Arab Journal of
Mathematical Sciences

Arab J Math Sci **19**(1) (2013), 73–83

On a multi point boundary value problem for a fractional order differential inclusion

Aurelian Cernea
Faculty of Mathematics and Informatics, University of Bucharest, Academiei 14, 010014 Bucharest, Romania

Received 4 June 2012; accepted 12 July 2012
Available online 21 July 2012

Abstract. The existence of solutions for a multi point boundary value problem of a fractional order differential inclusion is investigated. Several results are obtained by using suitable fixed point theorems when the right hand side has convex or non convex values.

Mathematics subject classification: 34A60; 34B18; 34B15

Keywords: Fractional derivative; Differential inclusion; Boundary value problem; Fixed point

1. Introduction

Differential equations with fractional order have recently proved to be strong tools in the modeling of many physical phenomena; for a good bibliography on this topic we refer to [18]. As a consequence there was an intensive development of the theory of differential equations of fractional order [2,16,22] etc.. The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim [13]. Very recently several qualitative results for fractional differential inclusions were obtained in [1,3,6–11,15,20] etc.

In this paper we study the following problem

$$D^{\alpha}x(t) \in F(t, x(t), x'(t)) \quad a.e.\ [0, 1], \tag{1.1}$$

$$x(0) = x'(0) = 0, \quad x(1) - \sum_{i=1}^{m} a_i x(\xi_i) = \lambda, \tag{1.2}$$

Tel.: +40 785810358.
E-mail address: acernea@fmi.unibuc.ro

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2012.07.001

where D^{α} is the standard Riemann–Liouville fractional derivative, $\alpha \in (2,3]$, $m \geqslant 1$, $0 < \xi_1 < \cdots < \xi_m < 1$, $\sum_{i=1}^{m} a_i \xi_i^{\alpha-1} < 1$, $\lambda > 0$, $a_i > 0$, $i = \overline{1,m}$ and $F : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

The present paper is motivated by a recent paper of Nyamoradi [19], where it is considered problem (1.1) and (1.2) with F single valued and several existence results are provided.

The aim of our paper is to extend the study in [19] to the set-valued framework and to present some existence results for problem (1.1) and (1.2). Our results are essentially based on a nonlinear alternative of Leray–Schauder type, on Bressan–Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Covitz and Nadler set-valued contraction principle. The methods used are known ([1,8,9] etc.), however their exposition in the framework of problem (1.1) and (1.2) is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space with the corresponding norm $|\cdot|$ and let $I \subset \mathbf{R}$ be a compact interval. Denoted by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu–Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x : I \to X$ endowed with the norm $|x|_1 = \int_I |x(t)| \mathrm{d}t$.

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

Consider $T : X \to \mathcal{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for T if $x \in T(x)$. T is said to be bounded on bounded sets if $T(B) := \cup_{x \in B} T(x)$ is a bounded subset of X for all bounded sets B in X. T is said to be compact if $\overline{T(B)}$ is relatively compact for any bounded sets B in X. T is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. T is said to be upper semicontinuous if for any open set $D \subset X$, the set $\{x \in X : T(x) \subset D\}$ is open in X. T is called completely continuous if it is upper semicontinuous and totally bounded on X.

It is well known that a compact set-valued map T with nonempty compact values is upper semicontinuous if and only if T has a closed graph.

We recall the following nonlinear alternative of Leray–Schauder type and its consequences.

Theorem 2.1 [21]. *Let D and $\overline{D}$ be open and closed subsets in a normed linear space X such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

(i) *the inclusion $x \in T(x)$ has a solution, or*
(ii) *there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.*

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

(i) *the inclusion $x \in T(x)$ has a solution, or*
(ii) *there exists $x \in X$ with $\|x\| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.*

Corollary 2.3. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

(i) *the equation $x = T(x)$ has a solution, or*
(ii) *there exists $x \in X$ with $\|x\| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.*

We recall that a multifunction $T : X \to \mathcal{P}(X)$ is said to be lower semicontinuous if for any closed subset $C \subset X$, the subset $\{s \in X: T(s) \subset C\}$ is closed.
If $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map with compact values and $x \in C(I,\mathbf{R})$ we define

$$S_F(x) := \{f \in L^1(I, \mathbf{R}) : \quad f(t) \in F(t, x(t), x'(t)) \quad a.e.\ I\}.$$

We say that F is of lower semicontinuous type if $S_F(.)$ is lower semicontinuous with closed and decomposable values.

Theorem 2.4 [4]. *Let S be a separable metric space and $G : S \to \mathcal{P}(L^1(I, \mathbf{R}))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then G has a continuous selection (i.e., there exists a continuous mapping g: $S \to L^1(I,\mathbf{R})$ such that $g(s) \in G(s) \quad \forall s \in S$).

A set-valued map $G : I \to \mathcal{P}(\mathbf{R})$ with nonempty compact convex values is said to be measurable if for any $x \in \mathbf{R}$ the function $t \to d(x,G(t))$ is measurable.

A set-valued map $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is said to be Carathéodory if $t \to F(t,x,y)$ is measurable for all $x,y \in \mathbf{R}$ and $(x,y) \to F(t,x,y)$ is upper semicontinuous for almost all $t \in I$.

F is said to be L^1-Carathéodory if for any $l > 0$ there exists $h_l \in L^1(I,\mathbf{R})$ such that $\sup\{\|v\| : v \in F(t,x,y)\} \leqslant h_l(t)$ a.e. I, $\forall x, y \in \overline{B_l(0)}$.

Theorem 2.5 [17]. *Let X be a Banach space, let $F : I \times X \to \mathcal{P}(X)$ be a L^1-Carathéodory set-valued map with $S_F \neq \emptyset$ and let Γ: $L^1(I,X) \to C(I,X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I, X) \to \mathcal{P}(C(I, X))$ *defined by*

$$(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$$

has compact convex values and has a closed graph in $C(I,X) \times C(I,X)$.
Note that if $dimX < \infty$, *and* F *is as in* Theorem 2.5, *then* $S_F(x) \neq \emptyset$ *for any* $x \in C(I,X)$ *(e.g.,* [17]).

Consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that

$$d_H(T(x), T(y)) \leqslant \lambda d(x, y) \quad \forall x, y \in X.$$

The set-valued contraction principle [12] states that if X is complete, and $T : X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then T has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$.

Definition 2.6.

(a) *The fractional integral of order* $\alpha > 0$ of a Lebesgue integrable function f: $(0, \infty) \to \mathbf{R}$ is defined by

$$I_0^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and Γ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

(b) *The Riemann–Liouville fractional derivative of order* $\alpha > 0$ of a continuous function f: $(0, \infty) \to \mathbf{R}$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-\alpha+n-1} f(s)ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0, \infty)$.By $AC^1([0, 1],\mathbf{R})$ we denote the space of continuous real-valued functions whose first derivative exists and it is absolutely continuous on I. On $AC^1([0, 1],\mathbf{R})$ we consider the norm

$$\|x\| = \max\{\sup_{t\in[0,1]} |x(t)|, \sup_{t\in[0,1]} |x'(t)|\}.$$

Definition 2.7. A function $x \in AC^1([0, 1],\mathbf{R})$ is called a solution of problem (1.1) and (1.2) if there exists a function $v \in L^1([0, 1],\mathbf{R})$ with $v(t) \in F(t,x(t),x'(t))$, *a.e.* $[0, 1]$ such that $D^\alpha x(t) = v(t)$, *a.e.* $[0, 1]$ and conditions (1.2) are satisfied.

In what follows $I = [0, 1]$, $\alpha \in (2,3]$, and $\Delta = \sum_{i=1}^m a_i \xi_i^{\alpha-1} \in (0, 1)$. Next we need the following technical result proved in [19].

Lemma 2.8 19. *For any* $h \in L^1(I,\mathbf{R})$ *the problem*

$$D^\alpha x(t) = h(t) \quad a.e.\ [0, 1],$$

$$x(0) = x'(0) = 0, \quad x(1) - \sum_{i=1}^m a_i x(\xi_i) = \lambda$$

has a unique solution given by

$$x(t) = \frac{\lambda t^{\alpha-1}}{1-\Delta} + \int_0^1 G(t,s)h(s)ds + \frac{t^{\alpha-1}}{1-\Delta}\sum_{i=1}^{m} a_i \int_0^1 G(\xi_i,s)h(s)ds, \quad t \in [0,1],$$

where

$$G(t,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & \text{if} \quad 0 \leqslant s < t \leqslant 1, \\ [t(1-s)]^{\alpha-1}, & \text{if} \quad 0 \leqslant t < s \leqslant 1. \end{cases}$$

Note that $G(t,s) > 0$ $\forall t,s \in I$ and $G(t,s) \leqslant \frac{1}{\Gamma(\alpha)}$, (e.g., Lemma 5 in [19]). If we denote $G_1(t,s) = G(t,s) + \sum_{i=1}^{m} \frac{a_i t^{\alpha-1}}{1-\Delta} G(\xi_i,s)$ one has $|G_1(t,s)| \leqslant \frac{1}{\Gamma(\alpha)}\left(1 + \frac{\sum_{i=1}^{m} a_i}{1-\Delta}\right)$ and $\left|\frac{\partial G_1}{\partial t}(t,s)\right| \leqslant \frac{2(\alpha-1)}{\Gamma(\alpha)}\left(1 + \frac{\sum_{i=1}^{m} a_i}{1-\Delta}\right)$.

Let $K_1 := \sup_{t,s\in I} |G_1(t,s)|$ and $K_2 := \sup_{t,s\in I} |\frac{\partial G_1}{\partial t}(t,s)|$.

Finally, we denote $z(t) = \frac{\lambda t^{\alpha-1}}{1-\Delta}$ and $C_1 := \sup_{t\in I} \|z(t)\|$.

3. The main results

Now we are able to present the existence results for problem (1.1) and (1.2). We consider first the case when F is convex valued.

Hypothesis 3.1.

(i) $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.

(ii) There exist $\varphi \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. I and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|, \quad v \in F(t,x,y)\} \leqslant \varphi(t)\psi(\max\{|x|,|y|\}) \quad a.e.\ I, \quad \forall x,y \in \mathbf{R}.$$

Theorem 3.2. *Assume that* Hypothesis 3.1 *is satisfied and there exists $r > 0$ such that*

$$r > C_1 + \max\{K_1,K_2\}|\varphi|_1\psi(r). \tag{3.1}$$

Then problem (1.1) and (1.2) has at least one solution x such that $\|x\| < r$.

Proof. Let $X = AC^1(I,\mathbf{R})$ and consider $r > 0$ as in (3.1). It is obvious that the existence of solutions to problem (1.1) and (1.2) reduces to the existence of the solutions of the integral inclusion

$$x(t) \in z(t) + \int_0^1 G_1(t,s)F(s,x(s),x'(s))ds, \quad t \in I. \tag{3.2}$$

Consider the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(AC^1(I, \mathbf{R}))$ defined by

$$T(x) := \left\{ v \in AC^1(I, \mathbf{R});\ v(t) = z(t) + \int_0^1 G_1(t, s)f(s)ds, \quad f \in \overline{S_F(x)} \right\}. \tag{3.3}$$

We show that T satisfies the hypotheses of Corollary 2.2.

First, we show that $T(x) \subset AC^1(I,\mathbf{R})$ is convex for any $x \in AC^1(I,\mathbf{R})$. If $v_1,v_2 \in T(x)$ then there exist $f_1,f_2 \in S_F(x)$ such that for any $t \in I$ one has

$$v_i(t) = z(t) + \int_0^1 G_1(t, s)f_i(s)ds, \quad i = 1, 2.$$

Let $0 \leqslant \alpha \leqslant 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1 - \alpha)v_2)(t) = z(t) + \int_0^1 G_1(t, s)[\alpha f_1(s) + (1 - \alpha)f_2(s)]ds.$$

The values of F are convex, thus $S_F(x)$ is a convex set and hence $\alpha v_1 + (1 - \alpha)v_2 \in T(x)$.

Second, we show that T is bounded on bounded sets of $AC^1(I,\mathbf{R})$. Let $B \subset AC^1(I,\mathbf{R})$ be a bounded set. Then there exists $m > 0$ such that $\|x\| \leqslant m\ \forall x \in B$. If $v \in T(x)$ there exists $f \in S_F(x)$ such that $v(t) = \int_0^1 G_1(t, s)f(s)ds$. One may write for any $t \in I$

$$\begin{aligned} |v(t)| &\leqslant |z(t)| + \int_0^1 |G_1(t, s)| \cdot |f(s)|ds \\ &\leqslant |z(t)| + \int_0^1 |G_1(t, s)|\varphi(s)\psi(\max\{|x(s)|, |x'(s)|\})ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} |v'(t)| &\leqslant |z'(t)| + \int_0^1 |\frac{\partial G_1}{\partial t}(t, s)| \cdot |f(s)|ds \\ &\leqslant |z'(t)| + \int_0^1 |\frac{\partial G_1}{\partial t}(t, s)|\varphi(s)\psi(\max\{|x(s)|, |x'(s)|\})ds. \end{aligned}$$

and therefore

$$\begin{aligned} \|v\| &= \max_{t\in I}\{|v(t)|, |v'(t)|\} \\ &\leqslant \max_{t\in I} \max\{|z(t)|, |z'(t)|\} + \int_0^1 \max_{t,s\in I}\{|G_1(t, s)|, |\frac{\partial G_1}{\partial t} \\ &\quad \times (t, s)|\}\varphi(s)\psi(\max\{|x(s)|, |x'(s)|\})ds \\ &\leqslant C_1 + \max\{K_1, K_2\}|\varphi|_1\psi(m) \end{aligned}$$

$\forall v \in T(x)$, i.e., $T(B)$ is bounded.

We show next that T maps bounded sets into equi-continuous sets. Let $B \subset AC^1(I,\mathbf{R})$ be a bounded set as before and $v \in T(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = z(t) + \int_0^1 G_1(t,s)f(s)ds$. Then for any $t,\tau \in I$ we have

$$\begin{aligned}|v(t) - v(\tau)| &\leqslant |z(t) - z(\tau)| + |\int_0^1 G_1(t,s)f(s)ds - \int_0^1 G_1(\tau,s)f(s)ds| \\ &\leqslant |z(t) - z(\tau)| + \int_0^1 |G_1(t,s) \\ &\quad - G_1(\tau,s)|\varphi(s)\psi(\max\{|x(s)|,|x'(s)|\})ds \\ &\leqslant |z(t) - z(\tau)| + \int_0^1 |G_1(t,s) - G_1(\tau,s)|\varphi(s)\psi(m)ds.\end{aligned}$$

Similarly, we have

$$|v'(t) - v'(\tau)| \leqslant |z'(t) - z'(\tau)| + \int_0^1 |\frac{\partial G_1}{\partial t}(t,s) - \frac{\partial G_1}{\partial t}(\tau,s)|\varphi(s)\psi(m)ds.$$

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $T(B)$ is an equi-continuous set in $AC^1(I,\mathbf{R})$. We apply now Arzela–Ascoli's theorem we deduce that T is completely continuous on $AC^1(I,\mathbf{R})$.

In the next step of the proof we prove that T has a closed graph. Let $x_n \in AC^1(I,\mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)$ $\forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$. Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = z(t)+ \int_0^1 G_1(t,s)f_n(s)ds$. Define Γ: $L^1(I,\mathbf{R}) \to AC^1(I,\mathbf{R})$ by $(\Gamma(f))(t) := \int_0^1 G_1(t,s)f(s)ds$. One has

$$\begin{aligned}&\max_{t\in I}\{|v_n(t) - z(t) - (v^*(t) - z(t))|, |v_n'(t) - z'(t) - ((v^*)'(t) - z'(t))| \\ &\quad = \max_{t\in I}\{|v_n(t) - v^*(t)|, |v_n'(t) - (v^*)'(t)|\} = \|v_n - v^*\| \to 0\end{aligned}$$

as $n \to \infty$.

We apply Theorem 2.5 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) - z(t) = \int_0^1 G_1(t,s)f^*(s)ds$. Therefore, T is upper semicontinuous and compact on $\overline{B_r(0)}$.

We apply Corollary 2.2 to deduce that either (i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or (ii) there exists $x \in X$ with $\|x\| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Assume that (ii) is true. With the same arguments as in the second step of our proof we get $r = \|x\| \leqslant C_1 + \max\{K_1,K_2\}|\varphi|_1\psi(r)$ which contradicts (3.1). Hence only (i) is valid and theorem is proved.

We consider now the case when F is not necessarily convex valued. Our first existence result in this case is based on the Leray–Schauder alternative for single valued maps and on Bressan Colombo selection theorem. □

Hypothesis 3.3.

(i) $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has compact values, F is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ measurable and $(x,y) \to F(t,x,y)$ is lower semicontinuous for almost all $t \in I$.

(ii) There exist $\varphi \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. I and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|, \quad v \in F(t,x,y)\} \leqslant \varphi(t)\psi(\max\{|x|,|y|\}) \quad a.e.\ I, \quad \forall x, y \in \mathbf{R}.$$

Theorem 3.4. *Assume that* Hypothesis 3.3 *is satisfied and there exists r > 0 such that condition* (3.1) *is satisfied. Then problem* (1.1) and (1.2) *has at least one solution on I.*

Proof. We note first that if Hypothesis 3.3 is satisfied then F is of lower semicontinuous type (e.g., [14]). Therefore, we apply Theorem 2.4 to deduce that there exists f: $AC^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that $f(x) \in S_F(x)$ $\forall x \in AC^1(I,\mathbf{R})$.

We consider the corresponding problem

$$x(t) = z(t) + \int_0^1 G_1(t,s)f(x(s))ds, \quad t \in I \tag{3.4}$$

in the space $X = AC^1(I,\mathbf{R})$. It is clear that if $x \in AC^1(I,\mathbf{R})$ is a solution of the problem (3.4) then x is a solution to problem (1.1) and (1.2).

Let $r > 0$ that satisfies the condition (3.1) and define the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(AC^1(I,\mathbf{R}))$ by

$$(T(x))(t) := z(t) + \int_0^1 G_1(t,s)f(x(s))ds.$$

Obviously, the integral Eq. (3.4) is equivalent with the operator equation

$$x(t) = (T(x))(t), \quad t \in I. \tag{3.5}$$

It remains to show that T satisfies the hypotheses of Corollary 2.3.

We show that T is continuous on $\overline{B_r(0)}$. From Hypotheses 3.3. (ii) we have

$$|f(x(t))| \leqslant \varphi(t)\psi(\max\{|x(t)|,|x'(t)|\}) \quad a.e.\ I$$

for all $x \in AC^1(I,\mathbf{R})$. Let $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \leqslant \varphi(t)\psi(r) \quad a.e.\ I.$$

From Lebesgue's dominated convergence theorem and the continuity of f we obtain, for all $t \in I$

$$\lim_{n\to\infty}(T(x_n))(t) = z(t) + \lim_{n\to\infty}\int_0^1 G_1(t,s)f(x_n(s))ds = z(t) + \int_0^1 G_1(t,s)f(x(s))ds$$
$$= (T(x))(t)$$

and

$$\lim_{n\to\infty}(T(x_n))'(t) = z'(t) + \lim_{n\to\infty}\int_0^1 \frac{\partial G_1}{\partial t}(t,s)f(x_n(s))ds$$
$$= z'(t) + \int_0^1 \frac{\partial G_1}{\partial t}(t,s)f(x(s))ds = (T(x))'(t)$$

i.e., T is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.2 with corresponding modifications it follows that T is compact on $\overline{B_r(0)}$. We apply Corollary 2.3 and we find that either (i) the equation $x = T(x)$ has a solution in $\overline{B_r(0)}$, or (ii) there exists $x \in X$ with $\|x\| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

As in the proof of Theorem 3.2 if the statement (ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement (i) is true and problem (1.1) has a solution $x \in AC^1(I,\mathbf{R})$ with $\|x\| < r$.

In order to obtain an existence result for problem (1.1) and (1.2) by using the set-valued contraction principle we introduce the following hypothesis on F. □

Hypothesis 3.5.

(i) $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact values, is integrably bounded and for every $x,y \in \mathbf{R}$, $F(.,x,y)$ is measurable.
(ii) There exists $l_1,l_2 \in L^1(I,\mathbf{R}_+)$ such that for almost all $t \in I$,

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2)) \leqslant l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2|$$
$$\forall x_1,x_2,y_1,y_2 \in \mathbf{R}.$$

Theorem 3.6. *Assume that* Hypothesis 3.5. *is satisfied and* $(|l_1|_1 + |l_2|_1)\cdot\max\{K_1,K_2\} < 1$. *Then problem* (1.1) and (1.2) *has a solution.*

Proof. We transform the problem (1.1) and (1.2) into a fixed point problem. Consider the set-valued map $T : AC^1(I,\mathbf{R}) \to \mathcal{P}(AC^1(I,\mathbf{R}))$ defined by

$$T(x) := \{v \in AC^1(I,\mathbf{R}); \quad v(t) = z(t) + \int_0^1 G_1(t,s)f(s)ds, \quad f \in S_F(x)\}.$$

Note that since the set-valued map $F(.,x(.))$ is measurable with the measurable selection theorem (e.g., Theorem III. 6 in [5]) it admits a measurable selection $f: I \to \mathbf{R}$. Moreover, since F is integrably bounded, $f \in L^1(I,\mathbf{R})$. Therefore, $S_{F,x} \neq \emptyset$.

It is clear that the fixed points of T are solutions of problem (1.1) and (1.2). We shall prove that T fulfills the assumptions of Covitz Nadler contraction principle.

First, we note that since $S_{F,x} \neq \emptyset$, $T(x) \neq \emptyset$ for any $x \in AC^1(I,\mathbf{R})$.

Second, we prove that $T(x)$ is closed for any $x \in AC^1(I,\mathbf{R})$. Let $\{x_n\}_{n\geqslant 0} \in T(x)$ such that $x_n \to x^*$ in $AC^1(I,\mathbf{R})$. Then $x^* \in AC^1(I,\mathbf{R})$ and there exists $f_n \in S_{F,x}$ such that

$$x_n(t) = z(t) + \int_0^1 G_1(t,s)f_n(s)ds.$$

Since F has compact values and Hypothesis 3.5 is satisfied we may pass to a subsequence (if necessary) to get that f_n converges to $f \in L^1(I,\mathbf{R})$ in $L^1(I,\mathbf{R})$. In particular, $f \in S_{F,x}$ and for any $t \in I$ we have

$$x_n(t) \to x^*(t) = z(t) + \int_0^1 G_1(t,s)f(s)ds,$$

i.e., $x^* \in T(x)$ and $T(x)$ is closed.

Finally, we show that T is a contraction on $AC^1(I,\mathbf{R})$. Let $x_1,x_2 \in AC^1(I,\mathbf{R})$ and $v_1 \in T(x_1)$. Then there exist $f_1 \in S_{F,x_1}$ such that

$$v_1(t) = z(t) + \int_0^1 G(t,s)f_1(s)ds, \quad t \in I.$$

Consider the set-valued map

$$\begin{aligned} H(t) &:= F(t,x_2(t),x_2'(t)) \cap \{x \in \mathbf{R}; \quad |f_1(t) - x| \\ &\leqslant l_1(t)|x_1(t) - x_2(t)| + l_2(t)|x_1'(t) - x_2'(t)|\}, \quad t \in I. \end{aligned}$$

From Hypothesis 3.5 one has

$$d_H(F(t,x_1(t),x_1'(t)),F(t,x_2(t),x_2'(t))) \leqslant l_1(t)|x_1(t) - x_2(t)| + l_2(t)|x_1'(t) - x_2'(t)|,$$

hence H has nonempty closed values. Moreover, since H is measurable, there exists f_2 a measurable selection of H. It follows that $f_2 \in S_{F,x_2}$ and for any $t \in I$

$$|f_1(t) - f_2(t)| \leqslant l_1(t)|x_1(t) - x_2(t)| + l_2(t)|x_1'(t) - x_2'(t)|.$$

Define

$$v_2(t) = z(t) + \int_0^1 G_1(t,s)f_2(s)ds, \quad t \in I$$

and we have

$$\begin{aligned} |v_1(t) - v_2(t)| &\leqslant \int_0^1 |G_1(t,s)| \cdot |f_1(s) - f_2(s)|ds \\ &\leqslant \int_0^1 G_1(t,s)[l_1(s)|x_1(s) - x_2(s)| + l_2(s)|x_1'(s) - x_2'(s)|]ds \\ &\leqslant K_1(|l_1|_1 + |l_2|_1)\|x_1 - x_2\|. \end{aligned}$$

Similarly, we have

$$|v_1'(t) - v_2'(t)| \leqslant K_2(|l_1|_1 + |l_2|_1)\|x_1 - x_2\|.$$

So, $\|v_1 - v_2\| \leqslant (|l_1|_1 + |l_2|_1)\max\{K_1,K_2\}\|x_1 - x_2\|$. From an analogous reasoning by interchanging the roles of x_1 and x_2 it follows

$$d_H(T(x_1), T(x_2)) \leqslant (|l_1|_1 + |l_2|_1)\max\{K_1, K_2\}\|x_1 - x_2\|.$$

Therefore, T admits a fixed point which is a solution to problem (1.1) and (1.2). □

References

[1] B. Ahmad, S.K. Ntouyas, Arab J. Math. Sci. 18 (2012) 121–134.
[2] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, J. Math. Anal. Appl. 338 (2008) 1340–1350.
[3] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Fract. Calc. Appl. Anal. 11 (2008) 35–56.
[4] A. Bressan, G. Colombo, Studia Math. 90 (1988) 69–86.
[5] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Springer, Berlin, 1977.
[6] A. Cernea, Fract. Calc. Appl. Anal. 12 (2009) 433–442.
[7] A. Cernea, Nonlinear Anal. 72 (2010) 204–208.
[8] A. Cernea, Electronic J. Qual. Theory Differ. Equ. 78 (2010) 1–13.
[9] A. Cernea, J. Appl. Math. Comput. 38 (2012) 133–143.
[10] A. Cernea, Fract. Calc. Appl. Anal. 15 (2012) 183–194.
[11] Y.K. Chang, J.J. Nieto, Math. Comput. Modell. 49 (2009) 605–609.
[12] H. Covitz, S.B. Nadler Jr., Israel J. Math. 8 (1970) 5–11.
[13] A.M.A. El-Sayed, A.G. Ibrahim, Appl. Math. Comput. 68 (1995) 15–25.
[14] M. Frignon, A. Granas, C. R. Acad. Sci. Paris I 310 (1990) 819–822.
[15] J. Henderson, A. Ouahab, Nonlinear Anal. 70 (2009) 2091–2105.
[16] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[17] A. Lasota, Z. Opial, Bull. Acad. Polon. Sci. Math. Astronom. Phys. 13 (1965) 781–786.
[18] K. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[19] N. Nyamoradi, Arab J. Math. Sci. 18 (2012) 165–175.
[20] A. Ouahab, Nonlinear Anal. 69 (2009) 3871–3896.
[21] D. O' Regan, Arch. Math. (Brno) 34 (1998) 191–197.
[22] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.