error-estimates on the final data $t = T$, we give in the following section the general framework depending upon the uniform variation of variable t in $[0,T]$. Extending in this way most of the results in [2,5,20] to more general settings.

Theorem 7. *Let us assume that $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$. Then, the non-homogeneous (BCP) problem admits a solution u(t) if and only if the sequence $u_\alpha(0)$ is convergent in $\mathcal{H}$. Furthermore, we have that $u_\alpha(t)$ converges to u(t) uniformly in t.*

Proof. Assume that $\lim_{\alpha\to 0} u_\alpha(0) = u_0$ exists and belongs to $\mathcal{H}$ set

$$u(t) = \sum_{k\geqslant 1}\left(u_{0k} + \int_0^t e^{\lambda_k s} f_k(s)ds\right)e^{-\lambda_k t}\varphi_k,$$

where

$$u_0 = \sum_{k\geqslant 1} u_{0k}\varphi_k.$$

Let $t \in [0,T]$, from (4) we have

$$\begin{aligned}\|u_\alpha(t) - u(t)\|^2 &= \sum_{k\geqslant 1}(u_{\alpha k}(t) - u_k(t))^2\\ &= \sum_{k\geqslant 1}\left[\frac{e^{-\lambda_k t}}{\alpha\lambda_k^p + e^{-\lambda_k T}}(g_k - \int_t^T e^{\lambda_k(s-T)}f_k(s)ds) - u_{0k}e^{-\lambda_k t}\right.\\ &\quad \left. - e^{-\lambda_k t}\int_0^t e^{\lambda_k s}f_k(s)ds\right]^2,\end{aligned}$$

gathering terms together, we get

$$\begin{aligned}\|u_\alpha(t) - u(t)\|^2 = \sum_{k\geqslant 1}&\left[e^{-\lambda_k t}\left[\frac{1}{\alpha\lambda_k^p + e^{-\lambda_k T}}\left(g_k - \int_0^T e^{\lambda_k(s-T)}f_k(s)ds\right) - u_{0k}\right]\right.\\ &\left. - \frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\int_0^t e^{\lambda_k(s-t)}f_k(s)ds\right]^2.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\|u_\alpha(t) - u(t)\|^2 &\leqslant 2\|u_\alpha(0) - u(0)\|^2\\ &\quad + 2\sum_{k\geqslant 1}\left(\frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\right)^2\left(\int_0^t e^{\lambda_k(s-t)}f_k(s)ds\right)^2.\end{aligned}$$

In view of

$$\begin{aligned}\left(\int_0^t e^{\lambda_k(s-t)}f_k(s)ds\right)^2 &\leqslant \int_0^t e^{2\lambda_k(s-t)}ds\int_0^t f_k^2(s)ds = \frac{1}{2\lambda_k}[1 - e^{-2\lambda_k t}]\int_0^t f_k^2(s)ds\\ &\leqslant \frac{1}{2\lambda_k}\|f\|^2_{L^2(0,T)},\end{aligned}$$

we get

$$\sum_{k\geqslant 1}\left(\frac{\alpha\lambda_k^p}{\alpha\lambda_k^p + e^{-\lambda_k T}}\right)^2\left(\int_0^t e^{\lambda_k(s-t)}f_k(s)ds\right)^2 \leqslant \sum_{k\geqslant 1}\frac{\alpha^2\lambda_k^{2p-1}}{(\alpha\lambda_k^p + e^{-\lambda_k T})^2}\|f\|^2_{L^2(0,T)}.$$

Using (6) for $s = \frac{1}{2}$, we obtain

$$\|u_\alpha(t) - u(t)\|^2 \leqslant 2\|u_\alpha(0) - u(0)\|^2 + \left(\frac{T}{\log\frac{2T}{\alpha}}\right)\|f\|^2_{L^2((0,T),\mathcal{H})}.$$

This implies that $u_\alpha(t)$ converges uniformly to $u(t)$ in $\mathcal{H}$. Moreover if $t = T$, then $\lim_{\alpha\to 0}u_\alpha(T) = u(T)$, using Theorem 3 we have that $u(T) = g$. Conversely, let us assume the non-homogeneous (BCP)-problem admits a solution $u(t)$, in this case from the Lemma 1, we have

$$\sum_{k\geqslant 1}\left(g_k-\int_0^T e^{\lambda_k(s-T)}f_k(s)ds\right)^2 e^{2\lambda_k T}<+\infty.$$

Then we write

$$\begin{aligned}\|u_\alpha(0)-u_\gamma(0)\|^2&=\sum_{k\geqslant 1}\frac{(\alpha-\gamma)^2\lambda_k^{2p}}{\left(\alpha\gamma\lambda_k^{2p}+(\alpha+\gamma)\lambda_k^p e^{-\lambda_k T}+e^{-2\lambda_k T}\right)^2}\left(g_k-\int_0^T e^{\lambda_k(s-T)}f_k(s)ds\right)^2\\&\leqslant(\alpha-\gamma)^2\sum\nolimits_{k=1}^N\frac{e^{2\lambda_k T}}{(\alpha+\gamma)^2}\left(g_k-\int_0^T e^{\lambda_k(s-T)}f_k(s)ds\right)^2\\&\quad+\sum\nolimits_{k\geqslant N+1}\frac{(\alpha-\gamma)^2}{(\alpha+\gamma)^2}e^{2\lambda_k T}\left(g_k-\int_0^T e^{\lambda_k(s-T)}f_k(s)ds\right)^2,\end{aligned}$$

therefore $\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right)<1$ and $\sum_{k\geqslant 1}\left(g_k-\int_0^T e^{\lambda_k(s-T)}f_k(s)ds\right)^2 e^{2\lambda_k T}<+\infty$, implies that $\{u_\alpha(0)\}$ is a Cauchy sequence, hence convergent in $\mathcal{H}$ this ends the proof. □

Theorem 8. *Let us assume that* $g\in\mathcal{H}$ *and* $f\in L^2((0,T),\mathcal{H}),\alpha\in(0,eT)$. *Suppose the non-homogeneous (BCP) problem admits a solution* $u(t)$ *in the classical sense satisfying* $\|Au(t)\|_{\mathcal{H}}<+\infty$, *then*

$$\|u_\alpha(t)-u(t)\|\leqslant\frac{c}{\log\frac{T}{\alpha}},$$

$\forall t\in[0,T]$, where $c=T\sup_{t\in[0,T]}\|Au(t)\|_{\mathcal{H}}$.

Proof. Let the (BCP) problem admit a unique solution in the classical sense, from the expressions (2), (4) of $u(t)$ and $u_\alpha(t)$ we have

$$\|u_\alpha(t)-u(t)\|^2=\sum_{k\geqslant 1}\frac{\alpha^2\lambda_k^{2(p-1)}}{\left(\alpha\lambda_k^p+e^{-\lambda_k T}\right)^2}\lambda_k^2\left(g_k e^{\lambda_k(T-t)}-\int_t^T e^{\lambda_k(s-t)}f_k(s)ds\right)^2.$$

From the supremum estimate (6), with $s=1$ we obtain

$$\|u_\alpha(t)-u(t)\|\leqslant\frac{c}{\log\frac{T}{\alpha}},\quad\forall t\in[0,T],$$

where the constant c is $T\sup_{t\in[0,T]}\|Au(t)\|$. Which finishes the proof. □

In an analogous manner to Theorems 5 and 6, weaker conditions on the given data g in $\mathcal{H}$ also induce error-estimates. From the supremum estimates used in their proofs, we have the following results.

Theorem 9. *Let us assume that* $g\in\mathcal{H}$ *and* $f\in L^2((0,T),\mathcal{H}),\alpha\in(0,eT)$. *Assume that* $u(t)\in\mathcal{D}(A^s),s>0$, *then*

$$\|u_\alpha(t) - u(t)\| \leqslant \frac{c}{\left(\log \frac{T}{\alpha}\right)^s}, \text{ for any } t \in [0, T],$$

where c depends on T and $\sup_{t\in[0,T]} \|A^s u(t)\|_{\mathcal{H}}$.

Theorem 10. *Under the same setting as in Theorem* 9, *if* $u(t) \in \mathcal{D}(\log^s A)$ *for s > 0, then*

$$\|u_\alpha(t) - u(t)\| \leqslant \frac{c}{\log^s\left(\frac{1}{T} \log \frac{T}{s\alpha}\right)}, \text{ for } t \in [0, T];$$

the constant c depends upon s,T and $\sup_{t\in[0,T]} \|\log^s A u(t)\|_{\mathcal{H}}$.

Finally, for non-exact data we establish the following result.

Theorem 11. *Let* $f \in L^2((0, T), \mathcal{H}),\ g \in \mathcal{H}$. *Assume that the (BCP) problem admits a unique solution u(t). Let* g_α *be a measured data satisfying*

$$\|g_\alpha - g\| \leqslant \alpha,$$

then there exists a solution $v_\alpha(t)$ *associated with this data satisfying*

$$\|v_\alpha(t) - u(t)\| \leqslant \frac{c}{\log \frac{T}{\alpha}} + \left(\frac{T}{\log \frac{T}{p\alpha}}\right)^p;$$

that is for any t ∈ [0,T], where c is a constant depends upon T and $\sup_{t\in[0,T]} \|Au(t)\|_{\mathcal{H}}$.

Proof. Let $u_\alpha(t)$ and $v_\alpha(t)$ be the solutions of the (ABCP) problem corresponding respectively to g and g_α , then

$$\|v_\alpha(t) - u(t)\| \leqslant \|v_\alpha(t) - u_\alpha(t)\| + \|u_\alpha(t) - u(t)\|,$$

from the stability in Theorems 1 and 8 we deduce

$$\|v_\alpha(t) - u(t)\| \leqslant \frac{c}{\log \frac{T}{\alpha}} + \left(\frac{T}{\log \frac{T}{p\alpha}}\right)^p,$$

where $p \in \mathbb{R}_+^*, t \in [0, T]$. This finishes the proof. □

Remark 3. The parameter p in $\mathbb{R}_+^*$ improves the estimate of Theorem 4 in [5], given for the homogeneous case and $p = 1$, and Theorem 2.10 in [20], where also $p = 1$ and A is the periodic heat operator.

5. On the backward heat equation with Bessel operator

In this section, we apply the obtained results to the study of the inverse problem of the non-homogeneous heat equation involving the Bessel operator of the form:

$$\frac{\partial u(x,t)}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)u(x,t) = f(x,t), (x,t) \in]0,1[\times]0,T[, \tag{7}$$

$$u(1,t) = 0 \text{ and } \lim_{x\to 0} xu'(x,t) = 0,\ 0 < t < T, \tag{8}$$

$$u(x,T) = g(x),\ x \in]0,1[. \tag{9}$$

The corresponding homogeneous case to (7) and (8) was first studied in [12,13], using a hyperbolic regularization method based upon a perturbation of the Eq. (7) by adding a term $\epsilon\frac{\partial^2 u(x,t)}{\partial t^2}$. In these papers convergence rates and error-estimates are not treated.

To apply our parabolic model, we regularize the non-homogeneous problem (7) by the following approximate problem:

$$\frac{\partial u_\alpha(x,t)}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)u_\alpha(x,t) = \sum_{k\geqslant 1}\frac{\sqrt{2}e^{-\lambda_k T}}{J_0'(\sqrt{\lambda_k})(\alpha\lambda_k + e^{-\lambda_k T})}f_k(t)J_0(\sqrt{\lambda_k}x), \tag{10}$$

$$u_\alpha(1,t) = 0 \text{ and } \lim_{x\to 0} xu_\alpha'(x,t) = 0, \tag{11}$$

$$u_\alpha(x,T) = \sum_{k\geqslant 1}\frac{\sqrt{2}e^{-\lambda_k T}}{J_0'(\sqrt{\lambda_k})(\alpha\lambda_k + e^{-\lambda_k T})}g_k J_0(\sqrt{\lambda_k}x), \tag{12}$$

where $f_k(t)$ and g_k are respectively the Fourier–Bessel coefficients of $f(x,t)$ and $g(x)$ in the space $H_x[0,1] = L^2([0,1],xdx)$; $J_0(x)$ is the well known Bessel function of the first kind.

For $f \in L^2((0,T),H_x[0,1])$ and $g \in H_x[0,1]$, we have:

$$f(x,t) = \sum_{k\geqslant 1} f_k(t)\Phi_k(x) \text{ and } g(x) = \sum_{k\geqslant 1} g_k\Phi_k(x),$$

where

$$f_k(t) = \int_0^1 xf(x,t)\Phi_k(x)dx \text{ and } g_k = \int_0^1 xg(x)\Phi_k(x)dx.$$

We denote the Bessel operator by

$$A = -\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right),$$

defined on

$$\mathcal{D}(A) = \{u \in H_x[0,1],\ u(1) = 0,\ \lim_{x\to 0} xu'(x) = 0\}.$$

It is well known that A is a selfadjoint singular operator with a complete orthonormal set of eigenfunctions $\{\Phi_k\}_{k\in\mathbb{N}^*}$ in the Hilbert space $H_x[0,1]$, given by:

$$\Phi_k(x) = \frac{\sqrt{2}}{J_0'(\sqrt{\lambda_k})}J_0(\sqrt{\lambda_k}x), \quad k = \overline{1,\infty},\ 0 < x < 1,$$

corresponding respectively to the unbounded increasing positive sequence $\{\lambda_k\}_{k\in\mathbb{N}^*}$ of eigenvalues, which are the squares of the zeros of the Bessel function of the first kind $J_0(x)$.

Under these assumptions, the problem (7) may be written under the abstract backward Cauchy problem (BCP) form. Hence, applying the results of the preceding sections, we obtain the following results.

Theorem 12. *The approximate non-homogeneous Backward problem (10)–(12) admits a unique classical solution $u_\alpha(x,t)$ which depends continuously upon the data g in $H_x[0,1]$. Moreover, we have for any $\alpha \in (0,1), t \in [0,T]$:*

$$\int_0^1 x u_\alpha^2(x,t)dx \leqslant \frac{2}{\left(\alpha \log \frac{T}{\alpha}\right)^2}\left(\int_0^1 x g^2(x)dx + T\int_0^T \int_0^1 x f^2(x,t)dxdt\right).$$

Proof. We apply Theorem 1 with parameter $p = 1$ in the Hilbert space $\mathcal{H} = H_x[0,1]$. □

Theorem 13. *Let us assume that $g \in H_x[0,1]$ and $f \in L^2((0,T), H_x[0,1])$. Then, the non-homogeneous backward problem (7) admits a solution $u(x,t)$ if and only if the sequence*

$$u_\alpha(x,0) = \sum_{k\geqslant 1} \frac{\sqrt{2}}{J_0'(\sqrt{\lambda_k})(\alpha\lambda_k + e^{-\lambda_k T})}\left(g_k - \int_0^T e^{\lambda_k(s-T)} f_k(s)ds\right) J_0(x\sqrt{\lambda_k}),$$

is convergent in $H_x[0,1]$. Furthermore, $u_\alpha(x,t)$ converges to $u(x,t)$ uniformly in t. Furthermore, if g satisfies:

$$\int_0^1 \frac{1}{x}\left(\frac{d}{dx}\left(x\frac{d}{dx}\right)g\right)^2 dx < +\infty,$$

then:

$$\int_0^1 x(u_\alpha(x,T) - g(x))^2 dx \leqslant \frac{c}{\left(\log\frac{T}{\alpha}\right)^2},$$

where the constant c depends uniquely upon T and $\int_0^1 \frac{1}{x}\left(\frac{d}{dx}\left(x\frac{d}{dx}\right)g\right)^2 dx$.

Proof. Results from Theorems 7 and 5 for $s = 1$ and $p = 1$. □

Theorem 14. *Let assume that $g \in H_x[0,1]$ and $f \in L^2((0,T), H_x[0,1])$, $\alpha \in (0, eT)$. Suppose the non-homogeneous problem (7) admits a solution $u(x,t)$ in the classical sense satisfying*

$$\int_0^1 \frac{1}{x}\left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)u(x,t)\right)^2 dx < +\infty,$$

then

$$\int_0^1 x(u_\alpha(x,t) - u(x,t))^2 dx \leqslant \frac{c}{\left(\log\frac{T}{\alpha}\right)^2},$$

for any $t \in [0,T]$*, where* $c = T\sup_{t\in[0,T]} \int_0^1 \frac{1}{x}\left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)u(x,t)\right)^2 dx$ *, and* $u_\alpha(x,t)$ *is the unique solution of (10)–(12).*

Proof. Results directly by applying Theorem 8. □

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