ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 23 (2017) 109–123

On a type of almost Kenmotsu manifolds with nullity distributions

U.C. DE*, KRISHANU MANDAL

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol-700019, West Bengal, India

Received 8 December 2015; accepted 11 April 2016
Available online 27 April 2016

Abstract. The object of the present paper is to characterize Weyl semisymmetric almost Kenmotsu manifolds with its characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution respectively. Also we characterize almost Kenmotsu manifolds satisfying the curvature condition $C \cdot S = 0$, where C and S are the Weyl conformal curvature tensor and Ricci tensor respectively with its characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution. As a consequence of the main results we obtain several corollaries. Finally, we present an example to verify our results.

2010 Mathematics Subject Classification: 53C25; 53C35

Keywords: Almost Kenmotsu manifolds; Nullity distribution; Weyl semisymmetric; Conformally flat

1. INTRODUCTION

In the present time the study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of k-nullity distribution ($k \in \mathbb{R}$) was introduced by Gray [7] and Tanno [11] in the study of Riemannian manifolds (M, g), which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.1}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$.

* Corresponding author.
E-mail addresses: uc_de@yahoo.com (U.C. De), krishanu.mandal013@gmail.com (K. Mandal).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2016.04.001

Recently Blair, Koufogiorgos and Papantoniou [3] introduced a generalized notion of the k-nullity distribution named the (k,μ)-nullity distribution on a contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p\in M^{2n+1}$ and $k,\mu\in\mathbb{R}$ as follows:

$$N_p(k,\mu)=\{Z\in T_pM^{2n+1}:R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y] \\ +\mu[g(Y,Z)hX-g(X,Z)hY]\}, \quad (1.2)$$

where $h=\frac{1}{2}\pounds_{\xi}\phi$ and $\pounds$ denotes the Lie differentiation.

In [4], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, another generalized notion of the k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p\in M^{2n+1}$ and $k,\mu\in\mathbb{R}$ as follows:

$$N_p(k,\mu)'=\{Z\in T_pM^{2n+1}:R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y] \\ +\mu[g(Y,Z)h'X-g(X,Z)h'Y]\}, (1.3)$$

where $h'=h\circ\phi$.

On the other hand, Kenmotsu [9] introduced a new type of contact metric manifolds named Kenmotsu manifolds nowadays. Let us consider M^{2n+1} be an almost contact metric manifold with almost contact structure (ϕ,ξ,η,g) given by a $(1,1)$ tensor field ϕ, a characteristic vector field ξ, a 1-form η and a compatible metric g satisfying the conditions [1,2]

$$\phi^2=-I+\eta\otimes\xi,\qquad \phi(\xi)=0,\qquad \eta(\xi)=1,\qquad \eta\circ\phi=0,$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),$$

for any vector fields X and Y of T_pM^{2n+1}. The fundamental 2-form Φ is defined by $\Phi(X,Y)=g(X,\phi Y)$ for any vector fields X and Y of T_pM^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi=[\phi,\phi]+2d\eta\otimes\xi$, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ [1]. A normal almost Kenmotsu manifold is a Kenmotsu manifold such that $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$. Also Kenmotsu manifolds can be characterized by $(\nabla_X\phi)Y=g(\phi X,Y)\xi-\eta(Y)\phi X$, for any vector fields X,Y. It is well known [9] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I\times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f=ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D}=Ker(\eta)=Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

A Riemannian manifold (M^{2n+1},g) is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R=0$, where ∇ is the Levi-Civita connection. The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by $R(X,Y)\cdot R=0$, where $R(X,Y)$ acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabó in [10]. A Riemannian manifold is said to be Weyl semisymmetric if the Weyl conformal curvature tensor C satisfies $R\cdot C=0$. In a recent paper [8] Jun, De and Pathak studied Weyl semisymmetric Kenmotsu manifolds.

In [5], Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds. Moreover almost Kenmotsu manifolds satisfying some nullity conditions were also investigated by Dileo and Pastore [4]. We refer the reader to [5,4,6] for more related results on $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution on almost Kenmotsu manifolds. In recent papers [12–15] Wang and Liu study almost Kenmotsu manifolds with nullity distributions.

In [13], Wang and Liu study ξ-Riemannian semisymmetric almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution.

The paper is organized as follows:

In Section 2, we give a brief account on almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution and ξ belonging to the $(k,\mu)'$-nullity distribution. Section 3 deals with Weyl semisymmetric almost Kenmotsu manifolds and almost Kenmotsu manifolds satisfying the curvature condition $C \cdot S = 0$ with characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution. Section 4 is devoted to study conformally flat almost Kenmotsu manifolds and Weyl semisymmetric almost Kenmotsu manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution. As a consequence of the main results we obtain several corollaries. In the final section, we present an example to verify our results.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost Kenmotsu manifold. We denote $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The two $(1,1)$-type tensors l and h are symmetric and satisfy [4]

$$h\xi = 0, \qquad l\xi = 0, \qquad tr(h) = 0, \qquad tr(h\phi) = 0, \qquad h\phi + \phi h = 0. \tag{2.1}$$

Besides the above we have the following results [4]

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.2}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.3}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.4}$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h\circ\phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that

$$h = 0 \quad \Leftrightarrow \quad h' = 0, \qquad {h'}^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.5}$$

3. ξ belongs to the $(k,\mu)'$-nullity distribution

In this section we consider an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue λ. Then from (2.5) it is clear that $\lambda^2 = -(k+1)$, a constant. Hence $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote the eigenspaces associated with h' by $[\lambda]'$ and $[-\lambda]'$ corresponding to the non-zero eigenvalues λ and $-\lambda$ of h' respectively. To prove our main theorem in this section we recall some results:

Lemma 3.1 (*Prop. 4.1 and Prop. 4.3 of [4]*). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigenvalue and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature is given as follows:*

(a) $K(X,\xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X,\xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$,
(b) $K(X,Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,
(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k-2n)$.

Lemma 3.2 (*Lemma 3 of [14]*). *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost Kenmotsu manifold with* ξ *belonging to the* $(k,\mu)'$*-nullity distribution. If* $h' \neq 0$, *then the Ricci operator* Q *of* M^{2n+1} *is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.1}$$

Moreover, the scalar curvature of M^{2n+1} *is* $2n(k-2n)$.

Lemma 3.3 (*Proposition 4.2 of [4]*). *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost Kenmotsu manifold such that* $h' \neq 0$ *and* ξ *belongs to the* $(k,-2)'$*-nullity distribution. Then for any* $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ *and* $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, *the Riemannian curvature tensor satisfies:*

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0,\\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0,\\
R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},\\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,\\
R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

From (1.3) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{3.2}$$

where $k, \mu \in \mathbb{R}$. Also we get from (3.2)

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{3.3}$$

Contracting Y in (3.2) we have

$$S(X,\xi) = 2nk\eta(X). \tag{3.4}$$

The Weyl conformal curvature tensor C on a $(2n+1)$-dimensional manifold is defined by [16]

$$\begin{aligned}
C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX\\
- g(X,Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\},
\end{aligned} \tag{3.5}$$

where X, Y, Z are any vector fields, S is the Ricci tensor of type $(0,2)$ and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$. Using the results (3.1)–(3.4) one can easily obtain the following:

$$C(\xi, Y)Z = \left(\mu + \frac{2n}{2n-1}\right)\{g(h'Y, Z)\xi - \eta(Z)h'Y\}, \tag{3.6}$$

$$C(X, Y)\xi = \left(\mu + \frac{2n}{2n-1}\right)\{\eta(Y)h'X - \eta(X)h'Y\}. \tag{3.7}$$

Now we are in a position to prove our main theorem.

Theorem 3.1. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ *be an almost Kenmotsu manifold with* ξ *belonging to the* $(k, \mu)'$*-nullity distribution and* $h' \neq 0$*. If the manifold* M^{2n+1} *is Weyl semisymmetric then* M^{2n+1} *is locally isometric to the Riemannian product of an* $(n+1)$*-dimensional manifold of constant sectional curvature* -4 *and a flat* n*-dimensional manifold.*

Proof. We suppose that the manifold is conformally semisymmetric, that is, $R \cdot C = 0$. Then $(R(X, Y) \cdot C)(U, V)W = 0$ for all vector fields X, Y, U, V, W, which implies

$$\begin{aligned} &R(X, Y)C(U, V)W - C(R(X, Y)U, V)W - C(U, R(X, Y)V)W \\ &\quad - C(U, V)R(X, Y)W = 0. \end{aligned} \tag{3.8}$$

Substituting $X = U = \xi$ in (3.8) we have,

$$\begin{aligned} &R(\xi, Y)C(\xi, V)W - C(R(\xi, Y)\xi, V)W \\ &\quad - C(\xi, R(\xi, Y)V)W - C(\xi, V)R(\xi, Y)W = 0. \end{aligned} \tag{3.9}$$

Making use of (3.3) and (3.6) we get

$$\begin{aligned} &R(\xi, Y)C(\xi, V)W \\ &\quad = k[g(Y, C(\xi, V)W)\xi - \eta(C(\xi, V)W)Y] \\ &\qquad + \mu[g(h'Y, C(\xi, V)W)\xi - \eta(C(\xi, V)W)h'Y] \\ &\quad = k\left(\mu + \frac{2n}{2n-1}\right)\{g(h'V, W)\eta(Y)\xi - g(h'V, W)Y - \eta(W)g(Y, h'V)\xi\} \\ &\qquad - \mu\left(\mu + \frac{2n}{2n-1}\right)\{g(h'Y, h'V)\eta(W)\xi + g(h'V, W)h'Y\}, \end{aligned} \tag{3.10}$$

for any vector fields Y, V, W on M^{2n+1}.

Similarly, it follows from (3.3) and (3.6) that

$$\begin{aligned} &C(R(\xi, Y)\xi, V)W \\ &\quad = k\eta(Y)C(\xi, V)W - kC(Y, V)W - \mu C(h'Y, V)W \\ &\quad = k\left(\mu + \frac{2n}{2n-1}\right)\{g(h'V, W)\eta(Y)\xi - \eta(W)\eta(Y)h'V\} \\ &\qquad - kC(Y, V)W - \mu C(h'Y, V)W, \end{aligned} \tag{3.11}$$

for any vector fields Y, V, W on M^{2n+1}.

With the help of (3.3) and (3.6) we obtain

$$
\begin{aligned}
&C(\xi, R(\xi, Y)V)W \\
&\quad = kg(Y, V)C(\xi, \xi)W - k\eta(V)C(\xi, Y)W \\
&\qquad + \mu g(h'Y, V)C(\xi, \xi)W - \mu\eta(V)C(\xi, h'Y)W \\
&\quad = -k\left(\mu + \frac{2n}{2n-1}\right)\{g(h'Y, W)\eta(V)\xi - \eta(W)\eta(V)h'Y\} \\
&\qquad + \mu(k+1)\left(\mu + \frac{2n}{2n-1}\right)\{g(Y, W)\eta(V)\xi - \eta(W)\eta(V)Y\}, \qquad (3.12)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

Again using (3.3), (3.6) and (3.7) we have

$$
\begin{aligned}
&C(\xi, V)R(\xi, Y)W \\
&\quad = kg(Y, W)C(\xi, V)\xi - k\eta(W)C(\xi, V)Y \\
&\qquad + \mu g(h'Y, W)C(\xi, V)\xi - \mu\eta(W)C(\xi, V)h'Y \\
&\quad = -\left(\mu + \frac{2n}{2n-1}\right)\{kg(Y, W)h'V + \mu g(h'Y, W)h'V + \mu g(h'V, h'Y)\eta(W)\xi\} \\
&\qquad - k\left(\mu + \frac{2n}{2n-1}\right)\{g(h'V, Y)\eta(W)\xi - \eta(Y)\eta(W)h'V\}, \qquad (3.13)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

Finally, substituting (3.10)–(3.13) in (3.9) yields

$$
\begin{aligned}
&kC(Y, V)W + \mu C(h'Y, V)W + \left(\mu + \frac{2n}{2n-1}\right)\{-kg(h'V, W)Y \\
&\quad - \mu g(h'V, W)h'Y + kg(h'Y, W)\eta(V)\xi - k\eta(V)\eta(W)h'Y \\
&\quad - \mu(k+1)g(Y, W)\eta(V)\xi + \mu(k+1)\eta(V)\eta(W)Y \\
&\quad + kg(Y, W)h'V + \mu g(h'Y, W)h'V\} = 0, \qquad (3.14)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

Substituting $Y = h'Y$ in (3.14) and using the fact ${h'}^2 = (k+1)\phi^2$ of (2.5) we get

$$
\begin{aligned}
&kC(h'Y, V)W - \mu(k+1)C(Y, V)W + \left(\mu + \frac{2n}{2n-1}\right)\{-kg(h'V, W)h'Y \\
&\quad + \mu(k+1)g(h'V, W)Y - k(k+1)g(Y, W)\eta(V)\xi + k(k+1)\eta(V)\eta(W)Y \\
&\quad - \mu(k+1)g(h'Y, W)\eta(V)\xi + \mu(k+1)\eta(V)\eta(W)h'Y \\
&\quad + kg(h'Y, W)h'V - \mu(k+1)g(Y, W)h'V\} = 0, \qquad (3.15)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

Subtracting μ multiple of (3.15) from k multiple of (3.14) implies

$$
\begin{aligned}
&(k+2)^2 C(Y, V)W + (k+2)^2\left(\mu + \frac{2n}{2n-1}\right)\{g(h'Y, W)\eta(V)\xi \\
&\quad - \eta(V)\eta(W)h'Y - g(h'V, W)Y + g(Y, W)h'V\} = 0, \qquad (3.16)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}. In [4], Dileo and Pastore proved that if ξ belongs to the $(k,\mu)'$-nullity distribution then $\mu = -2$. Using this result, Lemmas 3.2 and 3.3 and letting $Y, V, W \in [-\lambda]'$ we have

$$C(Y,V)W = \frac{2nk - 2\lambda + 2n}{2n-1}\{g(V,W)Y - g(Y,W)V\}, \tag{3.17}$$

for any vector fields Y, V, W on M^{2n+1}.

With the help of (3.17) and noticing $Y, V, W \in [-\lambda]'$ we obtain from (3.16)

$$(k+2)^2(k+1-\lambda)\{g(V,W)Y - g(Y,W)V\} = 0. \tag{3.18}$$

Making use of (3.18) and the fact $\lambda = \pm\sqrt{-k-1}$ yields

$$\lambda(\lambda-1)^2(\lambda+1)^3 = 0. \tag{3.19}$$

Since $h' \neq 0$, (2.5) implies $k \neq -1$ and hence $\lambda \neq 0$. Then it follows from (3.19) that $\lambda^2 = 1$ and consequently $k = -2$. Without loss of any generality we may choose $\lambda = 1$. Then we have from Lemma 3.3

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_\lambda &= -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= 0, \end{aligned}$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$ it follows from Lemma 3.1 that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1 we see that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [4] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof of our theorem. □

Since conformally symmetric manifold $(\nabla C = 0)$ implies $R \cdot C = 0$, therefore from Theorem 3.1 we can state the following:

Corollary 3.1. *A conformally symmetric almost Kenmotsu manifold $M^{2n+1}(n > 1)$ with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Since $R \cdot R = 0$ implies $R \cdot C = 0$, we obtain the following:

Corollary 3.2. *A semisymmetric almost Kenmotsu manifold $M^{2n+1}(n > 1)$ with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

The above corollary has been proved by Wang and Liu [13]. Obviously, Theorem 3.1 generalizes the theorem of Wang and Liu [13].

Next we consider an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ satisfying the curvature condition $C \cdot S = 0$. Then $(C(X,Y) \cdot S)(U,V) = 0$ for all vector fields X, Y, U, V, which implies

$$S(C(X,Y)U,V) + S(U,C(X,Y)V) = 0, \tag{3.20}$$

for any vector fields X, Y, U, V on M^{2n+1}.

Substituting $X = U = \xi$ in (3.20) we have,

$$S(C(\xi,Y)\xi,V) + S(\xi,C(\xi,Y)V) = 0. \tag{3.21}$$

Making use of (3.4) and (3.6) we get from (3.21)

$$\left(\mu + \frac{2n}{2n+1}\right)\{S(h'Y,V) - 2nkg(h'Y,V)\} = 0, \tag{3.22}$$

for any vector fields Y, V on M^{2n+1}. Since ξ belongs to the $(k,\mu)'$-nullity distribution, therefore $\mu = -2$ [4].

Hence for $2n+1 \geq 5$,

$$S(h'Y,V) = 2nkg(h'Y,V), \tag{3.23}$$

for any vector fields Y, V on M^{2n+1}.

Replacing Y by $h'Y$ in (3.23) and using (2.5) yields

$$(k+1)\{S(Y,V) - 2nkg(Y,V)\} = 0, \tag{3.24}$$

for any vector fields Y, V on M^{2n+1}.

Suppose $k+1 = 0$, that is, $k = -1$. Dileo and Pastore [4] prove that in an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution if $k = -1$, then $h' = 0$ and the manifold M^{2n+1} is locally a warped product of an almost Kähler manifold and an open interval. Thus $k+1 = 0$ contradicts our hypothesis $h' \neq 0$. Therefore $S(V,Y) = 2nkg(V,Y)$, for any vector fields V, Y on M^{2n+1}. Thus the manifold is an Einstein manifold.

Conversely, if the manifold under consideration is an Einstein manifold, then from (3.20) it follows that $C \cdot S = 0$ holds identically.

By the above discussions we can state the following:

Theorem 3.2. *An almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, $n > 1$, with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ satisfies the curvature condition $C \cdot S = 0$ if and only if the manifold is an Einstein manifold.*

4. ξ BELONGS TO THE (k,μ)-NULLITY DISTRIBUTION

In this section we study the curvature properties $C = 0$ and $R \cdot C = 0$ on an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution, where C and R are the conformal curvature tensor and Riemannian curvature tensor respectively.

From (1.2) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{4.1}$$

where $k, \mu \in \mathbb{R}$.

Now we state the following:

Lemma 4.1 (*Theorem 4.1 of [4]*)**.** *Let M^{2n+1} be an almost Kenmotsu manifold of dimension $2n+1$. Suppose the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Then $k = -1,\ h = 0$ and M^{2n+1} is locally a warped product of an open interval and an almost Kähler manifold.*

From (4.1) and Lemma 4.1 we have the following:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.2}$$
$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{4.3}$$
$$S(X,\xi) = -2n\eta(X), \tag{4.4}$$

for any vector fields X, Y on M^{2n+1}. Moreover, we have the following:

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{4.5}$$

for any vector fields X, Y on M^{2n+1}.

Let us consider the manifold M^{2n+1} be conformally flat, that is,

$$C(X,Y)Z = 0, \tag{4.6}$$

for any vector fields X, Y, Z on M^{2n+1}.

From (3.5) and (4.6) we have

$$\begin{aligned} R(X,Y)Z = {} & \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & - g(X,Z)QY\} - \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{4.7}$$

Substituting $Z = \xi$ in (4.7) and using (4.2), (4.4) yields

$$\eta(Y)QX - \eta(X)QY = \left(1 + \frac{r}{2n}\right)\{\eta(Y)X - \eta(X)Y\}. \tag{4.8}$$

Putting $Y = \xi$ in (4.8) and using (4.4) we obtain

$$QX = \left(1 + \frac{r}{2n}\right)X - \left(1 + 2n + \frac{r}{2n}\right)\eta(X)\xi. \tag{4.9}$$

Taking inner product of (4.9) with Y we have

$$S(X,Y) = \left(1 + \frac{r}{2n}\right)g(X,Y) - \left(1 + 2n + \frac{r}{2n}\right)\eta(X)\eta(Y). \tag{4.10}$$

Now substituting the values of QX and $S(X,Y)$ in the expression of the conformal curvature tensor and considering the hypothesis $C(X,Y)Z = 0$, we get

$$\begin{aligned} R(X,Y)Z = {} & \left(\frac{r+4n}{2n(2n-1)}\right)\{g(Y,Z)X - g(X,Z)Y\} \\ & - \frac{1}{2n-1}\left(1+2n+\frac{r}{2n}\right)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ & + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{4.11}$$

In [4], Dileo and Pastore prove that in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the sectional curvature $K(X,\xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$.

Thus from (4.11) we obtain

$$R(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\},$$

which implies that the manifold is of constant curvature -1.

Conversely, if the manifold M^{2n+1} is of constant curvature -1, then it can be easily shown that the manifold under consideration is conformally flat.

Hence we can state the following:

Proposition 4.1. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution. Then M^{2n+1} is conformally flat if and only if the manifold is of constant curvature -1.*

Using (4.3), (4.4) and (3.5) one can easily verify the following:

$$\begin{aligned} C(\xi,Y)Z = {} & \left(\frac{r+2n}{2n(2n-1)}\right)\{g(Y,Z)\xi - \eta(Z)Y\} \\ & - \frac{1}{2n-1}\{S(Y,Z)\xi - \eta(Z)QY\}, \end{aligned} \tag{4.12}$$

for any vector field Y, Z on M^{2n+1}.

Now we prove the following:

Proposition 4.2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution. Then M^{2n+1} is Weyl semisymmetric if and only if the manifold is conformally flat.*

Proof. Let M^{2n+1} be a Weyl semisymmetric almost Kenmotsu manifold with ξ belongs to the (k,μ)-nullity distribution. Therefore $(R(X,Y)\cdot C)(U,V)W = 0$ for all vector fields X, Y, U, V, W, which implies

$$\begin{aligned} & R(X,Y)C(U,V)W - C(R(X,Y)U,V)W \\ & - C(U,R(X,Y)V)W - C(U,V)R(X,Y)W = 0. \end{aligned} \tag{4.13}$$

Substituting $X = U = \xi$ in (4.13) we have,

$$\begin{aligned} & R(\xi,Y)C(\xi,V)W - C(R(\xi,Y)\xi,V)W \\ & - C(\xi,R(\xi,Y)V)W - C(\xi,V)R(\xi,Y)W = 0. \end{aligned} \tag{4.14}$$

Making use of (4.3) and (4.12) we get

$$\begin{aligned} R(\xi,Y)C(\xi,V)W &= \left(\frac{r+2n}{2n(2n-1)}\right)\{g(V,Y)\eta(W)\xi - g(V,W)\eta(Y)\xi \\ &\quad + g(V,W)Y - \eta(W)\eta(V)Y\} \\ &\quad + \frac{1}{2n-1}\{S(V,W)\eta(Y)\xi - S(V,Y)\eta(W)\xi \\ &\quad - S(V,W)Y - 2n\eta(W)\eta(V)Y\}, \end{aligned} \tag{4.15}$$

for any vector field Y, V, W on M^{2n+1}.

Again using (4.3) and (4.12) we obtain

$$\begin{aligned} C(R(\xi,Y)\xi,V)W &= C(Y,V)W - \left(\frac{r+2n}{2n(2n-1)}\right) \\ &\quad \times \{g(V,W)\eta(Y)\xi - \eta(W)\eta(Y)V\} \\ &\quad + \frac{1}{2n-1}\{S(V,W)\eta(Y)\xi - \eta(Y)\eta(W)QV\}, \end{aligned} \tag{4.16}$$

for any vector field Y, V, W on M^{2n+1}.

Similarly, it follows from (4.3) and (4.12) that

$$\begin{aligned} C(\xi,R(\xi,Y)V)W &= \left(\frac{r+2n}{2n(2n-1)}\right)\{g(Y,W)\eta(V)\xi - \eta(V)\eta(W)Y\} \\ &\quad - \frac{1}{2n-1}\{S(Y,W)\eta(V)\xi - \eta(W)\eta(V)QY\}, \end{aligned} \tag{4.17}$$

for any vector field Y, V, W on M^{2n+1}.

With the help of (4.3) and (4.12) we have

$$\begin{aligned} C(\xi,V)R(\xi,Y)W &= \left(\frac{r+2n}{2n(2n-1)}\right)\{g(Y,W)V - g(Y,W)\eta(V)\xi \\ &\quad + g(Y,V)\eta(W)\xi - \eta(W)\eta(Y)V\} \\ &\quad - \frac{1}{2n-1}\{2ng(Y,W)\eta(V)\xi + g(Y,W)QV \\ &\quad + S(V,Y)\eta(W)\xi - \eta(W)\eta(Y)QV\}, \end{aligned} \tag{4.18}$$

for any vector field Y, V, W on M^{2n+1}.

Finally, substituting (4.15)–(4.18) in (4.14) gives

$$\begin{aligned} &\left(\frac{r+2n}{2n(2n-1)}\right)\{g(V,W)Y - g(Y,W)V\} - C(Y,V)W \\ &+ \frac{1}{2n-1}\{S(Y,W)\eta(V)\xi - \eta(W)\eta(V)QY + 2ng(Y,W)\eta(V)\xi \\ &+ g(Y,W)QV - S(V,W)Y - 2n\eta(V)\eta(W)Y\} = 0, \end{aligned} \tag{4.19}$$

for any vector field Y, V, W on M^{2n+1}.

Using (3.5) in (4.19) yields

$$\begin{aligned}R(Y,V)W = {} & \frac{1}{2n-1}\{g(V,W)Y - g(Y,W)V - 2n\eta(V)\eta(W)Y \\ & + S(Y,W)\eta(V)\xi - \eta(V)\eta(W)QY + 2ng(Y,W)\eta(V)\xi \\ & - S(Y,W)V + g(V,W)QY\}. \end{aligned} \tag{4.20}$$

Contracting Y in (4.20) it follows that

$$S(V,W) = \left(1 + \frac{r}{2n}\right) g(V,W) - \left(1 + 2n + \frac{r}{2n}\right) \eta(V)\eta(W), \tag{4.21}$$

for any vector field V, W on M^{2n+1}.

Taking inner product of (4.19) with respect to Z gives

$$\begin{aligned}& \left(\frac{r+2n}{2n(2n-1)}\right) \{g(V,W)g(Y,Z) - g(Y,W)g(V,Z)\} - g(C(Y,V)W,Z) \\ & + \frac{1}{2n-1}\{S(Y,W)\eta(V)\eta(Z) - \eta(W)\eta(V)S(Y,Z) + 2ng(Y,W)\eta(V)\eta(Z) \\ & + g(Y,W)S(V,Z) - S(V,W)g(Y,Z) - 2n\eta(V)\eta(W)g(Y,Z)\} = 0. \end{aligned} \tag{4.22}$$

Putting the value of $S(V,W)$ in (4.22) one can easily obtain

$$g(C(Y,V)W,Z) = 0, \tag{4.23}$$

that is, $C(Y,V)W = 0$, for any vector field Y, V, W on M^{2n+1}. Hence the manifold is conformally flat.

Conversely, if the manifold is conformally flat then obviously it is Weyl semisymmetric. This completes the proof of the proposition. □

From Propositions 4.1 and 4.2 we obtain the following:

Theorem 4.1. *An almost Kenmotsu manifold $M^{2n+1}(n > 1)$ with ξ belonging to the (k,μ)-nullity distribution is Weyl semisymmetric if and only if the manifold is of constant curvature -1.*

Since conformally symmetric manifold ($\nabla C = 0$) implies $R \cdot C = 0$, therefore from Theorem 4.1 we can state the following:

Corollary 4.1. *An almost Kenmotsu manifold $M^{2n+1}(n > 1)$ with ξ belonging to the (k,μ)-nullity distribution is conformally symmetric if and only if the manifold is of constant curvature -1.*

Since $R \cdot R = 0$ implies $R \cdot C = 0$, we obtain the following:

Corollary 4.2. *An almost Kenmotsu manifold $M^{2n+1}(n > 1)$ with ξ belonging to the (k,μ)-nullity distribution is semisymmetric if and only if the manifold is of constant curvature -1.*

The above corollary has been proved by Wang and Liu [13].

5. Example of a 5-dimensional almost Kenmotsu manifold

In this section, we construct an example of an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$, which is an Einstein manifold. We consider 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 be five vector fields in $\mathbb{R}^5$ which satisfies [4]

$$[\xi, e_2] = -2e_2, \qquad [\xi, e_3] = -2e_3, \qquad [\xi, e_4] = 0, \qquad [\xi, e_5] = 0,$$
$$[e_i, e_j] = 0, \quad \text{where } i, j = 2, 3, 4, 5.$$

Let g be the Riemannian metric defined by

$$g(\xi, \xi) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1$$
$$\text{and} \quad g(\xi, e_i) = g(e_i, e_j) = 0 \quad \text{for } i \neq j;\, i, j = 2, 3, 4, 5.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \xi),$$

for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0, \qquad \phi(e_2) = e_4, \qquad \phi(e_3) = e_5, \qquad \phi(e_4) = -e_2, \qquad \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g we have

$$\eta(\xi) = 1, \qquad \phi^2 Z = -Z + \eta(Z)\xi$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $Z, U \in \chi(M)$. Moreover,

$$h'\xi = 0, \qquad h'e_2 = e_2, \qquad h'e_3 = e_3, \qquad h'e_4 = -e_4, \qquad h'e_5 = -e_5.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\begin{aligned}
&\nabla_\xi \xi = 0, && \nabla_\xi e_2 = 0, && \nabla_\xi e_3 = 0, && \nabla_\xi e_4 = 0, && \nabla_\xi e_5 = \xi, \\
&\nabla_{e_2} \xi = 2e_2, && \nabla_{e_2} e_2 = -2\xi, && \nabla_{e_2} e_3 = 0, && \nabla_{e_2} e_4 = 0, && \nabla_{e_2} e_5 = 0, \\
&\nabla_{e_3} \xi = 2e_3, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = -2\xi, && \nabla_{e_3} e_4 = 0, && \nabla_{e_3} e_5 = 0, \\
&\nabla_{e_4} \xi = 0, && \nabla_{e_4} e_2 = 0, && \nabla_{e_4} e_3 = 0, && \nabla_{e_4} e_4 = 0, && \nabla_{e_4} e_5 = 0, \\
&\nabla_{e_5} \xi = 0, && \nabla_{e_5} e_2 = 0, && \nabla_{e_5} e_3 = 0, && \nabla_{e_5} e_4 = 0, && \nabla_{e_5} e_5 = 0.
\end{aligned}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h'X,$$

for any $X \in \chi(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$\begin{aligned}
&R(\xi, e_2)\xi = 4e_2, \qquad R(\xi, e_2)e_2 = -4\xi, \qquad R(\xi, e_3)\xi = 4e_3,\\
&R(\xi, e_3)e_3 = -4\xi,\\
&R(\xi, e_4)\xi = R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0,\\
&R(e_2, e_3)e_2 = 4e_3, \qquad R(e_2, e_3)e_3 = -4e_2, \qquad R(e_2, e_4)e_2 = R(e_2, e_4)e_4 = 0,\\
&R(e_2, e_5)e_2 = R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0,\\
&R(e_3, e_5)e_3 = R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0.
\end{aligned}$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution, with $k = -2$ and $\mu = -2$.

Using the expressions of the curvature tensor we find the values of the Ricci tensor S as follows:

$$S(\xi, \xi) = S(e_2, e_2) = S(e_3, e_3) = -8, \qquad S(e_4, e_4) = S(e_5, e_5) = 0.$$

Since $\{\xi, e_2, e_3, e_4, e_5\}$ forms a basis, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1\xi + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5$$

and

$$Y = b_1\xi + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5,$$

where $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 \in \mathbb{R} \setminus \{0\}$ such that $a_4b_4 + a_5b_5 = 0$. Hence,

$$g(X, Y) = a_1b_1 + a_2b_2 + a_3b_3$$

and

$$S(X, Y) = -8(a_1b_1 + a_2b_2 + a_3b_3).$$

Therefore, we see that $S(X, Y) = -8g(X, Y)$, that is, the manifold M is an Einstein manifold.

Thus, Theorem 3.2 is verified.

References

[1] D.E. Blair, Contact manifold in Riemannian geometry, in: Lecture Notes on Mathematics, vol. 509, Springer, Berlin, 1976.
[2] D.E. Blair, Riemannian geometry on contact and symplectic manifolds, Progr. Math., 203.
[3] D.E. Blair, T. Koufogiorgos, B.J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel. J. Math. 91 (1995) 189–214.
[4] G. Dileo, A.M. Pastore, Almost Kenmotsu manifolds and nullity distributions, J. Geom. 93 (2009) 46–61.
[5] G. Dileo, A.M. Pastore, Almost Kenmotsu manifolds and local symmetry, Bull. Belg. Math. Soc. Simon Stevin 14 (2007) 343–354.
[6] G. Dileo, A.M. Pastore, Almost Kenmotsu manifolds with a condition of η-parallelism, Differential Geom. Appl. 27 (2009) 671–679.
[7] A. Gray, Spaces of constancy of curvature operators, Proc. Amer. Math. Soc. 17 (1966) 897–902.

[8] J.-B. Jun, U.C. De, G. Pathak, On Kenmotsu manifolds, J. Korean Math. Soc. 42 (2005) 435–445.

[9] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J. (2) 24 (1972) 93–103.

[10] Z.I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, the local version, J. Differential Geom. 17 (1982) 531–582.

[11] S. Tanno, Some differential equations on Riemannian manifolds, J. Math. Soc. Japan 30 (1978) 509–531.

[12] Y. Wang, X. Liu, Second order parallel tensors on almost Kenmotsu manifolds satisfying the nullity distributions, Filomat 28 (2014) 839–847.

[13] Y. Wang, X. Liu, Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions, Ann. Polon. Math. 112 (2014) 37–46.

[14] Y. Wang, X. Liu, On ϕ-recurrent almost Kenmotsu manifolds, Kuwait J. Sci. 42 (2015) 65–77.

[15] Y. Wang, X. Liu, On a type of almost Kenmotsu manifolds with harmonic curvature tensors, Bull. Belg. Math. Soc. Simon Stevin 22 (2015) 15–24.

[16] K. Yano, M. Kon, Structures on Manifolds, Vol. 40, World Scientific Press, 1989.