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On asymptotic properties of Laguerre–Sobolev type orthogonal polynomials

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Abstract. In this paper, we consider the asymptotic behavior of the sequence of monic polynomials orthogonal with respect to the Sobolev inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)d\mu + Mp^{(m)}(\zeta)q^{(m)}(\zeta),$$

where $\zeta < 0$, $M \geqslant 0$ and $d\mu = e^{-x}x^\alpha dx$. We study the outer relative asymptotics of these polynomials with respect to the classical Laguerre polynomials, and we deduce a Mehler–Heine type formula and a Plancherel–Rotach type formula for the rescaled polynomials.

Keywords: Laguerre–Sobolev type orthogonal polynomials; Outer asymptotics; Mehler–Heine type formula; Plancherel–Rotach type formula

1. Introduction

Let $\mu_1, \mu_2, \ldots, \mu_k$ be k measures supported in the real line, such that

$$\int_{\mathbb{C}} |x|^{2n} d\mu_j(x) < \infty, \quad n \in \mathbb{N}, \quad j = 1, \ldots, k,$$

the support of μ_1 is infinite and μ_k is not the null measure. The elements of the sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ of polynomials orthogonal with respect to *Sobolev inner product*

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$$\langle p,q\rangle_S=\sum_{j=1}^{k}\int p^{(j)}(x)q^{(j)}(x)d\mu_j(x),$$

are called *Sobolev Type Orthogonal Polynomials*, where p and q are real polynomials. In particular, the present contribution is focused on the study of asymptotic properties of the sequence $\{S_n^{\alpha}(x)\}_{n\in\mathbb{N}}$, of *Laguerre–Sobolev type* monic polynomials, orthogonal with respect to the Sobolev-type inner product

$$\langle p,q\rangle_S=\int_0^{\infty}p(x)q(x)e^{-x}x^{\alpha}dx+Mp^{(m)}(\zeta)q^{(m)}(\zeta), \tag{1}$$

where $M\geqslant 0$, $\alpha>-1$ and $\zeta<0$. The asymptotic behavior is an important topic in the theory of orthogonal polynomials because it plays a key role in several applications in physics and engineering, (electrostatic, rational approximation, among others). We present an approach to the subject when μ_1 is the classical Laguerre measure (unbounded support), μ_2 is a Dirac mass, and $\mu_3=\cdots=\mu_k=0$. The case $\zeta=0$ has been studied extensively in [2,7], where are shown some asymptotic and analytic properties of the corresponding orthogonal polynomials, as well as interlacing properties of their zeros and a limit relation between these and the zeros of Bessel function $J_{\alpha}(x)$. The monotonicity of each individual zero in terms of the mass M can be seen in [13]. On the other hand, if the mass point ζ is located outside the support of the measure, that is, if $\zeta<0$, an analytic approach has been done in [14], where is presented a second order differential equation for these polynomials and a $2m+3$ term recurrence relation that they satisfy. In [3] some properties concerning the location and monotonicity of the zeros and their asymptotics are established when either M converges to zero or to infinity. The particular cases when $m=0$ and $m=1$, has been treated, respectively, in [5,6], where it is interesting to note that the same results concerning outer relative asymptotics are obtained, as well as the same Mehler–Heine type formula, that is, the asymptotic behavior of these polynomials is independent of m. Motivated by the above results, the structure of the paper is as follows. In Section 2 we present the basic background concerning the classical Laguerre polynomials. In Section 3 we study the outer relative asymptotic of the sequence $\{S_n^{\alpha}(x)\}_{n\in\mathbb{N}}$, in Section 4 we deduce the respective Mehler–Heine type formula and in Section 5 we give a Plancherel–Rotach type formula for the scaled polynomials.

2. Preliminaries

Let $\{L_n^{\alpha}(x)\}_{n\in\mathbb{N}}$ be the sequence of classical Laguerre monic polynomials, orthogonal with respect to the inner product

$$\langle p,q\rangle=\int_0^{\infty}p(x)q(x)d\mu,$$

where $d\mu=e^{-x}x^{\alpha}dx$, with $\alpha>-1$, and for every $n\in\mathbb{N}$. It is well known that these polynomials satisfy a three term recurrence relation, as well as their representation as ${}_1F_1$ hypergeometric function, their characterization as eigenfunction of a second order linear differential equation, and the behavior of their zeros, their electrostatic interpretation and interlacing properties, (see [1,4,10–12,16,17]). We will summarize the other properties of the classical Laguerre polynomials we will use in the sequel.

Proposition 1. *The sequence* $\left\{\widehat{L}_n^{\alpha}(x)\right\}_{n\in\mathbb{N}}$ *denotes the classical Laguerre polynomials with leading coefficients* $\frac{(-1)^n}{n!}$*, i.e. for every n*

$$\frac{(-1)^n}{n!}L_n^{\alpha}(x)=\widehat{L}_n^{\alpha}(x). \tag{2}$$

For every $n\in\mathbb{N}$

$$(1)\qquad \left(\widehat{L}_n^{\alpha}\right)^{(k)}(x)=(-1)^k\widehat{L}_{n-k}^{\alpha+k}(x), \tag{3}$$

$$(2)\qquad \widehat{L}_{n+k}^{\alpha+v}(x)=\widehat{L}_{n+k+1}^{\alpha+v}(x)-\widehat{L}_{n+k+1}^{\alpha+v-1}(x), \tag{4}$$

(3) For $z\in\mathbb{C}-[0,\infty)$

$$\frac{\widehat{L}_{n+j}^{\alpha}(z)}{\widehat{L}_{n+k}^{\beta}(z)}=(-z)^{-\alpha/2+\beta/2}n^{\alpha/2-\beta/2}\times\left(1+\frac{\sqrt{-z}}{\sqrt{n}}(j-k)\right.$$
$$\left.+\left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j-\left(\frac{\beta}{2}-\frac{1}{4}\right)k-\frac{z}{2}(j-k)^2\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})\right). \tag{5}$$

(see [6]).

(4) *(Mehler–Heine)*

$$\lim_{n\to\infty}\frac{\widehat{L}_n^{\alpha}(x/(n+k))}{n^{\alpha}}=x^{-\alpha/2}J_{\alpha}(2\sqrt{x}), \tag{6}$$

uniformly on compact subset of $\mathbb{C}$*, (see* [17, thm. 8.1.3]). $J_{\alpha}(x)$ *represents Bessel's function of the first kind defined by*

$$J_{\alpha}(x)=\sum_{j=0}^{\infty}\frac{(-1)^j(x/2)^{2j+\alpha}}{j!\Gamma(j+\alpha+1)},$$

and is known that if $\alpha>-1$, $J_{\alpha}(x)$ *has a countably infinite set of real and positive zeros, all simple except for the possible zero at the origin, (see* [18,19]).

(5) *(Plancherel–Rotach) Let* $\varphi(x)=x+\sqrt{x^2+1}$ *be the conformal mapping of* $\mathbb{C}-[-1,1]$ *onto the exterior of the unit circle. Then*

$$\lim_{n\to\infty}\frac{\widehat{L}_{n-1}^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)}=-\frac{1}{\varphi((x-2)/2)}, \tag{7}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$*, (see* [8]).

If

$$K_n(x,y)=\sum_{k=0}^{n}\frac{L_k^{\alpha}(x)L_k^{\alpha}(y)}{\left\|L_k^{\alpha}\right\|_{\alpha}^2}$$

denotes the *n-th* Kernel polynomial, then for every $n\in\mathbb{N}$, (see [4])

$$K_n(x,y) = \frac{L_{n+1}^{\alpha}(x)L_n^{\alpha}(y) - L_{n+1}^{\alpha}(y)L_n^{\alpha}(x)}{\|L_n^{\alpha}\|_{\alpha}^2(x-y)}, \tag{8}$$

called *Christoffel–Darboux formula*. As a consequence of (8), is well known the next confluent formula, (see [7])

$$K_{n-1}^{(0,m)}(x,y) = \frac{m!}{\|L_{n-1}^{\alpha}\|_{\alpha}^2(x-y)^{m+1}}\left[L_n^{\alpha}(x)T_m^{[n-1,\alpha]}(x;y) - L_{n-1}^{\alpha}(x)T_m^{[n,\alpha]}(x;y)\right], \tag{9}$$

where

$$T_m^{[i,\alpha]}(x;y) = \sum_{k=0}^{m}\frac{\left(L_i^{\alpha}\right)^{(k)}(y)}{k!}(x-y)^k \tag{10}$$

denotes the *Taylor Polynomial* of degree m, of the polynomial $L_i^{\alpha}(x)$ around $x = y$, and

$$K_n^{(i,j)}(x,y) \equiv \frac{\partial^{i+j}K_n(x,y)}{\partial x^i \partial y^j}.$$

As a consequence of (9):

$$K_n^{(m,m)}(\zeta,\zeta) = \frac{(m!)^2}{(2m+1)!\|L_n^{\alpha}\|_{\alpha}^2}\sum_{j=0}^{m}\binom{2m+1}{j}\left[\left(L_n^{\alpha}\right)^{(j)}(\zeta)\left(L_{n+1}^{\alpha}\right)^{(2m+1-j)}(\zeta) - \left(L_n^{\alpha}\right)^{(2m+1-j)}(\zeta)\left(L_{n+1}^{\alpha}\right)^{(j)}(\zeta)\right]. \tag{11}$$

3. Relative outer asymptotics

In this section we will find the relative asymptotic behavior of the Laguerre–Sobolev-type orthogonal polynomials in the exterior of the positive real semiaxis. Let $\{S_n^{\alpha}(x)\}_{n\in\mathbb{N}}$ be the sequence of orthogonal polynomials with respect to (1). By using of the classical Laguerre polynomials $\{L_n^{\alpha}(x)\}_{n\in\mathbb{N}}$ as a basis, for every $n \in \mathbb{N}$, $S_n^{\alpha}(x)$ can be written as

$$S_n^{\alpha}(x) = L_n^{\alpha}(x) + \sum_{k=0}^{n-1}a_{n,k}L_k^{\alpha}(x),$$

where for $0 \leqslant j \leqslant n-1$

$$a_{n,j} = -\frac{M(L_j)^{(m)}(\zeta)\left(S_n^{\alpha}\right)^{(m)}(\zeta)}{\left\|L_j^{\alpha}\right\|_{\alpha}^2},$$

then

$$S_n^{\alpha}(x) = L_n^{\alpha}(x) - M\left(S_n^{\alpha}\right)^{(m)}(\zeta)K_{n-1}^{(0,m)}(x,\zeta). \tag{12}$$

Remark 1. From (12), if $m \geqslant n$ then $S_n^{\alpha}(x) = L_n^{\alpha}(x)$.

In order to compute $\left(S_n^\alpha\right)^{(m)}(\zeta)$, we take the $m-th$ derivative in (12) and evaluate it at $x=\zeta$

$$\left(S_n^\alpha\right)^{(m)}(\zeta)=\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)}{1+MK_{n-1}^{(m,m)}(\zeta,\zeta)},$$

and replacing this value in (12), we have

$$S_n^\alpha(x)=L_n^\alpha(x)-M\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)K_{n-1}^{(0,m)}(x,\zeta)}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)}. \tag{13}$$

Now we are going to analyze how the quotient $\frac{S_n^\alpha(x)}{L_n^\alpha(x)}$ behaves when $n\to\infty$, and $x<0$. In connection with (9), we have that for $0\leqslant k\leqslant m-1$, and $y<0$

$$\frac{m!}{k!}\frac{\left(L_n^\alpha\right)^{(k)}(y)(x-y)^k}{\left(L_n^\alpha\right)^{(m)}(y)(x-y)^m}=\frac{m!}{k!}(x-y)^{k-m}\frac{\left(\widehat{L}_n^\alpha\right)^{(k)}(y)}{\left(\widehat{L}_n^\alpha\right)^{(m)}(y)},$$

and using (3) and (5)

$$\frac{\left(\widehat{L}_n^\alpha\right)^{(k)}(y)}{\left(\widehat{L}_n^\alpha\right)^{(m)}(y)}=(-1)^{k-m}|y|^{(m-k)/2}n^{(k-m)/2}\times\left(1+\frac{\sqrt{|y|}}{\sqrt{n}}(m-k)\right.$$
$$\left.+\left[\left(\frac{\alpha+m}{2}-\frac{1}{4}\right)m-\left(\frac{\alpha+k}{2}-\frac{1}{4}\right)k-\frac{y}{2}(m-k)^2\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})\right),$$

and given that $k-m<0$, $\frac{\left(\widehat{L}_n^\alpha\right)^{(k)}(y)}{\left(\widehat{L}_n^\alpha\right)^{(m)}(y)}\to 0$ when $n\to\infty$, as a consequence

$$T_m^{[n,\alpha]}(x;y)\sim\frac{\left(L_n^\alpha\right)^{(m)}(y)}{m!}(x-y)^m \tag{14}$$

Now, we divide by $L_n^\alpha(x)$ in (13) and thus

$$\frac{S_n^\alpha(x)}{L_n^\alpha(x)}=1-M\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)\frac{K_{n-1}^{(0,m)}(x,\zeta)}{L_n^\alpha(x)}}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)} \tag{15}$$

In order to estimate the behavior of the above expression, by using the formula (9) and (14), the numerator in the right hand side of (15) can be written as

$$\begin{aligned}\left(L_n^\alpha\right)^{(m)}(\zeta)\frac{K_{n-1}^{(0,m)}(x,\zeta)}{L_n^\alpha(x)}&=\frac{m!\left(L_n^\alpha\right)^{(m)}(\zeta)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2(x-\zeta)^{m+1}}\frac{\left[L_n^\alpha(x)T_m^{[n-1,\alpha]}(x;\zeta)-L_{n-1}^\alpha(x)T_m^{[n,\alpha]}(x;\zeta)\right]}{L_n^\alpha(x)}\\&\sim\frac{m!\left(L_n^\alpha\right)^{(m)}(\zeta)\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)(x-\zeta)^m}{\left\|L_{n-1}^\alpha\right\|_\alpha^2(x-\zeta)^{m+1}m!}\left[1-\frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)}{\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)}\right]\\&=\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2(x-\zeta)}\left[1-\frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)}{\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)}\right].\end{aligned}$$

Taking into account the normalization (2), we obtain

$$M\left(L_n^\alpha\right)^{(m)}(\zeta)\frac{K_{n-1}^{(0,m)}(x,\zeta)}{L_n^\alpha(x)}\sim -\frac{n!(n-1)!M\left(\widehat{L}_n^\alpha\right)^{(m)}(\zeta)\left(\widehat{L}_{n-1}^\alpha\right)^{(m)}(\zeta)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2(x-\zeta)}\left[1-\frac{\widehat{L}_{n-1}^\alpha(x)}{\widehat{L}_n^\alpha(x)}\frac{\left(\widehat{L}_n^\alpha\right)^{(m)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(m)}(\zeta)}\right]. \tag{16}$$

On the other hand, applying (11), we get

$$1+MK_{n-1}^{(m,m)}(\zeta,\zeta)=1+M\frac{(m!)^2}{(2m+1)!\left\|L_{n-1}^\alpha\right\|_\alpha^2}\sum_{j=0}^{m}\binom{2m+1}{j}$$
$$\times\left[\left(L_{n-1}^\alpha\right)^{(j)}(\zeta)\left(L_n^\alpha\right)^{(2m+1-j)}(\zeta)-\left(L_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta)\left(L_n^\alpha\right)^{(j)}(\zeta)\right].$$

By using (2), the sum in the above expression becomes

$$n!(n-1)!\sum_{j=0}^{m}b_{mj}\left[1-\frac{\left(\widehat{L}_{n-1}^\alpha\right)^{(j)}(\zeta)}{\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)}\frac{\left(\widehat{L}_n^\alpha\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta)}\right]\times\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta),$$

where $b_{mj}=\binom{2m+1}{j}$. Thereby, the denominator $1+MK_{n-1}^{(m,m)}(\zeta,\zeta)$ can be written as

$$1+M\frac{(m!)^2n!(n-1)!}{(2m+1)!\left\|L_{n-1}^\alpha\right\|_\alpha^2}$$
$$\times\sum_{j=0}^{m}b_{mj}\left[1-\frac{\left(\widehat{L}_{n-1}^\alpha\right)^{(j)}(\zeta)}{\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)}\frac{\left(\widehat{L}_n^\alpha\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta)}\right]\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta) \tag{17}$$

Taking the results obtained in (16) and (17) we get

$$M\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)K_{n-1}^{(0,m)}(x,\zeta)/L_n^\alpha(x)}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)}\sim\frac{-\frac{1}{(x-\zeta)}\left[1-\frac{\widehat{L}_{n-1}^\alpha(x)}{\widehat{L}_n^\alpha(x)}\frac{\left(\widehat{L}_n^\alpha\right)^{(m)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(m)}(\zeta)}\right]}{N_n+\frac{(m!)^2}{(2m+1)!}\sum_{j=0}^{m}b_{mj}\left[1-\frac{\left(\widehat{L}_{n-1}^\alpha\right)^{(j)}(\zeta)}{\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)}\frac{\left(\widehat{L}_n^\alpha\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta)}\right]\frac{\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)}{\left(\widehat{L}_n^\alpha\right)^{(m)}(\zeta)}\frac{\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(m)}(\zeta)}},$$
$$\sim\frac{-\frac{1}{(x-\zeta)}\left[1-\frac{\widehat{L}_{n-1}^\alpha(x)}{\widehat{L}_n^\alpha(x)}\frac{\left(\widehat{L}_n^\alpha\right)^{(m)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(m)}(\zeta)}\right]}{\frac{(m!)^2}{(2m+1)!}\sum_{j=0}^{m}\left(b_{mj}\left[1-\frac{\left(\widehat{L}_{n-1}^\alpha\right)^{(j)}(\zeta)}{\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)}\frac{\left(\widehat{L}_n^\alpha\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta)}\right]\frac{\left(\widehat{L}_n^\alpha\right)^{(j)}(\zeta)}{\left(\widehat{L}_n^\alpha\right)^{(m)}(\zeta)}\frac{\left(\widehat{L}_{n-1}^\alpha\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^\alpha\right)^{(m)}(\zeta)}\right)}.$$

Taking into account that

$$N_n=\frac{\left\|L_{n-1}^\alpha\right\|_\alpha^2}{Mn!(n-1)!\left(\widehat{L}_n^\alpha\right)^{(m)}(\zeta)\left(\widehat{L}_{n-1}^\alpha\right)^{(m)}(\zeta)}\to 0,$$

when $n \to \infty$ is faster than $\mathcal{O}(n^{-3/2})$, therefore we can remove it from the computations. Now we are going to estimate the rate of convergence of the ratios in the above result, outside the support of the measure. Using (5)

$$\frac{\widehat{L}_{n-1}^{\alpha}(x)}{\widehat{L}_{n}^{\alpha}(x)} = 1 - \frac{\sqrt{-x}}{\sqrt{n}} - \left(\left(\frac{\alpha}{2} - \frac{1}{4}\right) + \frac{x}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2}),$$

and applying (3)

$$\frac{\left(\widehat{L}_{n}^{\alpha}\right)^{(m)}(\zeta)}{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(m)}(\zeta)} = 1 + \frac{\sqrt{|\zeta|}}{\sqrt{n}} + \left(\left(\frac{\alpha+m}{2} - \frac{1}{4}\right) - \frac{\zeta}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2}),$$

then the term $-\frac{1}{(x-\zeta)}\left[1 - \frac{\widehat{L}_{n-1}^{\alpha}(x)}{\widehat{L}_{n}^{\alpha}(x)} \frac{\left(\widehat{L}_{n}^{\alpha}\right)^{(m)}(\zeta)}{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(m)}(\zeta)}\right]$ can be written as

$$-\frac{1}{(x-\zeta)}\left[1 - \left(1 + \frac{\sqrt{|\zeta|}}{\sqrt{n}} - \frac{\sqrt{-x}}{\sqrt{n}} - \frac{\sqrt{-x}\sqrt{|\zeta|}}{n} + \frac{m-\zeta+x}{2n} + \mathcal{O}(n^{-3/2})\right)\right]$$

$$\sim \frac{1}{(x-\zeta)} \frac{\sqrt{|\zeta|} - \sqrt{-x}}{\sqrt{n}}. \tag{18}$$

On the other hand, if $0 \leqslant j \leqslant m$, denoting $\beta = 2m + 1 - j$ and applying (5) we obtain

$$\frac{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(j)}(\zeta)}{\left(\widehat{L}_{n}^{\alpha}\right)^{(j)}(\zeta)} = 1 - \frac{\sqrt{|\zeta|}}{\sqrt{n}} - \left(\left(\frac{\alpha+j}{2} - \frac{1}{4}\right) + \frac{\zeta}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2}),$$

and

$$\frac{\left(\widehat{L}_{n}^{\alpha}\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(2m+1-j)}(\zeta)} = 1 + \frac{\sqrt{|\zeta|}}{\sqrt{n}} + \left(\left(\frac{\alpha+\beta}{2} - \frac{1}{4}\right) - \frac{\zeta}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2}).$$

Then we conclude that

$$\begin{aligned} 1 - \frac{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(j)}(\zeta)}{\left(\widehat{L}_{n}^{\alpha}\right)^{(j)}(\zeta)} \frac{\left(\widehat{L}_{n}^{\alpha}\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(2m+1-j)}(\zeta)} &= 1 - \left(1 - \frac{\sqrt{|\zeta|}}{\sqrt{n}} - \left(\left(\frac{\alpha+j}{2} - \frac{1}{4}\right) + \frac{\zeta}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \\ &\quad \times \left(1 + \frac{\sqrt{|\zeta|}}{\sqrt{n}} + \left(\left(\frac{\alpha+\beta}{2} - \frac{1}{4}\right) - \frac{\zeta}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \\ &= \left(j - m - \frac{1}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2}) \sim \left(j - m - \frac{1}{2}\right)\frac{1}{n}. \end{aligned} \tag{19}$$

Moreover

$$\begin{aligned} \frac{\left(\widehat{L}_{n}^{\alpha}\right)^{(j)}(\zeta)}{\left(\widehat{L}_{n}^{\alpha}\right)^{(m)}(\zeta)} = (-1)^{j-m}|\zeta|^{(m-j)/2} n^{(j-m)/2} \times \Bigg(1 + \frac{\sqrt{|\zeta|}}{\sqrt{n}}(m-j) \\ + \left[-\left(\frac{\alpha}{2} + \frac{j}{2} - \frac{1}{4}\right)j + \left(\frac{\alpha}{2} + \frac{m}{2} - \frac{1}{4}\right)m - \frac{\zeta}{2}(m-j)^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})\Bigg) \end{aligned}$$

and

$$\frac{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(m)}(\zeta)} = (-1)^{m+1-j}|\zeta|^{(j-1-m)/2}n^{(m+1-j)/2} \times \Bigg(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}(j-m-1) + \left[-\left(\frac{\alpha}{2}+\frac{\beta}{2}-\frac{1}{4}\right)(j-2m-2)+\left(\frac{\alpha}{2}+\frac{m}{2}-\frac{1}{4}\right)(m+1) - \frac{\zeta}{2}(j-m-1)^2\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})\Bigg)$$

thus

$$\begin{aligned}\frac{\left(\widehat{L}_{n}^{\alpha}\right)^{j)}(\zeta)}{\left(\widehat{L}_{n}^{\alpha}\right)^{(m)}(\zeta)}\frac{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(2m+1-j)}(\zeta)}{\left(\widehat{L}_{n-1}^{\alpha}\right)^{(m)}(\zeta)} &= (-1)^{j-m}|\zeta|^{\frac{(m-j)}{2}}n^{\frac{(j-m)}{2}}\times\Bigg(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}(m-j) \\ &\quad +\left[-\left(\frac{\alpha}{2}+\frac{j}{2}-\frac{1}{4}\right)j+\left(\frac{\alpha}{2}+\frac{m}{2}-\frac{1}{4}\right)m-\frac{\zeta}{2}(m-j)^2\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})\Bigg) \\ &\quad \times(-1)^{m+1-j}|\zeta|^{\frac{(j-1-m)}{2}}n^{\frac{(m+1-j)}{2}}\times\Bigg(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}(j-m-1) \\ &\quad +\left[-\left(\frac{\alpha}{2}+\frac{\beta}{2}-\frac{1}{4}\right)(j-2m-2)+\left(\frac{\alpha}{2}+\frac{m}{2}-\frac{1}{4}\right)(m+1)-\frac{\zeta}{2}(j-m-1)^2\right]\frac{1}{n} \\ &\quad +\mathcal{O}(n^{-3/2})\Bigg) = -|\zeta|^{-1/2}\sqrt{n}\left(1-\frac{\sqrt{|\zeta|}}{\sqrt{n}}+\frac{K(m,\alpha,j)}{n}+\mathcal{O}(n^{-3/2})\right) \\ &\sim -|\zeta|^{-1/2}\sqrt{n}\left(1-\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right), \end{aligned} \tag{20}$$

where $K(m,\alpha,j)$ is a term that does not depend on n. Then from (18)–(20) we get

$$M\frac{\left(L_n^{\alpha}\right)^{(m)}(\zeta)K_{n-1}^{(0,m)}(x,\zeta)/L_n^{\alpha}(x)}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)} \sim -\frac{\frac{1}{(x-\zeta)}\frac{\sqrt{|\zeta|}-\sqrt{-x}}{\sqrt{n}}}{\left(\frac{1}{\sqrt{n}\sqrt{|\zeta|}}\left(1-\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right)\right)\left[\frac{(m!)^2}{(2m+1)!}\sum_{j=0}^{m}b_{mj}\left(j-m-\frac{1}{2}\right)\right]}$$

In the next proposition, we are going to compute the sum of the above result.

Proposition 2. *For every* $m\in\mathbb{N}\cup\{0\}$

$$\frac{(m!)^2}{(2m+1)!}\sum_{j=0}^{m}\binom{2m+1}{j}\left(j-m-\frac{1}{2}\right) = -\frac{1}{2} \tag{21}$$

Proof. It can be shown that

$$\sum_{j=0}^{m}\binom{2m+1}{j}(2m+1-2j) = \binom{2m+1}{m+1}(m+1) \qquad \square \tag{22}$$

Finally, by using (21) we get

$$\frac{S_n^\alpha(x)}{L_n^\alpha(x)} = 1 - \frac{M\left(L_n^\alpha\right)^{(m)}(\zeta)\,\frac{K_{n-1}^{(0,m)}(x,\zeta)}{L_n^\alpha(x)}}{\left(1 + MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)} \sim 1 - \frac{2\sqrt{|\zeta|}\left(\sqrt{|\zeta|} - \sqrt{-x}\right)}{(x-\zeta)} = \frac{\sqrt{-x} - \sqrt{|\zeta|}}{\sqrt{-x} + \sqrt{|\zeta|}},$$

thus, we have proved the next

Theorem 3.

$$\lim_{n\to\infty} \frac{S_n^\alpha(x)}{L_n^\alpha(x)} = \frac{\sqrt{-x} - \sqrt{|\zeta|}}{\sqrt{-x} + \sqrt{|\zeta|}}, \tag{23}$$

uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$.

Note that the above result has been obtained in [5] , ($m = 0$), and in [6], ($m = 1$). It means that the sequence of analytic functions $\{\phi_n(x)\}_{n\in\mathbb{N}}$, with $\phi_n(x) = \frac{S_n^\alpha(x)}{L_n^\alpha(x)}$, converges uniformly on compact subsets of the exterior of the positive axis to $\phi(x) = \frac{\sqrt{-x}-\sqrt{|\zeta|}}{\sqrt{-x}+\sqrt{|\zeta|}}$. Thus thanks to the well known Hurwitz's theorem, [17, Thm. 1.91.3], given that $\zeta$is a simple zero of $\phi(x)$, then ζ is an accumulation point of roots of $\phi_n(x)$, that is, given that the zeros of every $S_n^\alpha(x)$ are the same as those of every $\phi_n(x)$, then ζ attracts exactly one zero of $S_n^\alpha(x)$ for n large enough. As a consequence, for n large enough, $S_n^\alpha(x)$ has a negative zero, but this fact is well known of [15].

4. Mehler–Heine type formula

In this section we will focus our attention on Mehler–Heine type formula for the polynomials $\left\{S_n^\alpha(x)\right\}_{n\in\mathbb{N}}$. Taking into account the connection formula

$$S_n^\alpha(x) = L_n^\alpha(x) - M\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)K_{n-1}^{(0,m)}(x,\zeta)}{\left(1 + MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)},$$

multiplying both hand sides by $\frac{(-1)^n}{n!}$, and applying (2), we get

$$\widehat{S}_n^\alpha(x) = \widehat{L}_n^\alpha(x) - M\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)\left[\frac{(-1)^n}{n!}K_{n-1}^{(0,m)}(x,\zeta)\right]}{\left(1 + MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)}.$$

Making the change of variable $x \to x/n$, and taking into account the results of the previous section, it is convenient to introduce the factor $\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)$ in the next way

$$\widehat{S}_n^\alpha\left(\frac{x}{n}\right) = \widehat{L}_n^\alpha\left(\frac{x}{n}\right) - M\frac{\left(L_n^\alpha\right)^{(m)}(\zeta)\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)\frac{\left[\frac{(-1)^n}{n!}\;K_{n-1}^{(0,m)}\left(\frac{x}{n},\zeta\right)\right]}{\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)}}{\left(1 + MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)}. \tag{24}$$

In order to estimate the behavior of the numerator in (24) and by using (9), we get

$$\left(L_n^{\alpha}\right)^{(m)}(\zeta)\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)\frac{\left[\frac{(-1)^n}{n!}K_{n-1}^{(0,m)}\left(\frac{x}{n},\zeta\right)\right]}{\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)}=\frac{m!\left(L_n^{\alpha}\right)^{(m)}(\zeta)\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^{2}\left(\frac{x}{n}-\zeta\right)}$$

$$\times\frac{\left[\widehat{L}_n^{\alpha}\left(\frac{x}{n}\right)T_m^{[n-1,\alpha]}\left(\frac{x}{n};\zeta\right)+\frac{\widehat{L}_{n-1}^{\alpha}\left(\frac{x}{n}\right)}{n}T_m^{[n,\alpha]}\left(\frac{x}{n};\zeta\right)\right]}{-\frac{(n-1)!}{(-1)^n}(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m},$$

Now we will analyze the behavior of the terms

$$\frac{T_m^{[n-1,\alpha]}\left(\frac{x}{n};\zeta\right)}{-\frac{(n-1)!}{(-1)^n}(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m} \tag{25}$$

and

$$\frac{\frac{1}{n}T_m^{[n,\alpha]}\left(\frac{x}{n};\zeta\right)}{-\frac{(n-1)!}{(-1)^n}(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}. \tag{26}$$

Beginning with (25), we will use (10) as follows

$$\frac{T_m^{[n-1,\alpha]}\left(\frac{x}{n};\zeta\right)}{-\frac{(n-1)!}{(-1)^n}(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}=\frac{\sum_{k=0}^{m}(-1)^k\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)}{k!}\left(\frac{x}{n}-\zeta\right)^k}{(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}.$$

Then, if $0\leqslant k\leqslant m-1$, we get

$$\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)\left(\frac{x}{n}-\zeta\right)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}=\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)}\left(\frac{x}{n}-\zeta\right)^{k-m},$$

and by using (5) as our main tool to analyze the above ratios, we get

$$\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)}=(|\zeta|)^{\frac{(m-k)}{2}}n^{\frac{k-m}{2}}\times\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}(m-k)\right.$$
$$\left.-\left[\left(\frac{(\alpha+k)}{2}-\frac{1}{4}\right)(k+1)+\left(\frac{(\alpha+m)}{2}-\frac{1}{4}\right)(m+1)-\frac{\zeta}{2}(m-k)^2\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})\right).$$

Given that $\frac{k-m}{2}<0$ and $\lim_{n\to\infty}\left(\frac{x}{n}-\zeta\right)^{k-m}=|\zeta|^{k-m}$, we have $0\leqslant k\leqslant m-1$

$$\lim_{n\to\infty}\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)\left(\frac{x}{n}-\zeta\right)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}=0.$$

When $k=m$ we obtain

$$\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)\left(\frac{x}{n}-\zeta\right)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}=\frac{1}{m!}.$$

As a consequence

$$\frac{T_m^{[n-1,\alpha]}\left(\frac{x}{n};\zeta\right)}{-\frac{(n-1)!}{(-1)^n}(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}\sim\frac{1}{m!}. \tag{27}$$

On the other hand, (26) can be written as

$$-\frac{\sum_{k=0}^{m}\frac{(-1)^{(k)}\widehat{L}_{n-k}^{\alpha+k}(\zeta)}{k!}\left(\frac{x}{n}-\zeta\right)^k}{(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}$$

then if $0\leqslant k\leqslant m-1$, using (5) we get

$$\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-k}^{\alpha+k}(\zeta)\left(\frac{x}{n}-\zeta\right)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}=(|\zeta|)^{\frac{m-k}{2}}n^{\frac{k-m}{2}}\times\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}((m+1)-k)\right.$$
$$\left.-\left[\left(\frac{(\alpha+k)}{2}-\frac{1}{4}\right)k-\left(\frac{(\alpha+m)}{2}-\frac{1}{4}\right)(m+1)+\frac{\zeta}{2}((m+1)-k)^2\frac{1}{n}+\right]+\mathcal{O}(n^{-3/2})\right).$$

Given that $k-m<0$ and taking into account that $\lim_{n\to\infty}\left(\frac{x}{n}-\zeta\right)^{k-m}=|\zeta|^{k-m}$, we have

$$\lim_{n\to\infty}\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-k}^{\alpha+k}(\zeta)\left(\frac{x}{n}-\zeta\right)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}=0.$$

In the case where $k=m$, we obtain

$$\frac{1}{m!}\frac{\widehat{L}_{n-m}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}=\frac{1}{m!}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}+\left[\left(\frac{(\alpha+m)}{2}-\frac{1}{4}\right)-\frac{\zeta}{2}\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})\right)$$
$$\sim\frac{1}{m!}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right).$$

Now taking into account the above process, we conclude that

$$\frac{\frac{1}{n}T_m^{[n,\alpha]}\left(\frac{x}{n};\zeta\right)}{-\frac{(n-1)!}{(-1)^n}(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}\sim-\frac{1}{m!}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right). \tag{28}$$

By using (27), (28) and (6) and knowing that $\frac{\widehat{L}_{n-1}^{\alpha}\left(\frac{x}{n}\right)}{n^\alpha}\sim\frac{\widehat{L}_{n}^{\alpha}\left(\frac{x}{n}\right)}{n^\alpha}$, if the expression (24) is divided by n^α we have

$$\frac{\widehat{S}_n^\alpha\left(\frac{x}{n}\right)}{n^\alpha}=\frac{\widehat{L}_n^\alpha\left(\frac{x}{n}\right)}{n^\alpha}-\frac{M\frac{m!\left(L_n^\alpha\right)^{(m)}(\zeta)\left(L_{n-1}^\alpha\right)^{(m)}(\zeta)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2\left(\frac{x}{n}-\zeta\right)}\frac{\left[\frac{\widehat{L}_n^\alpha\left(\frac{x}{n}\right)}{n^\alpha}T_m^{[n-1,\alpha]}\left(\frac{x}{n};\zeta\right)+\frac{\widehat{L}_{n-1}^\alpha\left(\frac{x}{n}\right)}{n^\alpha}\frac{1}{n}T_m^{[n,\alpha]}\left(\frac{x}{n};\zeta\right)\right]}{-\frac{(n-1)!}{(-1)^n}(-1)^m\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)}}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)}$$
$$\sim x^{-\alpha/2}J_\alpha(2\sqrt{x})-\frac{\frac{2}{|\zeta|}\left[\frac{\sqrt{|\zeta|}}{\sqrt{n}}x^{-\alpha/2}J_\alpha(2\sqrt{x})\right]}{\left(\frac{1}{\sqrt{n}\sqrt{|\zeta|}}\right)}=-x^{-\alpha/2}J_\alpha(2\sqrt{x}).$$

We summarize the last result in the next

Theorem 4.

$$\lim_{n\to\infty}\frac{\widehat{S}_n^{\alpha}\left(\frac{x}{n}\right)}{n^{\alpha}} = -x^{-\alpha/2}J_{\alpha}(2\sqrt{x}), \tag{29}$$

uniformly on every compact subset of $\mathbb{C}$.

Note that the same result is obtained in [5,6], for the particular cases $m = 0$ and $m = 1$. On the other hand, given that 0 is a zero of $\psi(x) = -x^{-\alpha/2}J_{\alpha}(2\sqrt{x})$, then for Hurwit's theorem, for n large enough, it attracts at least one zero of the polynomial $S_n^{\alpha}(x)$. Let $\{\eta_{n,k}\}_{k=1}^{n}$ be the zeros in the increasing order of this one, then if $j_{\alpha,i}$ represents the $i-th$ positive zero of the Bessel function $J_{\alpha}(x)$, again by Hurwitz's theorem, the number $j_{\alpha,i}^2/4$ attracts to $n\eta_{n,i}^*$, for n large enough, where $\eta_{n,i}^*$ represents the $i-th$ positive zero of the polynomial $S_n^{\alpha}(x)$.

5. Plancherel–Rotach type formula

Our next purpose is to determine the Plancherel–Rotach type formula of the Laguerre–Sobolev-type for the scaled polynomials $\left\{S_n^{\alpha}(nx)\right\}_{n\in\mathbb{N}}$. Taking into account the formula

$$S_n^{\alpha}(x) = L_n^{\alpha}(x) - \frac{Mm!}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^2 (x-\zeta)^{m+1}} \times \frac{\left(L_n^{\alpha}\right)^{(m)}(\zeta)\left[L_n^{\alpha}(x)T_m^{[n-1,\alpha]}(x;\zeta) - L_{n-1}^{\alpha}(x)T_m^{[n,\alpha]}(x;\zeta)\right]}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)},$$

multiplying both sides by $\frac{(-1)}{n!}$ we get

$$\widehat{S}_n^{\alpha}(x) = \widehat{L}_n^{\alpha}(x) - \frac{Mm!}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^2 (x-\zeta)^{m+1}} \times \frac{\left(L_n^{\alpha}\right)^{(m)}(\zeta)\left[\widehat{L}_n^{\alpha}(x)T_m^{[n-1,\alpha]}(x;\zeta) + \widehat{L}_{n-1}^{\alpha}(x)\frac{T_m^{[n,\alpha]}(x;\zeta)}{n}\right]}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)},$$

and scaling the variable $x \to nx$, divide by $\widehat{L}_n^{\alpha}(nx)$, and introducing the factor $\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)$, we have

$$\frac{\widehat{S}_n^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} = 1 - \frac{Mm!}{\left\|L_{n-1}^{\alpha}\right\|_{\alpha}^2 (nx-\zeta)^{m+1}} \times \frac{\left(L_n^{\alpha}\right)^{(m)}(\zeta)\frac{\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)}{\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)}\left(T_m^{[n-1,\alpha]}(nx;\zeta) + \frac{\widehat{L}_{n-1}^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)}\frac{T_m^{[n,\alpha]}(nx;\zeta)}{n}\right)}{\left(1+MK_{n-1}^{(m,m)}(\zeta,\zeta)\right)}. \tag{30}$$

We will analyze the behavior of the terms

$$\frac{T_m^{[n-1,\alpha]}(nx;\zeta)}{\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)}, \tag{31}$$

and

$$\frac{T_m^{[n,\alpha]}(nx;\zeta)}{n\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)}.$$

As before, beginning with (31), when $k \leqslant m-1$

$$\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)(nx-\zeta)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)(nx-\zeta)^m} = \frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)}n^{k-m}\left(x-\frac{\zeta}{n}\right)^{k-m} \to 0,$$

when $n \to \infty$. When $k = m$ the quotient $\frac{(-1)^{k-m}}{k!}\frac{\widehat{L}_{n-(1+k)}^{\alpha+k}(\zeta)(nx-\zeta)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)(nx-\zeta)^m}$ becomes $\frac{1}{m!}$ and then as a consequence

$$\frac{T_m^{[n-1,\alpha]}(nx;\zeta)}{\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)} \sim \frac{1}{m!}. \tag{32}$$

On the other hand, if $0 \leqslant k \leqslant m-1$,

$$\frac{(-1)^{k-m}}{k!}\lim_{n\to\infty}\frac{\widehat{L}_{n-k}^{\alpha+k}(\zeta)\left(\frac{x}{n}-\zeta\right)^k}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m} = 0,$$

and if $k = m$

$$\frac{1}{m!}\frac{\widehat{L}_{n-m}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m}{\widehat{L}_{n-(1+m)}^{\alpha+m}(\zeta)\left(\frac{x}{n}-\zeta\right)^m} \sim \frac{1}{m!}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right),$$

and thus

$$\frac{T_m^{[n,\alpha]}(nx;\zeta)}{n\left(L_{n-1}^{\alpha}\right)^{(m)}(\zeta)} \sim -\frac{1}{m!}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right) \tag{33}$$

Then, by using (32), (33) and (7) we have

$$\begin{aligned}\frac{\widehat{S}_n^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} &\sim 1 - \frac{-\frac{m!}{(nx-\zeta)}\left(\frac{1}{m!}+\frac{1}{\varphi((x-2)/2)}\frac{1}{m!}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right)\right)}{\frac{1}{2\sqrt{n}\sqrt{|\zeta|}}} \\ &= 1 + \frac{2\sqrt{n}\sqrt{|\zeta|}}{(nx-\zeta)}\left(1+\frac{1}{\varphi((x-2)/2)}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right)\right).\end{aligned}$$

Note that $\left|\frac{\sqrt{n}}{(nx-\zeta)}\right| = \left|\frac{\sqrt{n}}{(nx+|\zeta|)}\right| \leqslant \frac{1}{|x|\sqrt{n}}$, so that

$$\frac{2\sqrt{n}\sqrt{|\zeta|}}{(nx-\zeta)}\left(1+\frac{1}{\varphi((x-2)/2)}\left(1+\frac{\sqrt{|\zeta|}}{\sqrt{n}}\right)\right) \to 0,$$

when $n \to \infty$. We summarize the main result of this section in the next

Theorem 5.

$$\lim_{n\to\infty} \frac{\widehat{S}_n^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} = 1. \tag{34}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,4]$

Note that the value of m does not modify the result obtained in [6], in the case $m = 1$. The same result is given for $m = 0$ in [9].

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