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On q-extension of Laurent expansion with applications

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Abstract. In this article, Cauchy's integral formula for nth q-derivative of analytic functions is established and used to introduce a new proof to q-Taylor series by means of using the residue calculus in the complex analysis. Some theorems related to this formula are presented. A q-extension of a Laurent expansion is derived and proved by means of using Cauchy's integral formula for a function, which is analytic on a ring-shaped region bounded by two concentric circles. Three illustrative examples are presented to be as applications for a q-Laurent expansion.

Mathematical Subject Classification: 30B10; 30G35

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1. Introduction and preliminaries

Two important problems in complex function theory are the problems of expanding a function in a series of polynomials and the interpolation problem of finding an entire function from its values. The Taylor series and the Laurent expansion play a very important role to solve these problems. The Taylor series is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point and the Laurent series of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

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The Laurent expansion has no real-variable counterpart and is a key in the discussion of singularities and residues.

The problem of expanding a function with respect to a given polynomial basis has many implications in the q-analysis. The simplest example of this kind is q-Taylor expansion theorem. The q-Taylor series was introduced at the first time by Jackson [8]. Al-Salam and Verma introduced the q-type interpolation series to deriving the q-Leibniz rule for the q-fractional Riemann–Liouville operator [1]. Ismail and Stanton established q-analogs of the Taylor series expansions in special polynomial bases for analytic functions in bounded domains and for entire functions [9,10].

We believe that the q-Taylor series has drawn the attention of many authors, but it seems so far that no one has developed a formula corresponding to the formula for the Laurent expansion series in q-calculus. Therefore we give, in this paper, a q-extension of the Laurent series expansion via using Cauchy's integral formula. Also in the present article, Cauchy's integral formula for analytic function is used to derive a contour integral representation for nth q-derivative of analytic function. This formula (contour integral representation for nth q-derivative) is used to identify a bound for nth q-derivative and to introduce a new proof to q-Taylor series.

Due to the difference of the definitions and notations used in the q-calculus, we devote the rest of this section to list all the definitions and notations needed throughout this work. These definitions will be taken from the well known books in this field [3,4,7].

For any complex number a, the basic number and the q-factorial are defined as

$$[a]_q = \frac{1-q^a}{1-q}, \quad q\neq 1; \quad [n]_q! = [n]_q[n-1]_q\ldots[1]_q, \quad n=1,2,\ldots; \quad [0]_q! = 1 \tag{1.1}$$

and the scalar q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n=1,2,\ldots. \tag{1.2}$$

The limit, $\lim_{n\to\infty}(a;q)_n$, is denoted by $(a;q)_\infty$ provided $|q| < 1$. This implies that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad n=0,1,2,\ldots. \tag{1.3}$$

Recall the q-analog of Newtons' binomial formula

$$(z;q)_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} z^k, \qquad n=0,1,2,\ldots \tag{1.4}$$

and its dual

$$z^n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+1)/2-nk}(z;q)_k, \qquad n=0,1,2,\ldots. \tag{1.5}$$

The q-binomial coefficients are defined for positive integer n,k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{1.6}$$

We shall make use of the (equivalent by symmetry) q-Pascal recurrences

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q - q^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{1.7}$$

The q-analogs of the exponential functions are defined as

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q;q)_n} = (-z;q)_\infty, \tag{1.8}$$

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}. \tag{1.9}$$

For the convergence of the second series, we need $|z| < 1$.

The q-derivative $D_q f(z)$ of a function f is given as

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad q \neq 1, \ z \neq 0, \quad (D_q f)(0) = f'(0), \tag{1.10}$$

provided $f'(0)$ exists. If f is differentiable then $D_q f(z)$ tends to $f'(z)$ as $q \to 1$. Furthermore, we define

$$D_q^0 f(z) = f(z) \qquad \text{and} \qquad D_q^n f(z) = D_q(D_q^{n-1} f(z)), \qquad n = 1, 2, \ldots.$$

Then, $D_q^n f(z)$ can be expressed as

$$D_q^n f(z) = (1-q)^{-n} z^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+1)/2 - nk} f(zq^k). \tag{1.11}$$

As tool to provide some of results in our paper, we need the following lemma

Lemma 1. *For all complex variables z, and q and positive integer n, we have*

$$\sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+1)/2 - nk} (z;q)_{k-1} = -q^{1-n} \frac{z^n - q^n}{z - q}, \qquad z \neq q. \tag{1.12}$$

In particular, when z tends to q, we have

$$\sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+1)/2 - nk} (q;q)_{k-1} = -n. \tag{1.13}$$

Proof. Let the function

$$f(n) = \sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+1)/2 - nk} (z;q)_{k-1} = -\sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)/2 - (n-1)(k+1)} (z;q)_k$$

to establish the recursive equation

$$
\begin{aligned}
f(n)-f(n+1) &= \sum_{k=0}^{n-1}(-1)^k\left\{\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q - q^{k+1}\begin{bmatrix} n \\ k \end{bmatrix}_q\right\} q^{k(k+1)/2-(k+1)n}(z;q)_k \\
&\quad + (-1)^n q^{n(n+1)/2-n(n+1)}(z;q)_n \\
&= \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+1)/2-(k+1)n}(z;q)_k + (-1)^n q^{n(n+1)/2-n(n+1)}(z;q)_n \\
&= q^{-n}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k+1)/2-nk}(z;q)_k = q^{-n}z^n.
\end{aligned}
$$

Here, we used the formulas (1.5) and (1.7). The previous equation can be solved with the initial condition $f(1) = -1$ by using iterative method and geometric sequence rule to complete the proof. $\square$

2. Cauchy's integral formula for nth q-derivative

Let $f(z)$ be an analytic function everywhere on and inside a closed contour C containing the point a. Then the Cauchy's integral formula states [12]

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz. \tag{2.1}$$

For a non-negative integer n and a closed contour C surrounding the points aq^k, $k = 0,1,2,\ldots,n$, where a is any point in the complex plane apart from the origin and q is a complex number apart from 1, and situated entirely within the region in which $f(z)$ is analytic, the following result can be established.

Theorem 2.1 (*Cauchy's integral formula for nth q-derivative*). *Let $f(z)$ be analytic function on a simply connected subset U of the complex z-plane. Let C be a closed contour completely contained in U. Let the points aq^k, $k = 0,1,2,\ldots,n$ lie inside the contour C for all non-negative integers n where $a \neq 0$ and $q \neq 1$ are complex. Then the Cauchy's integral formula for nth q-derivative can be derived as*

$$D_q^n f(a) = \frac{[n]_q!}{2\pi i}\int_C \frac{f(z)}{z^{n+1}(a/z;q)_{n+1}}dz, \qquad n = 0,1,2,\ldots, \tag{2.2}$$

where the contour integral is taken counter-clockwise. In particular, when $a \to 0$

$$\lim_{a\to 0} D_q^n f(a) = \frac{[n]_q!}{n!} f^{(n)}(0). \tag{2.3}$$

Proof. The proof of theorem has been provided in [4]. $\square$

Theorem 2.2. *Let C be the positively oriented circle of center at the origin and radius $r > 0$. If f is analytic function everywhere on and inside C and n is a non-negative integer, then the Cauchy's integral formula (2.2) can be expressed as*

$$D_q^n f(z) = [n]_q! \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{f^{(n+k)}(0)}{(n+k)!} z^k, \qquad |q| < 1. \tag{2.4}$$

Proof. It is obvious that if z is inside C and $|q| < 1$, then zq^k, $k = 0,1,2,\ldots,n$ are also inside C.

By using the relation (1.3) and the q-binomial theorem

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z| < 1, \tag{2.5}$$

the relation (2.2) can be rewritten in the form

$$\begin{aligned} D_q^n f(z) &= \frac{[n]_q!}{2\pi i} \int_C \frac{f(t)}{t^{n+1}} \frac{(zq^{n+1}/t;q)_\infty}{(z/t;q)_\infty} dt \\ &= \frac{[n]_q!}{2\pi i} \int_C \sum_{k=0}^{\infty} \frac{f(t)(q^{n+1};q)_k}{t^{n+1}(q;q)_k} \left(\frac{z}{t}\right)^k dt. \end{aligned}$$

Now we are interested in the permutation of the sum and the integral in the last expression. To legitimate this process, note from the q-binomial theorem that the previous summation is absolutely convergent for $|z/t| < 1$ which is true inside the circle C. By the dominated convergence theorem [6, p. 83], the series and the integral can be permuted and we can write

$$D_q^n f(z) = [n]_q! \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q z^k \frac{1}{2\pi i} \int_C \frac{f(t)}{t^{n+k+1}} dt.$$

Since the circle C centered at the origin point,

$$\frac{f(t)}{t^{n+k+1}}$$

is a function of t, which is analytic at all points within the circle C except the point $t = 0$ and so the Cauchy's differentiation formula would yield

$$\frac{1}{2\pi i} \int_C \frac{f(t)}{t^{n+k+1}} dt = \frac{f^{(n+k)}(0)}{(n+k)!}.$$

This ends the proof. □

Theorem 2.3 (*Cauchy's inequality for nth q-derivative*). *Let C be the positively oriented circle of center a and radius $r > 0$ such that $|a| \leqslant r$. If f is analytic function everywhere on and inside the circle C, then for $0 < q < 1$, we have*

$$|D_q^n f(a)| \leqslant \frac{[n]_q! M}{r^n q^{n(n+1)/2}}, \qquad n = 0, 1, 2, \ldots, \tag{2.6}$$

where M is the upper bound of $f(z)$ on the circle C.

Proof. Since the circle C centered at a with radius $r \geqslant |a|$ and $0 < q < 1$, then the points $aq^k, k = 0,1,2,\ldots,n$ are inside the circle C. Using the triangle inequality with noting that $|a| \leqslant r$, would yield

$$|z - aq^k| \geqslant r - |a - aq^k| = r - |a|(1 - q^k) \geqslant rq^k, \quad k = 0, 1, 2, \ldots, n.$$

It follows from Cauchy's integral formula (2.2) for nth q-derivative for $0 < q < 1$, that

$$\begin{aligned} |D_q^n f(a)| &\leqslant \frac{[n]_q! M}{2\pi} \int_C \frac{1}{\prod_{k=0}^{n} |(z - aq^k)|} |dz| \\ &\leqslant \frac{[n]_q! Mr}{\prod_{k=0}^{n} (r - |a|(1 - q^k))} \leqslant \frac{[n]_q! M}{r^n \prod_{k=1}^{n} q^k} = \frac{[n]_q! M}{r^n q^{n(n+1)/2}}. \end{aligned}$$

This completes the proof. □

3. q-Taylor series

Jackson [8] introduced the following q-counterpart of Taylor series

$$f(z) = \sum_{k=0}^{\infty} \frac{z^k (a/z; q)_k}{[k]_q!} D_q^k f(a). \tag{3.1}$$

Also Al-Salam and Verma [1] introduced the q-type interpolation series

$$f(z) = \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k}{2}} \frac{a^k (z/a; q)_k}{[k]_q!} D_q^k f(aq^{-k}). \tag{3.2}$$

Neither Jackson nor Al-Salam and Verma gave proofs of their q-analogs. However Al-Salam and Verma verified the validity of (3.2) under the condition of the expandability of f in the from (see [5])

$$f(z) = \sum_{k=0}^{\infty} c_k a^k (z/a; q)_k. \tag{3.3}$$

Annaby and Mansour [5] gave analytic proof of Jackson and Al-Salam–Verma q-Taylor series based on the q-Cauchy integral formula of Al-Salam [2]. They proved the convergence of the q-Taylor series to the original functions if these latter ones are analytic in some complex domain. In the following theorem, we will give analytic proof of Jackson q-Taylor series based on the Cauchy's integral formula (2.2) for nth q-derivative of analytic functions.

Theorem 3.1 (q-*Taylor series*). *Let C be the positively oriented circle containing the point a apart from the origin. If f is analytic function everywhere inside C, then the q-Taylor series (3.1) holds for $|q| < 1$.*

Proof. It is easy to prove by mathematical induction that

$$\frac{1}{t-z}=\sum_{k=0}^{n-1}\frac{(a/z;q)_k z^k}{(a/t;q)_{k+1}t^{k+1}}+\frac{(a/z;q)_n z^n}{(t-z)(a/t;q)_n t^n},\qquad n=1,2,\ldots. \tag{3.4}$$

Substituting into Cauchy's integral formula (2.1) with using (2.2) would yield

$$f(z)=\sum_{k=0}^{n-1}\frac{z^k(a/z;q)_k}{[k]_q!}D_q^k f(a)+R_n(z),\qquad n=1,2,\ldots,$$

where

$$R_n(z)=\frac{(a/z;q)_n z^n}{2\pi i}\int_C\frac{f(t)}{(t-z)(a/t;q)_n t^n}dt,\qquad n=1,2,\ldots. \tag{3.5}$$

Now we are interested in proving that $|R_n(z)|\to 0$ as $n\to\infty$. Notice that the function

$$g(t)=\frac{f(t)}{t-z} \tag{3.6}$$

is a function of t, which is a bounded function on the circle C (as it is continuous) and so will not exceed a finite number M. Therefore

$$|R_n(z)|\leqslant\frac{2\pi rM}{2\pi}\prod_{k=0}^{n-1}\frac{|z|z+|a||q|^k}{r-|a||q|^k}=Mr\left(\frac{|z|}{r}\right)^n\frac{(-|a|/|z|;|q|)_n}{(|a|/r;|q|)_n},$$

where r is the radius of the circle C, so that $2\pi r$ is the length of the path of integration in the last integral, and $r=|t|$ for point t on the circumference of C. As $n\to\infty$, the two products $(-|a|/|z|;|q|)_n$ and $(|a|/r;|q|)_n$ converge absolutely due to $|q|<1$, and so the right hand side of the last inequality tends to zero. This ends the proof. □

Remark 3.2. In the previous theorem, $R_n(z)$ is a remainder term, denoting the difference between the q-Taylor polynomial of degree $(n-1)$ and the original function. The remainder term $R_n(z)$ depends on z and is small if z is closed enough to a. Note that $g(t)$ in (3.4) is analytic function within and on the circle C which allows us to estimate the remainder term $R_n(z)$ by using relation (2.2) as

$$R_n(z)=\frac{(a/z;q)_n z^n}{[n-1]_q!}D_q^{n-1}g(a),\qquad g(a)=\frac{f(a)}{a-z},\qquad n=1,2,\ldots.$$

4. q-Laurent series expansion

The Laurent series of a complex function is a representation of that function as a power series which includes terms of negative degree. The Laurent series for a complex function $f(z)$ about a point c is given as

$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-c)^k, \tag{4.1}$$

where the a_k are constants defined by a line integral which is a generalization of Cauchy's integral formula

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-c)^{k+1}}dz, \tag{4.2}$$

where the path of integration γ is counterclockwise around a closed, rectifiable path containing no self-intersections, enclosing the point c and lying in an annulus in which $f(z)$ is holomorphic. The expansion for $f(z)$ will then be valid anywhere inside the annulus.

In this section, a q-analog of the Laurent expansion is derived and proved with identifying the domains in which this expansion is convergent. The q-Laurent expansion is applied to expand a complex function as a power q-series to which q-Taylor series cannot be applied.

Theorem 4.1 (q-*Laurent expansion*). *Let Ω_0 and Ω be closed and open subsets, respectively, of the complex z-plane, and $0 \in \Omega_0 \subset \Omega$. Let $f(z)$ be an analytic function on Ω/Ω_0 and $z_0 \in \Omega_0$. Then, for any $z \in \Omega/\Omega_0$, $f(z)$ admits the q-Laurent expansion*

$$f(z) = \sum_{k=0}^{n-1} a_k (z_0/z;q)_k z^k + \sum_{k=1}^{n} \frac{b_k z^{-k}}{(z_0/z;q)_k} + R_n(z,z_0) + R'_n(z,z_0) \tag{4.3}$$

where the coefficients a_k and b_k of the expansion are given, respectively, by the Cauchy integrals

$$a_k = \frac{1}{2\pi i}\int_\ell \frac{f(w)}{w^{k+1}(z_0/w;q)_{k+1}}dw, \tag{4.4}$$

$$b_k = \frac{1}{2\pi i}\int_{\ell'} w^{k-1}(z_0/w;q)_{k-1} f(w)dw. \tag{4.5}$$

The remainder terms $R_n(z,z_0)$ and $R'_n(z,z_0)$ are given by the Cauchy integral formulas

$$R_n(z,z_0) = \frac{z^n (z_0/z;q)_n}{2\pi i}\int_\ell \frac{f(w)}{w^n (w-z)(z_0/w;q)_n}dw, \tag{4.6}$$

$$R'_n(z,z_0) = \frac{1}{z^n (z_0/z;q)_n}\frac{1}{2\pi i}\int_{\ell'} \frac{w^n (z_0/w;q)_n f(w)}{z-w}dw, \tag{4.7}$$

where ℓ and ℓ' are simple closed loops contained in Ω/Ω_0 which encircle the point z_0 in the counterclockwise direction. Moreover, ℓ' does not contain the point z inside, whereas ℓ encircles ℓ' and the point z. The q-Laurent expansion (4.3) is convergent on the annulus

$$O = \{z \in \Omega/\Omega_0, \quad r_2 < |z| < r_1\}, \tag{4.8}$$

where

$$r_1 = \inf\{|w| : w \in /\Omega\}, \qquad r_2 = \sup\{|w| : w \in \Omega_0\} \tag{4.9}$$

and Ç is the complex plane.

Proof. By Cauchy theorem [12]

$$f(z) = \frac{1}{2\pi i}\int_\ell \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\ell'} \frac{f(w)}{w-z}dw. \tag{4.10}$$

Inserting the expansion (3.4) after replacing t and a by w and z_0, respectively, into the Eq. (4.10) would yield (4.3)–(4.7) after straightforward calculations. Now we are interested in proving that both $|R_n(z,z_0)|$ and $|R'_n(z,z_0)|$ tend to 0 as $n \to \infty$. Notice that the function $g(w)$ (defined by (3.6)) is a function of w, which is a bounded function on the contours ℓ and ℓ' (as it is continuous) and so will not exceed some finite numbers M and M' on them, respectively. Based upon the definitions (4.8) and (4.9), the annulus O centered at the origin, $|w| \geqslant r_1$ for any point w on the contour ℓ and $|w| > |z_0|$. Thus

$$|w - z_0 q^k| \geqslant ||w| - |z_0||q|^k| = |w| - |z_0||q|^k \geqslant r_1 - |z_0||q|^k, \quad k = 0, 1, 2, \ldots, n-1.$$

Therefore

$$|R_n(z, z_0)| \leqslant \frac{M \cdot L}{2\pi} \prod_{k=0}^{n-1} \frac{|z| + |z_0||q|^k}{r_1 - |z_0||q|^k} = \frac{M \cdot L}{2\pi} \left(\frac{|z|}{r_1}\right)^n \frac{(-|z_0|/|z|; |q|)_n}{(-|z_0|/r_1; |q|)_n},$$

where L is the length of the path of integration in (4.6). As $n \to \infty$, the two infinite products $(-|z_0|/|z|;|q|)_\infty$ and $(|z_0|/r_1;|q|)_\infty$ converge absolutely due to $|q| < 1$, and so the right hand side of the last inequality tends to zero. Similarly, one can easily prove that $|R'_n(z, z_0)| \to 0$ as $n \to \infty$. This finishes the proof. □

This result is q-Laurent theorem; changing the notations, it can be expressed in the following form: If $f(z)$ is analytic function on the annulus surrounded by two concentric circles ℓ and ℓ' of center at the origin point with the point z_0 lies inside ℓ' where ℓ' is completely contained in ℓ, then at any point z of the annulus, $f(z)$ can be expanded in the form

$$f(z) = \sum_{k=0}^{\infty} a_k (z_0/z; q)_k z^k + \sum_{k=1}^{\infty} \frac{b_k z^{-k}}{(z_0/z; q)_k}, \tag{4.11}$$

where the coefficients a_k and b_k are defined as in (4.4) and (4.5), respectively.

Theorem 4.2. *Let $f(z)$ be analytic function on Ω/Ω_0 and $z_0 \in \Omega_0$. Then the function f has an expansion as a q-Laurent series at z_0 on this region. This series expansion of f is unique.*

Proof. Suppose the function $f(z)$ holomorphic on Ω/Ω_0 has two q-Laurent series

$$f(z) = \sum_{k=0}^{\infty} a_k (z_0/z; q)_k z^k + \sum_{k=1}^{\infty} \frac{b_k z^{-k}}{(z_0/z; q)_k} = \sum_{k=0}^{\infty} a'_k (z_0/z; q)_k z^k + \sum_{k=1}^{\infty} \frac{b'_k z^{-k}}{(z_0/z; q)_k}.$$

Multiplying both sides with $z^{-n-1}(z_0/z; q)_{n+1}^{-1}$, where n is an arbitrary non-negative integer, followed by integrating on a path γ inside the region Ω/Ω_0 yield

$$\int_\gamma \sum_{k=0}^{\infty} a_k (z_0/z; q)_k z^{k-n-1} (z_0/z; q)_{n+1} dz + \int_\gamma \sum_{k=1}^{\infty} \frac{b_k z^{-k-n-1}}{(z_0/z; q)_k (z_0/z; q)_{n+1}} dz$$
$$= \int_\gamma \sum_{k=0}^{\infty} a'_k \frac{(z_0/z; q)_k z^{k-n-1}}{(z_0/z; q)_{n+1}} dz + \int_\gamma \sum_{k=1}^{\infty} \frac{b_k z^{-k-n-1}}{(z_0/z; q)_k (z_0/z; q)_{n+1}} dz.$$

The series converge uniformly on Ω/Ω_0 for γ to be contained in the constricted closed region, so the integrations and summations can be interchanged. Substituting the identities

$$\int_\gamma \frac{(z_0/z;q)_k z^{k-n-1}}{(z_0/z;q)_{n+1}}dz = 2\pi i\delta_{nk} \quad \text{and} \quad \int_\gamma \frac{b_k z^{-k-n-1}}{(z_0/z;q)_k(z_0/z;q)_{n+1}}dz = 0$$

into the summations yields $a_k = a'_k$. Again, multiplying both sides with $z^{n-1}(z_0/z;q)_{n-1}$ followed by integrating on the path γ inside the region Ω/Ω_0, this yields $b_k = b'_k$. Hence the q-Laurent series is unique. □

Remark 4.3. The q-Laurent series (4.11) is equivalent to the classical Laurent series (4.1) when $z_0 = 0$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k. \tag{4.12}$$

This can be verified as

$$b_{-k} = \int_\gamma \frac{f(w)}{w^{k+1}}dw = a_k, \quad k = -1, -2, \ldots,$$

where the path of integration γ lies in the annulus Ω/Ω_0.

Proposition 4.4. *In Theorem* 4.1*, if $f(z)$ is analytic function on Ω_0, then the q-Laurent expansion (*4.11*) tends to the q-Taylor series (*3.1*) with replacing a by z_0.*

Proof. Since $f(z)$ is analytic function on Ω_0, then $f(z)$ is analytic function within and on the contour ℓ' and by Cauchy's integral theorem, we have $b_k = 0, k = 1,2,\ldots$. Also $f(z)$ is analytic function within and on the contour ℓ and the point $z_0q^i, i = 0,1,2,\ldots,k$ are inside ℓ due to finding the origin point inside ℓ and $|q| < 1$. Therefore, these points are simple poles inside the path of integration ℓ of the coefficients a_k. Theorem 2.1 can be used to compute the coefficient a_k as $a_k = 1/[k]_q!D_q^k f(z_0)$. Therefore the q-Taylor series (3.1) holds. □

Definition 4.5. Let n be a positive integer and $|q| < 1$, then the analytic function $f(z)$ is said to have a q-pole of order n at the point z_0 if, in the q-Laurent expansion (4.11), $b_k = 0$ for $k \geqslant n + 1$ and $b_n \neq 0$. In the case of $n = 1$, the analytic function $f(z)$ is said to have a simple q-pole at a point z_0. The analytic function $f(z)$ is said to have a q-essential singularity if $\lim_{n\to\infty} b_n \neq 0$.

Equivalently, $f(z)$ is said to have a q-pole of order n at a point z_0 if n is the smallest positive integer for which $z^n(z_0/z;q)_n f(z)$ is analytic function at the points $z_0q^k, k = 0,1,2,\ldots,n-1$.

Theorem 4.6. *Let n be a positive integer and the analytic function $f(z)$ has just a q-pole of order n at the point z_0, then we have*

$$\int_{\gamma} f(z)dz = 2\pi i b_1, \tag{4.13}$$

where the path of integration γ is a simple closed curve surrounding the points $z_0q^i, i = 0,1,2,\ldots,n-1$ and is taken counter-clockwise.

Proof. According to the Definition 4.5 and the q-Laurent series (4.11), we have

$$\int_{\gamma'} f(z)dz = \int_{\gamma'} \sum_{k=0}^{\infty} a_k (z_0/z;q)_k z^k dz + \int_{\gamma'} \frac{b_1}{z-z_0} dz + \int_{\gamma'} \sum_{k=2}^{\infty} \frac{b_k z^{-k}}{(z_0/z;q)_k} dz,$$

where the path of integration γ' is a circle centered at the origin with radius greater than $|z_0|$ and is taken counter-clockwise. Due to the convergence of the q-Laurent expansion (4.11) on the region $\{z:|z| > |z_0|\}$, the integral and summation can be permuted to obtain

$$\begin{aligned}\int_{\gamma'} f(z)dz &= 0 + 2\pi i b_1 + 2\pi i \sum_{k=2}^{n} \frac{b_k}{z_0^{k-1}} \sum_{i=0}^{k-1} \frac{(-1)^i q^{i(i+1)/2 - i(k-1)}}{(q;q)_i (q;q)_{k-i-1}} \\ &= 2\pi i b_1 + 2\pi i \sum_{k=2}^{n} \frac{b_k}{z_0^{k-1}} \frac{(q^{2-k};q)_{k-1}}{[k-1]_q!} \\ &= 2\pi i b_1 + 0 = 2\pi i b_1.\end{aligned}$$

The function $f(z)$ is analytic except at the points $z_0q^k, k = 0,1,2,\ldots,n-1$ and so the circle γ' can be deformed to any contour γ surrounds these points. □

Remark 4.7. In the previous theorem, if $n \to \infty$, the point z_0 converts to q-essential singularity and the Eq. (4.13) is also identified.

Definition 4.8. Let U be a simply connected open subset of the complex z-plane surrounding by γ and $z_0q^i \in U, i = 0,1,2,\ldots,n-1; n = 1,2,\ldots$. Let $f(z)$ be an analytic function on $U/\{z_0q^i : i = 0,1,2,\ldots,n-1\}$. Define the q-residue of f at z_0 (q-pole of order n) as

$$\mathrm{Res}(f, z_0; q) = \frac{1}{2\pi i} \int_{\gamma} f(z)dz = b_1, \tag{4.14}$$

where b_1 is the first coefficient of the second series in q-Laurent series (4.11) for $f(z)$.

Theorem 4.9. *Under the notations in the Definition* 4.8, *we have*

$$\mathrm{Res}(f, z_0; q) = \frac{1}{[n-1]_q!} \lim_{z \to z_0} D_q^{n-1} [z^n (z_0/z;q)_n f(z)], \qquad n = 1,2,\ldots. \tag{4.15}$$

Proof. We can choose $g(z) = z^n(z_0/z;q)_n f(z)$ which is analytic function in U, consequently within and on the contour γ. Substituting into the Eq. (4.5) with using Theorem 2.1 would yield

$$b_k = \begin{cases} \frac{1}{[n-k]_q!} \lim\limits_{z \to z_0 q^{k-1}} D_q^{n-k}[z^n (z_0/z;q)_n f(z)], & k = 1, 2, \ldots, n \\ 0 & k \geqslant n+1 \end{cases}, \tag{4.16}$$

when $k = 1$, we obtain (4.15) and thus the proof is completed.

5. Illustrative examples

The Laurent expansion has no real-variable counterpart and is a key in the discussion of singularities and residues. Laurent series with complex coefficients is an important tool in complex analysis, especially to investigate the behavior of functions near singularities. The residue theorem, sometimes called Cauchy's residue theorem, in complex analysis is a powerful tool to evaluate line integrals of analytic functions over closed curves and can often be used to compute real integrals as well.

In the present section, we apply our results to expand some functions in q-series on various domains and to evaluate line integral of these function over closed curves that surrounds its singular points.

Example 5.1. Consider the functions $E_q(1/z)$ and $e_q(1/z)$ defined as in (1.8) and (1.9), respectively. As complex functions, they have a singularity at $z = 0$ which do not allow to expand them in a q-Taylor series. Nevertheless, by replacing z by $1/z$ in the power series for the q-exponential functions, we obtain the q-Laurent series which converge and are equal to them for all complex numbers z for $E_q(1/z)$ except at the singularity $z = 0$ and for $e_q(1/z)$ we have to choose $|z| > 1$

$$E_q(1/z) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k z^k}, \qquad |z| > 0, \tag{5.1}$$

$$e_q(1/z) = 1 + \sum_{k=1}^{\infty} \frac{1}{(q;q)_k z^k}, \qquad |z| > 1. \tag{5.2}$$

Since q-Laurent series expansions are unique, and so these must be the q-Laurent series representations for $E_q(1/z)$ and $e_q(1/z)$. In particular, we know that if γ and γ' are simple closed contours centered at the origin and radius greater than 1 for γ', with positive orientation, then the coefficient of $1/z$ is

$$b_1 = \frac{1}{2\pi i} \int_\gamma E_q(1/z) dz = \frac{1}{2\pi i} \int_{\gamma'} e_q(1/z) dz = \frac{1}{1-q}. \tag{5.3}$$

Example 5.2. Let the function

$$f(z) = \frac{1}{z^2 - 1}. \tag{5.4}$$

This function has singularities at $z = \pm 1$, where the denominator of the expression is zero and the expression is therefore undefined. A Taylor series (or q-Taylor series) about $z = 0$ (which yields a power series) will only converge in an open disk of radius 1, since it "hits" the singularities at ± 1. However, there are many possible Laurent expansions (q-Laurent expansions) about $z = 0$, depending on the region z is in, since the Laurent expansion and q-Laurent expansion are equivalent about the origin. Here, we will show how this function can be expanded at $z_0 = \pm 1$ in q-Laurent expansion. The Eq. (3.4) can be used to give the following expressions

$$\frac{1}{z-1} = \sum_{k=0}^{\infty} \frac{(-q;q)_k}{z^{k+1}(-q/z;q)_{k+1}} \quad \text{and} \quad \frac{1}{z+1} = \sum_{k=0}^{\infty} \frac{(-1)^k(-q;q)_k}{z^{k+1}(q/z;q)_{k+1}}, \quad |z| > 1 \tag{5.5}$$

which can be used to obtain a q-Laurent expansions for this function at $z_0 = -1$ as

$$\begin{aligned} \frac{1}{z^2-1} &= \frac{1}{z+1}\sum_{k=0}^{\infty} \frac{(-q;q)_k}{z^{k+1}(-q/z;q)_{k+1}} = \sum_{k=0}^{\infty} \frac{(-q;q)_k}{z^{k+2}(-1/z;q)_{k+2}} \\ &= \sum_{k=2}^{\infty} \frac{(-q;q)_{k-2}}{z^k(-1/z;q)_k}, \qquad |z| > 1 \end{aligned} \tag{5.6}$$

and at $z_0 = 1$ as

$$\begin{aligned} \frac{1}{z^2-1} &= \frac{1}{z-1}\sum_{k=0}^{\infty} \frac{(-1)^k(-q;q)_k}{z^{k+1}(q/z;q)_{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k(-q;q)_k}{z^{k+2}(1/z;q)_{k+2}} \\ &= \sum_{k=2}^{\infty} \frac{(-1)^k(-q;q)_{k-2}}{z^k(1/z;q)_k}, \qquad |z| > 1. \end{aligned} \tag{5.7}$$

Since there always exists a unique q-Laurent series represents the function $f(z)$ at the point z_0 inside the circle $\gamma = \{z:|z| = 1 + \varepsilon, \varepsilon > 0\}$ and converges on and outside the circle γ, so these must be the q-Laurent series representations for $f(z)$. Therefore, with using Eqs. (4.4) and (4.5), we find that

$$\int_\gamma \frac{1}{w^{k+1}(\pm 1/w;q)_{k+1}(w^2-1)} dw = 0, \qquad k = 0, 1, 2, \ldots$$

and

$$\int_\gamma \frac{w^{k-1}(z_0/w;q)_{k-1}}{w^2-1} dw = \begin{cases} 0 & \text{if} \quad z_0 = \pm 1, k = 1 \\ (-q;q)_{k-2} & \text{if} \quad z_0 = -1, k = 2, 3, \ldots \\ (-1)^k(-q;q)_{k-2} & \text{if} \quad z_0 = 1, k = 2, 3, \ldots . \end{cases}$$

Although the singular points for the function $f(z)$ in this example are simple poles in the classical case but in this case are not simple q-pole, due to the domain of convergence of q-Laurent expansion where required to be centered at the origin point which does not allow to establish a separate domain for each point. This means that each simple q-pole is a simple pole and not vice versa. This example also shows that we are in need to establish (in the future) a new version corresponding to Laurent expansions in two points [11].

Example 5.3. Suppose the function

$$f(z) = e_q(z) = \frac{1}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}, \qquad |z| < 1. \tag{5.8}$$

This function is an analytic function on the open unit disk $|z| < 1$ and has a q-Taylor series as the previous series. We are trying here to expand $f(z)$ outside this disk. It obvious that the function $f(z)$ has singular points at $q^{1-n}, n = 1,2,\ldots$ and so it can be expanded in q-Laurent expansion (4.11) on the annuli

$$A_n = \{z : |q|^{1-n} < |z| < |q|^{-n}, 0 < |q| < 1\}, \qquad n = 1, 2, \ldots. \tag{5.9}$$

In this case, the function $f(z)$ has a q-pole of order n at $z = q^{1-n}$ and according to our results obtained in the previous section, the q-Laurent expansion for $f(z)$ has the form

$$f(z) = \sum_{k=0}^{\infty} a_k (q^{1-n}/z; q)_k z^k + \sum_{k=1}^{n} \frac{b_k z^{-k}}{(q^{1-n}/z; q)_k}, \qquad z \in A_n; n = 1, 2, \ldots, \tag{5.10}$$

where the coefficients a_k and b_k of the expansion are given, respectively, by the Cauchy integrals

$$a_k = \frac{1}{2\pi i} \int_{\ell_n} \frac{1}{z^{k+1} (q^{1-n}/z; q)_{k+1} (z; q)_\infty} dz, \quad k = 0, 1, 2, \ldots, \tag{5.11}$$

$$b_k = \frac{1}{2\pi i} \int_{\ell_n} \frac{z^{k-1} (q^{1-n}/z; q)_{k-1}}{(z; q)_\infty} dz, \qquad k = 1, 2, \ldots, n \tag{5.12}$$

where the contour ℓ_n is a simple closed curve lies inside the annulus A_n. The coefficients b_k can be evaluated explicitly by using (4.16) as

$$b_k = \frac{1}{[n-k]_q!} \lim_{z \to q^{k-n}} D_q^{n-k} \left[\frac{z^n (q^{1-n}/z; q)_n}{(z; q)_\infty} \right], \quad k = 1, 2, \ldots, n. \tag{5.13}$$

The above coefficients can be computed more explicitly by using the Eqs. (1.11) and (1.5) after some calculations to be

$$b_k = \frac{(-1)^n q^{n(n+1)/2 - nk}}{(q^k; q)_\infty (q; q)_{n-k}}, \qquad k = 1, 2, \ldots, n. \tag{5.14}$$

In particular, when $k = 1$

$$b_1 = \frac{1}{2\pi i} \int_{\ell_n} \frac{1}{(z; q)_\infty} dz = \frac{(-1)^n q^{n(n-1)/2}}{(q; q)_\infty (q; q)_{n-1}}, \qquad \ell_n \subset A_n; n = 1, 2, \ldots. \tag{5.15}$$

Due to the difficult of the calculations in the general case, we will suffice to calculate the coefficients a_k when $n = 1$

$$a_0 = -\lim_{z \to 1} \frac{d}{dz} \left\{ \frac{1}{(zq; q)_\infty} \right\} = \frac{1}{(q; q)_\infty} \sum_{r=1}^{\infty} \frac{q^r}{1 - q^r}.$$

For $k = 1,2,\ldots$, the coefficients a_k can be evaluate by using the relation (1.13) as

$$a_k = -\lim_{z \to 1} \frac{d}{dz} \left\{ \frac{1}{(zq; q)_\infty \prod_{r=1}^{k} (z - q^r)} \right\} + \sum_{r=1}^{k} \lim_{z \to q^r} \frac{z - q^r}{(z; q)_\infty \prod_{i=0}^{k} (z - q^i)}$$

$$= \frac{1}{(q;q)_\infty (q;q)_k}\left\{\sum_{r=1}^{k}\frac{1}{1-q^r} - \sum_{r=1}^{\infty}\frac{q^r}{1-q^r} + \sum_{r=1}^{k}(-1)^r \begin{bmatrix} k \\ r \end{bmatrix}_q q^{r(r+1)/2-rk}(q;q)_{r-1}\right\}$$

$$= \frac{1}{(q;q)_\infty (q;q)_k}\left\{k - \sum_{r=k+1}^{\infty}\frac{q^r}{1-q^r} - k\right\} = \frac{-1}{(q;q)_\infty (q;q)_k}\sum_{r=k+1}^{\infty}\frac{q^r}{1-q^r}$$

and so

$$a_k = \frac{-1}{(q;q)_\infty (q;q)_k}\sum_{r=k+1}^{\infty}\frac{q^r}{1-q^r}, \qquad k = 0, 1, 2, \ldots. \tag{5.16}$$

Therefore, the q-Laurent expansion of the function $f(z)$ about $z = 1$ has the form

$$\frac{1}{(z;q)_\infty} = \frac{-1}{(q;q)_\infty}\left\{\frac{1}{z-1} + \sum_{k=0}^{\infty}\frac{z^k(1/z;q)_k}{(q;q)_k}\sum_{r=k+1}^{\infty}\frac{q^r}{1-q^r}\right\}. \tag{5.17}$$

Note that the previous expansion converges absolutely on the annulus $\{z{:}1 < |z| < |q|^{-1}\}$ where $|q| < 1$ and can be rewritten in the form

$$\frac{1}{(z;q)_\infty} = \frac{-1}{(q;q)_\infty}\left\{\frac{1}{z-1} + \frac{q}{1-zq}\sum_{r=0}^{\infty}\frac{(q;q)_r q^r}{(zq^2;q)_r}\right\}.$$

This means that

$$\sum_{r=0}^{\infty}\frac{(q;q)_r q^{r+1}}{(zq^2;q)_r} = \frac{1}{1-z}\left\{1 - zq + \frac{(q;q)_\infty}{(zq^2;q)_\infty}\right\}, \qquad 1 < |z| < |q|^{-1}.$$

6. Conclusion

The Cauchy's integral formula is used to establish a contour integral representation for nth q-derivative which plays a principal role to arrive at the results of this paper. This formula is used to identify a bound for nth q-derivative and to introduce a new analytic proof for q-Taylor series. Also, we used it to establish q-extension of Laurent expansion. Some illustrative examples are derived to be as applications of q-Laurent expansion. In fact, our results tend to the classical case when $q \to 1^-$. In Theorems 3.1 and 4.1 we consider the domains of convergence are centered at the origin point but in the classical case are not and this perhaps confuses the readers. Nevertheless, we emphasize that our results tend to the classical results when $q \to 1^-$ due to the fact that all singular points will be merged and converted to just a pole of a certain order at the expansion point. Moreover, the contours will be deformed to be as in the classical case due to the convergence of the analytic functions except at its singularities.

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