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On connections on principal bundles

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Abstract. A new construction of a universal connection was given in Biswas, Hurtubise and Stasheff (2012). The main aim here is to explain this construction. A theorem of Atiyah and Weil says that a holomorphic vector bundle E over a compact Riemann surface admits a holomorphic connection if and only if the degree of every direct summand of E is zero. In Azad and Biswas (2002), this criterion was generalized to principal bundles on compact Riemann surfaces. This criterion for principal bundles is also explained.

Keywords: Principal bundle; Universal connection; Holomorphic connection; Real Higgs bundle

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1. Introduction

A connection ∇^0 on a C^∞ principal G-bundle $\mathcal{E}_G \longrightarrow \mathcal{X}$ is called *universal* if given any C^∞ principal G-bundle E_G on a finite dimensional C^∞ manifold M, and any connection ∇ on E_G, there is a C^∞ map

$$\xi : M \longrightarrow \mathcal{X}$$

such that

- the pulled back principal G-bundle $\xi^*\mathcal{E}_G$ is isomorphic to E_G, and
- the isomorphism between $\xi^*\mathcal{E}_G$ and E_G can be so chosen that it takes the pulled back connection $\xi^*\nabla^0$ on $\xi^*\mathcal{E}_G$ to the connection ∇ on E_G.

In [8] and [9] universal connections were constructed. In [4] a very simple, in fact quite tautological, universal connection was constructed.

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2. ATIYAH BUNDLE

All manifolds considered here will be C^∞, second countable and Hausdorff. Later we will impose further conditions such as complex structure.

Let G be a finite dimensional Lie group. Take a connected C^∞ manifold M. A principal G-bundle over M is a triple of the form

$$(E_G,\, p,\, \psi), \tag{2.1}$$

where

(1) E_G is a C^∞ manifold,

(2)

$$p\,:\, E_G \longrightarrow M \tag{2.2}$$

is a C^∞ surjective submersion, and

(3)

$$\psi\,:\, E_G \times G \longrightarrow E_G \tag{2.3}$$

is a C^∞ map defining a right action of G on E_G, such that the following two conditions hold:

- the two maps $p \circ \psi$ and $p \circ p_1$ from $E_G \times G$ to M coincide, where p_1 is the natural projection of $E_G \times G$ to E_G, and
- the map to the fiber product

$$\mathrm{Id}_{E_G} \times \psi\,:\, E_G \times G \longrightarrow E_G \times_M E_G$$

is a diffeomorphism; note that the first condition $p \circ \psi = p \circ p_1$ implies that the image of $\mathrm{Id}_{E_G} \times \psi$ is contained in the submanifold $E_G \times_M E_G \subset E_G \times E_G$ consisting of all points $(z_1,\, z_2) \in E_G \times E_G$ such that $p(z_1) = p(z_2)$.

Therefore, the first condition implies that G acts on E_G along the fibers of p, while the second condition implies that the action of G on each fiber of p is both free and transitive.

Take a C^∞ principal G-bundle $(E_G,\, p,\, \psi)$ over M. The tangent bundle of the manifold E_G will be denoted by TE_G. Take a point $x \in M$. Let

$$(TE_G)^x \,:=\, (TE_G)|_{p^{-1}(x)} \longrightarrow p^{-1}(x)$$

be the restriction of the vector bundle TE_G to the fiber $p^{-1}(x)$ of p over the point x. As noted above, the action ψ of G on E_G preserves $p^{-1}(x)$, and the resulting action of G on $p^{-1}(x)$ is free and transitive. Therefore, the action of G on TE_G given by ψ restricts to an action of G on $(TE_G)^x$. Let $\mathrm{At}(E_G)_x$ be the space of all G-invariant sections of $(TE_G)^x$. Since the action of G on the fiber $p^{-1}(x)$ is transitive, it follows that any G-invariant section of $(TE_G)^x$ is automatically smooth. More precisely, any G-invariant sections of $(TE_G)^x$ is uniquely determined by its evaluation of some fixed point of $p^{-1}(x)$. Therefore, $\mathrm{At}(E_G)_x$ is a real vector space whose dimension coincides with the dimension of E_G.

There is a natural vector bundle over M, which was introduced in [1], whose fiber over any $x \in M$ is $\mathrm{At}(E_G)_x$. This vector bundle is known as the *Atiyah bundle*, and it is denoted by $\mathrm{At}(E_G)$. We now recall the construction of $\mathrm{At}(E_G)$.

As before, consider the action of G to TE_G given by the action ψ of G on E_G. Since the action of G is free and transitive on each fiber of p, it follows that this action of G on TE_G is free and proper. Therefore, we have a quotient manifold

$$\mathrm{At}(E_G) := (TE_G)/G \tag{2.4}$$

for this action of G on TE_G. Since the natural projection $TE_G \longrightarrow E_G$ is G-equivariant, it produces a projection

$$\mathrm{At}(E_G) := (TE_G)/G \longrightarrow E_G/G = M. \tag{2.5}$$

This projection in (2.5) is clearly surjective. Furthermore, it is a submersion because the projection $TE_G \longrightarrow E_G$ is so. It is now straight-forward to check that the projection in (2.5) makes $\mathrm{At}(E_G)$ a C^∞ vector bundle over M. Its rank coincides with the rank of the tangent bundle TE_G, so its rank is $\dim G + \dim M$. From (2.4) it follows immediately that we have a natural diffeomorphism

$$\mu : p^*\mathrm{At}(E_G) \longrightarrow TE_G. \tag{2.6}$$

It is straight-forward to check that μ is a C^∞ isomorphism of vector bundles over E_G.

Let

$$dp : TE_G \longrightarrow p^*TM \tag{2.7}$$

be the differential of the projection p in (2.2). Consider the surjective C^∞ homomorphism of vector bundles

$$dp \circ \mu : p^*\mathrm{At}(E_G) \longrightarrow p^*TM, \tag{2.8}$$

where μ is constructed in (2.6). Since $p^*\mathrm{At}(E_G)$ and p^*TM are pulled back to E_G from $M = E_G/G$, they are naturally equipped with an action of G. The homomorphism $dp \circ \mu$ in (2.8) is clearly G-equivariant. Therefore, it descends to a surjective C^∞ homomorphism of vector bundles

$$\eta : \mathrm{At}(E_G) \longrightarrow TM. \tag{2.9}$$

The kernel of the differential dp in (2.7) is clearly preserved by the action of G on TE_G. The quotient $\mathrm{kernel}(dp)/G$ will be denoted by $\mathrm{ad}(E_G)$. It is a C^∞ vector bundle on M whose rank is $\dim G$. The inclusion of $\mathrm{kernel}(dp)$ in TE_G produces a fiberwise injective C^∞ homomorphism of vector bundles

$$\iota_0 : \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G).$$

The kernel of the homomorphism η in (2.9) coincides with the image of ι_0. Therefore, we have a short exact sequence of C^∞ vector bundles over M

$$0 \longrightarrow \mathrm{ad}(E_G) \xrightarrow{\iota_0} \mathrm{At}(E_G) \xrightarrow{\eta} TM \longrightarrow 0, \tag{2.10}$$

which is known as the *Atiyah exact sequence* for E_G. Using the Lie bracket operation of vector fields on E_G, the fibers of $\mathrm{ad}(E_G)$ are Lie algebras; this will be elaborated below.

The Lie algebra of G will be denoted by $\mathfrak{g}$. Consider the action of G on itself defined by $\mathrm{Ad}(g)(h) = g^{-1}hg$. This action defines an action of G on $\mathfrak{g}$, which is known as the *adjoint action*; this adjoint action of G on $\mathfrak{g}$ will also be denoted by Ad. Consider the quotient of

$E_G \times \mathfrak{g}$ where two points $(z, v), (z', v') \in E_G \times \mathfrak{g}$ are identified if there is some $g_0 \in G$ such that $z' = zg_0$ and $v' = \text{Ad}(g_0^{-1})(v)$. This quotient space coincides with the total space of the adjoint vector bundle $\text{ad}(E_G)$ in (2.10). Note that the projection

$$\text{ad}(E_G) \longrightarrow M \tag{2.11}$$

sends the equivalence class of any $(z, v) \in E_G \times \mathfrak{g}$ to $p(z)$ (it is clearly independent of the choice of the element in the equivalence class). The fibers of $\text{ad}(E_G)$ are identified with $\mathfrak{g}$ up to conjugation. Since the adjoint action of G on $\mathfrak{g}$ preserves its Lie algebra structure, the fibers of $\text{ad}(E_G)$ are in fact Lie algebras isomorphic to $\mathfrak{g}$. This Lie algebra structure of a fiber of $\text{ad}(E_G)$ coincides with the one constructed earlier using the Lie bracket operation of vector fields. The pulled back vector bundle $p^*\text{ad}(E_G)$ on E_G is identified with the trivial vector bundle $E_G \times \mathfrak{g}$ with fiber $\mathfrak{g}$. This identification sends any vector $(z, v) \in (p^*\text{ad}(E_G))_z$ in the fiber over z of the pulled back bundle to the element (z, v) of the trivial vector bundle $E_G \times \mathfrak{g}$.

A *connection* on E_G is a C^∞ splitting of the Atiyah exact sequence for E_G [1]. In other words, a connection on E_G is a C^∞ homomorphism of vector bundles

$$D : TM \longrightarrow \text{At}(E_G) \tag{2.12}$$

such that $\eta \circ D = \text{Id}_{TM}$, where η is the projection in (2.9).

Let

$$D : TM \longrightarrow \text{At}(E_G) \tag{2.13}$$

be a homomorphism defining a connection on E_G. Consider the composition homomorphism

$$p^*TM \xrightarrow{p^*D} p^*\text{At}(E_G) \xrightarrow{\mu} TE_G,$$

where μ is the isomorphism in (2.6). Its image

$$\mathcal{H}(D) := (\mu \circ p^*D)(p^*TM) \subset TE_G \tag{2.14}$$

is known as the *horizontal subbundle* of TE_G for the connection D. Since μ is an isomorphism, and the splitting homomorphism D in (2.13) is uniquely determined by its image $D(TM) \subset \text{At}(E_G)$, it follows immediately that the horizontal subbundle $\mathcal{H}(D)$ determines the connection D uniquely.

The composition

$$\text{kernel}(dp) \hookrightarrow TE_G \longrightarrow TE_G/\mathcal{H}(D)$$

is an isomorphism. Hence we have

$$TE_G = \mathcal{H}(D) \oplus (E_G \times \mathfrak{g});$$

it was noted earlier that $p^*\text{ad}(E_G)$ is identified with the trivial vector bundle $E_G \times \mathfrak{g}$. The projection of TE_G to the second factor of the above direct sum decomposition defines a $\mathfrak{g}$-valued smooth one-form on E_G. The connection D is clearly determined uniquely by this $\mathfrak{g}$-valued one-form on E_G.

See [4, p. 370, Lemma 2.2] for a proof of the following lemma:

Lemma 2.1. *Any principal G-bundle $E_G \longrightarrow M$ admits a connection.*

The space of all connections on a principal G-bundle E_G is an affine space for the vector space $C^\infty(M; \text{Hom}(TM, \text{ad}(E_G)))$.

3. A universal connection

3.1. A tautological connection

As before, let $p : E_G \longrightarrow M$ be a C^∞ principal G-bundle. Consider the Atiyah exact sequence in (2.10). Tensoring it with the cotangent bundle $T^*M = (TM)^*$ we get the following short exact sequence of vector bundles on M

$$0 \longrightarrow \mathrm{ad}(E_G)\otimes T^*M \longrightarrow \mathrm{At}(E_G)\otimes T^*M \stackrel{\eta\otimes \mathrm{Id}_{T^*M}}{\longrightarrow} TM\otimes T^*M =: \mathrm{End}(TM) \longrightarrow 0. \tag{3.1}$$

Let Id_{TM} denote the identity automorphism of TM. It defines a C^∞ section of the endomorphism bundle $\mathrm{End}(TM)$. Let

$$\delta : \mathcal{C}(E_G) := (\eta\otimes \mathrm{Id}_{T^*M})^{-1}(\mathrm{Id}_{TM}) \subset \mathrm{At}(E_G)\otimes T^*M \longrightarrow M \tag{3.2}$$

be the fiber bundle over M, where $\eta\otimes \mathrm{Id}_{T^*M}$ is the surjective homomorphism in (3.1).

We recall that a connection on E_G is a C^∞ splitting of the Atiyah exact sequence.

See [4, p. 371, Lemma 3.1] for a proof of the following:

Lemma 3.1. *The space of all connections on E_G is in bijective correspondence with the space of all smooth sections of the fiber bundle*

$$\delta : \mathcal{C}(E_G) \longrightarrow M$$

constructed in (3.2).

Combining Lemma 2.1 with Lemma 3.1, the following is obtained.

Corollary 3.2. *The fiber bundle δ in (3.2) is an affine bundle over M for the vector bundle* $\mathrm{Hom}(TM, \mathrm{ad}(E_G))$. *In particular, if we fix a connection on E_G (which exists by Lemma 2.1), then the fiber bundle in (3.2) gets identified with the total space of the vector bundle* $\mathrm{Hom}(TM, \mathrm{ad}(E_G))$.

See [4, p. 372, Proposition 3.3] for a proof of the following:

Proposition 3.3. *There is a tautological connection on the principal G-bundle δ^*E_G over $\mathcal{C}(E_G)$.*

The key observations in the construction of the tautological connection in Proposition 3.3 are the following:

There is a tautological homomorphism

$$\beta : \delta^*\mathrm{At}(E_G) \longrightarrow \delta^*\mathrm{ad}(E_G) = \mathrm{ad}(\delta^*E_G).$$

On the other hand, there is a tautological projection

$$\beta' : \mathrm{At}(\delta^*E_G) \longrightarrow \delta^*\mathrm{At}(E_G)$$

such that the diagram

$$\begin{array}{ccc} \mathrm{At}(\delta^* E_G) & \xrightarrow{\beta'} & \delta^* \mathrm{At}(E_G) \\ \downarrow & & \downarrow \delta^*\eta \\ T\mathcal{C}(E_G) & \xrightarrow{d\delta} & \delta^* TM \end{array}$$

where the projection $\mathrm{At}(\delta^* E_G) \longrightarrow T\mathcal{C}(E_G)$ is constructed as in (2.9) for the principal G-bundle $\delta^* E_G$. Finally, the composition

$$\beta \circ \beta' : \mathrm{At}(\delta^* E_G) \longrightarrow \mathrm{ad}(\delta^* E_G)$$

gives a splitting of the Atiyah exact sequence for $\delta^* E_G$. This splitting $\beta \circ \beta'$ defines the tautological connection on $\delta^* E_G$.

The above tautological connection on the principal G-bundle $\delta^* E_G$ will be denoted by $\mathcal{D}_0$.

In Lemma 3.1 we noted that the connections on E_G are in bijective correspondence with the smooth sections of $\mathcal{C}(E_G)$. Take any smooth section

$$\sigma : M \longrightarrow \mathcal{C}(E_G) \tag{3.3}$$

of the fiber bundle $\mathcal{C}(E_G) \longrightarrow M$. Let $D(\sigma)$ be the corresponding connection on the principal G-bundle E_G. We note that $\sigma^*\delta^* E_G = E_G$ because $\delta \circ \sigma = \mathrm{Id}_M$.

The following lemma is a consequence of the construction of the tautological connection $\mathcal{D}_0$.

Lemma 3.4. *The connection $D(\sigma)$ on E_G coincides with the pulled back connection $\sigma^*\mathcal{D}_0$ on the principal G-bundle $\sigma^*\delta^* E_G = E_G$.*

3.2. Construction of universal connection

All infinite dimensional manifolds will be modeled on the direct limit $\mathbb{R}^\infty$ of the sequence of vector spaces $\{\mathbb{R}^n\}_{n>0}$ with natural inclusions $\mathbb{R}^i \hookrightarrow \mathbb{R}^{i+1}$.

Let

$$p_0 : E_G \longrightarrow B_G \tag{3.4}$$

be a universal principal G-bundle in the C^∞ category; see [7] for the construction of a universal principal G-bundle. So, B_G is a C^∞ manifold, the projection p_0 is smooth, and E_G is contractible. Define

$$\mathcal{B}_G := B_G \times \mathbb{R}^\infty .$$

Define

$$\mathcal{E}_G := p_{B_G}^* E_G = E_G \times \mathbb{R}^\infty ,$$

where $p_{B_G} : B_G \times \mathbb{R}^\infty \longrightarrow B_G$ is the natural projection.

See [4, p. 374, Lemma 4.1] for a proof of the following:

Lemma 3.5. *The principal G-bundle*

$$p := p_0 \times \mathrm{Id}_{\mathbb{R}^\infty} : \mathcal{E}_G \longrightarrow \mathcal{B}_G$$

is universal.

Set the principal G-bundle $E_G \longrightarrow M$ in Section 3.1 to be $\mathcal{E}_G \longrightarrow \mathcal{B}_G$. Construct $\mathcal{C}(\mathcal{E}_G)$ as in (3.2). Let

$$\delta : \mathcal{C}(\mathcal{E}_G) \longrightarrow \mathcal{B}_G \tag{3.5}$$

be the natural projection (see Lemma 3.1). Let $\mathcal{D}_0$ be the tautological connection on $\delta^*\mathcal{E}_G$ constructed in Proposition 3.3.

The following theorem is proved in [4, p. 375, Lemma 4.2].

Theorem 3.6. *The connection $\mathcal{D}_0$ on the principal G-bundle $\delta^*\mathcal{E}_G$ is universal.*

In Theorem 3.6, we took a special type of universal G-bundle, namely we took the Cartesian product of a universal G-bundle with $\mathbb{R}^\infty$. It should be mentioned that Theorem 3.6 is not valid if we do not take this Cartesian product. For example, take G to be the additive group $\mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, the projection $\mathbb{R}^n \longrightarrow \{\text{point}\}$ is a universal $\mathbb{R}^n$-bundle. Note that $\mathcal{C}(\mathbb{R}^n)$ is a point. But the trivial principal $\mathbb{R}^n$ bundle on any manifold X of dimension at least two admits connections with nonzero curvature.

4. Holomorphic connections

Assume that M is a complex manifold and G is a complex Lie group. A *holomorphic* principal G-bundle on M is a triple (E_G, p, ψ) as in (2.1) such that E_G is a complex manifold, and both the maps p and ψ are holomorphic.

Let (E_G, p, ψ) be a holomorphic principal G-bundle on M. Consider the holomorphic tangent bundle $T^{1,0}E_G$, which is a holomorphic vector bundle on E_G. The real tangent bundle TE_G gets identified with $T^{1,0}E_G$ in the obvious way. More precisely, the isomorphism $T^{1,0}E_G \longrightarrow TE_G$ sends a tangent vector to its real part. Using this identification between $T^{1,0}E_G$ and TE_G, the complex structure on the total space of $T^{1,0}E_G$ produces a complex structure on the total space of TE_G. This complex structure on TE_G produces a complex structure on the quotient $\text{At}(E_G)$ in (2.4), because the action of G on TE_G is holomorphic.

The differential dp in (2.7) is holomorphic, which makes the projection η in (2.9) holomorphic. The exact sequence in (2.10) becomes an exact sequence of holomorphic vector bundles. The holomorphic structure on E_G produces a holomorphic structure on any fiber bundle associated to E_G for a holomorphic action of G. In particular, the adjoint vector bundle $\text{ad}(E_G)$ has a holomorphic structure, because the adjoint action of G on $\mathfrak{g}$ is holomorphic. The homomorphism ι_0 in (2.10) is holomorphic with respect to this holomorphic structure on $\text{ad}(E_G)$.

A connection

$$D : TM \longrightarrow \text{At}(E_G)$$

on E_G as in (2.12) is called *holomorphic* if the homomorphism D is holomorphic.

4.1. Holomorphic connection on principal bundles over a compact Riemann surface

Now take M to be a compact connected Riemann surface. It is natural to ask the question when a holomorphic vector bundle on M admits a holomorphic connection. Note that any holomorphic connection on a Riemann surface is automatically flat because there are no nonzero $(2,\, 0)$ forms on a Riemann surface. A well-known theorem of Atiyah and Weil says that a holomorphic vector bundle E over M admits a holomorphic connection if and only if each direct summand of E is of degree zero (see [1,11]). We will describe a generalization of it to principal bundles.

Let G be a complex connected reductive affine algebraic group. A parabolic subgroup of G is a Zariski closed connected subgroup $P \subset G$ such that the quotient G/P is compact. A Levi subgroup of G is a Zariski closed connected subgroup

$$L \subset G$$

such that there is a parabolic subgroup $P \subset G$ containing L that satisfies the following condition: L contains a maximal torus of P, and moreover L is a maximal reductive subgroup of P. Given a holomorphic principal G-bundle E_G on M and a complex Lie subgroup $H \subset G$, a holomorphic reduction of E_G to H is given by a holomorphic section of the holomorphic fiber bundle E_G/H over M. Let

$$q_H : E_G \longrightarrow E_G/H$$

be the quotient map. If $\nu : M \longrightarrow E_G/H$ is a holomorphic section of the fiber bundle E_G/H, then note that $q_H^{-1}(\nu(M)) \subset E_G$ is a holomorphic principal H-bundle on M. If E_H is a holomorphic principal H-bundle on M, and χ is a holomorphic character of H, then the associated holomorphic line bundle $E_H(\lambda) = (E_H \times \mathbb{C})/H$ is the quotient of $E_H \times \mathbb{C}$, where $(z_1, c_1), (z_2, c_2) \in E_H \times \mathbb{C}$ are identified if there is an element $g \in H$ such that

- $z_2 = z_1 g$, and
- $c_2 = \frac{c_1}{\lambda(g)}$.

The following theorem is proved in [2] (see [2, Theorem 4.1]).

Theorem 4.1. *A holomorphic G-bundle E_G over M admits a holomorphic connection if and only if for every triple of the form (H, E_H, λ), where*

(1) *H is a Levi subgroup of G,*
(2) *$E_H \subset E_G$ is a holomorphic reduction of structure group to H, and*
(3) *λ is a holomorphic character of H,*

the associated line bundle $E_H(\lambda) = (E_H \times \mathbb{C})/H$ over M is of degree zero.

Note that setting $G = \mathrm{GL}(n, \mathbb{C})$ in Theorem 4.1 the above mentioned criterion of Atiyah and Weil is recovered.

We will describe a sketch of the proof of Theorem 4.1.

Let E_G be a holomorphic G-bundle over M equipped with a holomorphic connection ∇. Take any triple (H, E_H, λ) as in Theorem 4.1. We will first show that the connection ∇ produces a holomorphic connection on the principal H-bundle E_H.

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. The group H has adjoint actions on both $\mathfrak{h}$ and $\mathfrak{g}$. To construct the connection on E_H, fix a splitting of the injective homomorphism of H-modules

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g}.$$

Since a holomorphic connection on E_G is given by a holomorphic splitting of the Atiyah exact sequence for E_G, a holomorphic connection ∇ on E_G produces a $\mathfrak{g}$-valued holomorphic 1-form ω on E_G satisfying the following two conditions:

- ω is G-equivariant (G acts on $\mathfrak{g}$ by inner automorphism), and
- the restriction of ω to any fiber of E_G is the Maurer–Cartan form on the fiber.

Using the chosen splitting homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 0,$$

the connection form ω on E_G defines a $\mathfrak{h}$-valued holomorphic one-form ω' on E_G. The restriction of ω' to the complex submanifold $E_H \subset E_G$ satisfies the two conditions needed for a holomorphic $\mathfrak{h}$-valued 1-form on E_H to define a holomorphic connection on E_H.

Therefore, E_H admits a holomorphic connection. A holomorphic connection on E_H induces a holomorphic connection on the associated line bundle $E_H(\lambda)$. Any line bundle admitting a holomorphic connection must be of degree zero [1]. Therefore, if E_G admits a holomorphic connection then we know that the degree of $E_H(\lambda)$ is zero.

To prove the converse, let E_G be a holomorphic G-bundle over M such that

$$\text{degree}(E_H(\lambda)) = 0$$

for all triples $(H,\, E_H,\, \lambda)$ of the above type. We need to show that the Atiyah exact sequence for E_G in (2.10) splits holomorphically.

As the first step, in [2] the following is proved: it is enough to prove that the Atiyah exact sequence for E_G splits holomorphically under the assumption that E_G does not admit any holomorphic reduction of structure group to any proper Levi subgroup of G. Therefore, we assume that E_G does not admit any holomorphic reduction of structure group to any proper Levi subgroup of G.

Let Ω^1_M denote the holomorphic cotangent bundle of M. The obstruction for splitting of the Atiyah exact sequence for E_G is an element

$$\tau(E_G) \in H^1(M,\, \Omega^1_M \otimes \text{ad}(E_G)).$$

By Serre duality,

$$H^1(M,\, \Omega^1_M \otimes \text{ad}(E_G)) = H^0(M,\, \text{ad}(E_G))^*.$$

So we have

$$\tau(E_G) \in H^1(M,\, \text{ad}(E_G))^*. \tag{4.1}$$

Any homomorphic section f of $\text{ad}(E_G)$ has a Jordan decomposition

$$f = f_s + f_n,$$

where f_s is pointwise semisimple and f_n is pointwise nilpotent. From the assumption that E_G does not admit any holomorphic reduction of structure group to any proper Levi subgroup of G it follows that the semisimple section f_s is given by some element of the center of $\mathfrak{g}$. Using this, from the assumption on E_G it can be deduced that

$$\tau(E_G)(f_s) = 0,$$

where $\tau(E_G)$ is the element in (4.1).

The nilpotent section f_n of $\text{ad}(E_G)$ gives a holomorphic reduction of structure group $E_P \subset E_G$ of E_G to a proper parabolic subgroup P of G. This reduction E_P has the property that f_n lies in the image

$$H^0(M,\, \mathrm{ad}(E_P)) \hookrightarrow H^0(M,\, \mathrm{ad}(E_G)),$$

where $\mathrm{ad}(E_P)$ is the adjoint bundle of E_P. Using this reduction it can be shown that $\tau(E_G)(f_n) = 0$.

Hence $\tau(E_G)(f) = 0$ for all f, which implies that $\tau(E_G) = 0$. Therefore, the Atiyah exact sequence for E_G splits holomorphically, implying that E_G admits a holomorphic connection.

5. Real Higgs bundles

As before, let M be a compact connected Riemann surface. Let

$$\sigma \,:\, M \longrightarrow M$$

be an anti-holomorphic automorphism of order two. Take a holomorphic vector bundle E on M of rank r. Let $\overline{E}$ denote the $C^\infty\mathbb{C}$-vector bundle on M of rank r whose underlying $C^\infty\mathbb{R}$-vector bundle is the $\mathbb{R}$-vector bundle underlying E, while the multiplication by $\sqrt{-1}$ on the fibers of $\overline{E}$ coincides with the multiplication by $-\sqrt{-1}$ on the fibers of E. We note that the pullback $\sigma^*\overline{E}$ has a natural structure of a holomorphic vector bundle. Indeed, a C^∞ section s of $\sigma^*\overline{E}$ defined over an open subset $U \subset M$ is holomorphic if the section σ^*s of E over $\sigma(U)$ is holomorphic; this condition uniquely defines the holomorphic structure on $\sigma^*\overline{E}$. We use the terminology "$\mathbb{R}$-vector bundles" because the terminology "real vector bundles" will be used for something else.

If $\alpha \,:\, A \longrightarrow B$ is a C^∞ homomorphism of holomorphic vector bundles on M, then $\overline{\alpha}$ will denote the homomorphism $\overline{A} \longrightarrow \overline{B}$ defined by α using the identifications of A and B with $\overline{A}$ and $\overline{B}$ respectively. A *real structure* on E is a holomorphic isomorphism of vector bundles

$$\phi \,:\, E \longrightarrow \sigma^*\overline{E}$$

over the identity map of M such that the composition

$$E \xrightarrow{\phi} \sigma^*\overline{E} \xrightarrow{\sigma^*\overline{\phi}} \sigma^*\overline{\sigma^*\overline{E}} = E \tag{5.1}$$

is the identity map of E.

A *quaternionic structure* on E is a holomorphic isomorphism of vector bundles

$$\phi \,:\, E \longrightarrow \sigma^*\overline{E}$$

over the identity map of M such that the composition $E \longrightarrow E$ in (5.1) is $-\mathrm{Id}_E$.

A *real vector bundle* on $(M,\, \sigma)$ is a pair of the form $(E,\, \phi)$, where E is a holomorphic vector bundle on M and ϕ is a real structure on E.

A *quaternionic vector bundle* on $(M,\, \sigma)$ is a pair of the form $(E,\, \phi)$, where E is a holomorphic vector bundle on M and ϕ is a quaternionic structure on E.

Consider the differential $d\sigma \,:\, T^{\mathbb{R}}M \longrightarrow \sigma^*T^{\mathbb{R}}M$ of the automorphism σ. Since σ is anti-holomorphic, it produces an isomorphism

$$\sigma'' \,:\, T^{1,0}M \longrightarrow \sigma^*T^{0,1}M = \sigma^*\overline{T^{1,0}M}.$$

It is easy to check that σ'' is holomorphic and it is a real structure on the holomorphic tangent bundle $T^{1,0}M$. Let

$$\sigma' : K_M := (T^{1,0}M)^* \longrightarrow \sigma^*\overline{K_M} \tag{5.2}$$

be the real structure on the holomorphic cotangent bundle K_M obtained from σ''.

We recall that a Higgs field on E is a holomorphic section of $\text{Hom}(E, E \otimes K_M) = \text{End}(E) \otimes K_M$ [6,10]. A Higgs field θ on a real or quaternionic vector bundle (E, ϕ) is called *real* if the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E \otimes K_M \\ \downarrow \phi & & \downarrow \phi \otimes \sigma' \\ \sigma^*\overline{E} & \xrightarrow{\sigma^*\overline{\theta}} & \sigma^*\overline{E \otimes K_M} = \sigma^*\overline{E} \otimes \sigma^*\overline{K_M} \end{array}$$

where σ' is the isomorphism in (5.2). A *real* (respectively, quaternionic) Higgs bundle on (M, σ) is a triple of the form $((E, \phi), \theta)$, where (E, ϕ) is a real (respectively, quaternionic) vector bundle on (M, σ) and θ is a real Higgs field on (E, ϕ).

We recall that the *slope* of a holomorphic vector bundle W on M is the rational number $\text{degree}(W)/\text{rank}(W) := \mu(W)$. A real or quaternionic Higgs bundle $((E, \phi), \theta)$ on (M, σ) is called *semistable* (respectively, *stable*) if for all nonzero holomorphic subbundles $F \subsetneq E$ with

(1) $\phi(F) \subset \sigma^*\overline{F} \subset \sigma^*\overline{E}$, and
(2) $\theta(F) \subset F \otimes K_M$,

we have $\mu(F) \leq \mu(E)$ (respectively, $\mu(F) < \mu(E)$). A semistable real (respectively, quaternionic) Higgs bundle is called *polystable* if it is a direct sum of stable real (respectively, quaternionic) Higgs bundles.

It is known that a real Higgs bundle $((E, \phi), \theta)$ is semistable (respectively, polystable) if and only if the Higgs bundle (E, θ) is semistable (respectively, polystable) [3, p. 2555, Lemma 5.3]. Similarly, a quaternionic Higgs bundle $((E, \phi), \theta)$ is semistable (respectively, polystable) if and only if the Higgs bundle (E, θ) is semistable (respectively, polystable).

A polystable Higgs vector bundle (E, θ) of degree zero on M admits a harmonic metric h that satisfies the Yang–Mills–Higgs equation [10,5,6]. If $((E, \phi), \theta)$ is real or quaternionic polystable of degree zero, then E admits a harmonic metric h because (E, θ) is polystable of degree zero. The harmonic metric h on E can be so chosen that the isomorphism ϕ is an isometry (note that h induces a Hermitian structure on $\overline{E}$) [3, p. 2557, Proposition 5.5].

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