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Numerical and theoretical treatment for solving linear and nonlinear delay differential equations using variational iteration method

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Abstract. This article is devoted to use the variational iteration method (VIM) established by J.H. He for solving linear and nonlinear delay differential equations (DDEs). This method is based on the use of Lagrange multiplier for identification of optimal value of a parameter in a functional. This procedure is a powerful tool for solving large amount of problems. Using VIM, it is possible to find the exact solution or an approximate solution of the proposed problem. This technique provides a sequence of functions which converges to the exact solution of the problem. Convergence analysis is reliable enough to estimate the maximum absolute error of the approximate solution given by VIM. A comparison with the Adomian decomposition method is given.

Keywords: Delay differential equation; Variational iteration method; Convergence analysis

1. Introduction

Many different methods have recently been introduced to solve nonlinear problems, such as, VIM [1,13–16,19,26–28,30,31], Adomian decomposition method (ADM) [2,11], homotopy perturbation method [20,24,29] and others [18]. The VIM is strongly and simply capable for solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also it reduces the size of calculations besides, its interactions are direct and straightforward.

A delay differential equation (DDE) is a differential equation in which the derivative of the function at any time depends on the solution at previous time. Introduction of

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delay in the model enriches its dynamics and allows a precise description of the real life phenomena. DDEs are proved to be useful in control systems [12], lasers, traffic models [4], metal cutting, epidemiology, neuroscience, population dynamics [21], chemical kinetics [9], etc. In DDE one has to provide history of the system over the delay interval $[\tau, 0]$ as the initial condition. Due to this reason delay systems are infinite dimensional in nature. Because of infinite dimensionality the DDEs are difficult to analyze analytically [3] and hence the numerical solutions play an important role.

The main aim in this work is to effectively employ VIM to establish exact solutions of DDEs and study the convergence of the method. To guarantee this study we present five examples of linear and nonlinear delay differential equations. Many papers have been interested to study the numerical solutions of DDEs. In [22] the author studied the first order delay differential equations, using spline functions, and studied the stability and the error analysis. Also, in [23] the authors studied the system of first order delay differential equations, using spline functions, and studied the stability and the error analysis.

2. Analysis of VIM

To illustrate the analysis of VIM, we limit ourselves to consider the following nonlinear delay differential equation in the type

$$Lu(t) = f(t, u(t), u(\alpha(t))), \quad 0 \leqslant t \leqslant T, \tag{1}$$

with the following initial conditions

$$u^{(k)}(0) = u_0^k, \quad k = 0, 1, \ldots, n-1, \quad u(t) = \phi(t), \quad t \leqslant 0, \tag{2}$$

where the differential operator L is defined by $L(\cdot) = \frac{d^n(\cdot)}{dt^n}$.

Now, to illustrate the analysis of VIM and study the analysis of convergence, we rewrite Eq. (1) in the following form

$$Lu + Ru + N(u) = 0, \tag{3}$$

with specified initial conditions, where L and R are linear bounded operators, i.e., it is possible to find numbers $m_1, m_2 > 0$ such that $\|Lu\| \leqslant m_1\|u\|$, $\|Ru\| \leqslant m_2\|u\|$. The nonlinear term $N(u)$ is Lipschitz continuous with $|N(u) - N(v)| \leqslant m|u - v|$, $\forall t \in J = [0, T]$, for arbitrary constant $m > 0$. The VIM gives the possibility to write the solution of Eq. (3) with the aid of the correction functional

$$u_p = u_{p-1} + \int_0^t \lambda(\tau)[Lu_{p-1} + R\tilde{u}_{p-1} + N(\tilde{u}_{p-1})]d\tau, \quad p \geqslant 1. \tag{4}$$

It is obvious that the successive approximations u_p, $p \geqslant 0$ (the subscript p denotes the pth order approximation), can be established by determining λ, a general Lagrange multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_p$ is a restricted variation, which means $\delta\tilde{u}_p = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations u_p, $p \geqslant 1$, of the solution u will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. With λ

determined, then several approximations u_p, $p \geqslant 1$, follow immediately. Consequently, the exact solution may be obtained by using

$$u = \lim_{p \to \infty} u_p. \tag{5}$$

In what follows, we apply VIM to some examples of linear and nonlinear delay differential equations to illustrate the strength of the method and to establish exact solutions for these problems.

Now, to illustrate how to find the value of the Lagrange multiplier λ, we will consider the following case, which is dependent on the order of the operator L in Eq. (3), we will study the case of the operator $L = \frac{d}{dt}$ (without loss of generality).

Making the above correction functional stationary, and noticing that $\delta \tilde{u}_p = 0$, we obtain

$$\begin{aligned}\delta u_p &= \delta u_{p-1} + \delta \int_0^t \lambda(\tau)\left[\frac{du_{p-1}}{d\tau} + R\ \tilde{u}_{p-1} + N(\tilde{u}_{p-1})\right] d\tau \\ &= \delta u_{p-1} + [\lambda(\tau)\delta u_{p-1}]_{\tau=t} - \int_0^t \dot{\lambda}(\tau)[\delta u_{p-1}] d\tau = 0,\end{aligned}$$

where $\delta \tilde{u}_p$ is considered as a restricted variation i.e., $\delta \tilde{u}_p = 0$, yields the following stationary conditions

$$\dot{\lambda}(\tau) = 0, \quad 1 + \lambda(\tau)|_{\tau=t} = 0. \tag{6}$$

The first equation in (6) is called Lagrange–Euler equation and the second equation in (6) is called natural boundary condition. The solution of this equation gives the Lagrange multiplier $\lambda(\tau) = -1$. Now, the following variational iteration formula can be obtained

$$u_p = u_{p-1} - \int_0^t [Lu_{p-1} + Ru_{p-1} + N(u_{p-1})] d\tau. \tag{7}$$

We start with an initial approximation, and by using the above iteration formula (7), we can obtain directly the other components of the solution.

For more details about VIM and its advantages, see [8,30–32]

3. Convergence analysis of VIM

In this section, the sufficient conditions are presented to guarantee the convergence of VIM, when applied to solve nonlinear DDEs, where the main point is that we prove the convergence of the recurrence sequence, which is generated by using VIM.

Definition 1. The variation of the functional $v[u(x)]$ is defined as follows [10]

$$\delta v[u(x)] = \left[\frac{\partial}{\partial \alpha} v[u(x) + \alpha \delta u]\right]_{\alpha=0}, \tag{8}$$

where $v[u(x)]$ is a functional dependent on the function $u(x), \alpha \in \Re$.

Theorem 1. *If a functional $v[u(x)]$; has a variation, achieves a maximum or a minimum at $u = u_0$, then at $u = u_0$*

$$\delta v = 0, \tag{9}$$

where $u(x)$ is an interior point of the domain of definition of the functional.

Lemma 1. *Let $A : U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V.*

Proof. Since

$$\|Au_p - Au\|_V = \|A(u_p - u)\|_V \leqslant \|A\| \; \|u_p - u\|_U,$$

hence

$$\lim_{p\to\infty} \|Au_p - Au\|_V \leqslant \|A\| \lim_{p\to\infty} \|u_p - u\|_U = 0$$

implies that $A(u_p) \to A(u)$. $\square$

3.1. Uniqueness theorem

Theorem 2. *The nonlinear problem (3) has a unique solution, whenever $0 < \alpha < 1$, where $\alpha = (m_2 + m)T$ and the constants m_2 and m are defined above.*

Proof. Since, the solution of Eq. (3) can be written in the following form

$$u = f(t) - L^{-1}[Ru + N(u)],$$

where $f(t)$ is the solution of the homogenous equation $Lu = 0$, and the inverse operator L^{-1} is defined by $L^{-1}(\cdot) = \int_0^t (\cdot) dt$.

Now let, u and u^* be two different solutions to (3) then by using the above equation, we get

$$\begin{aligned} |u - u^*| &= \left| -\int_0^t [R\;(u - u^*) + N(u) - N(u^*)]dt \right| \\ &\leqslant \int_0^t [|R(u - u^*)| + |N(u) - N(u^*)|]dt \leqslant (m_2|u - u^*| + m|u - u^*|)T \\ &\leqslant \alpha|u - u^*| \end{aligned}$$

from which we get $(1 - \alpha)|u - u^*| \leqslant 0$. Since $0 < \alpha < 1$, then $|u - u^*| = 0$ implies, $u = u^*$ and this completes the proof. $\square$

Now, to prove the convergence of the variational iteration method, we will rewrite the Eq. (7) in the operator form as follows

$$u_p = A[u_{p-1}], \tag{10}$$

where the operator A takes the following form

$$A[u] = -\int_0^t [Lu + R\,u + N(u)]d\tau. \tag{11}$$

3.2. Convergence theorem

Theorem 3 (*Banach's fixed point theorem*). *Assume that X be a Banach space and* $A : X \to X$ *is a nonlinear mapping, and suppose that*

$$\|A[u] - A[v]\| \leqslant \gamma\|u - v\|, \quad \forall u, v \in X, \tag{12}$$

for some constant $\gamma = (\alpha + m_1 T) < 1$. *Then A has a unique fixed point. Furthermore, the sequence* (10) *using VIM with an arbitrary choice of* $u_0 \in X$, *converges to the fixed point of A and*

$$\|u_p - u_q\| \leqslant \frac{\gamma^q}{1-\gamma}\|u_1 - u_0\|. \tag{13}$$

Proof. Denoting $(C[J], \|\cdot\|)$ Banach space of all continuous functions on J with the norm defined by

$$\|f(t)\| = \max_{t \in J}|f(t)|.$$

We are going to prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space

$$\begin{aligned}
\|u_p - u_q\| &= \max_{t\in J}|u_p - u_q| \\
&= \max_{t\in J}\left| -\int_0^t [L(u_{p-1} - u_{q-1}) + R(u_{p-1} - u_{q-1}) + N(u_{p-1}) - N(u_{q-1})]d\tau\right| \\
&\leqslant \max_{t\in J}\int_0^t [|L(u_{p-1} - u_{q-1})| + |R(u_{p-1} - u_{q-1})| + |N(u_{p-1}) - N(u_{q-1})|]d\tau \\
&\leqslant \max_{t\in J}\int_0^t [(m_1 + m_2 + m)(u_{p-1} - u_{q-1})]d\tau \leqslant \gamma\|u_{p-1} - u_{q-1}\|.
\end{aligned}$$

Let, $p = q + 1$ then

$$\|u_{q+1} - u_q\| \leqslant \gamma\|u_q - u_{q-1}\| \leqslant \gamma^2\|u_{q-1} - u_{q-2}\| \leqslant \ldots \leqslant \gamma^q\|u_1 - u_0\|.$$

From the triangle inequality we have

$$\begin{aligned}
\|u_p - u_q\| &\leqslant \|u_{q+1} - u_q\| + \|u_{q+2} - u_{q+1}\| + \cdots + \|u_p - u_{p-1}\| \\
&\leqslant [\gamma^q + \gamma^{q+1} + \cdots + \gamma^{p-1}]\|u_1 - u_0\| \\
&\leqslant \gamma^q[1 + \gamma + \gamma^2 + \cdots + \gamma^{p-q-1}]\|u_1 - u_0\| \leqslant \gamma^q \frac{1-\gamma^{p-q-1}}{1-\gamma}]\|u_1 - u_0\|.
\end{aligned}$$

Since $0 < \gamma < 1$ so, $(1 - \gamma^{p-q}) < 1$ then

$$\|u_p - u_q\| \leqslant \frac{\gamma^q}{1-\gamma}\|u_1 - u_0\|.$$

But $\|u_1 - u_0\| < \infty$ so, as $q \to \infty$ then $\|u_p - u_q\| \to 0$. We conclude that $\{u_p\}$ is a Cauchy sequence in $C[J]$ so, the sequence is convergent. And this ends the proof of the theorem. $\square$

3.3. Error estimate

Theorem 4. *The maximum absolute error of the approximate solution u_p to problem (3) is estimated to be*

$$\max_{t\in J}|u_{exact} - u_p| \leqslant \beta, \tag{14}$$

where $\beta = \frac{\gamma^q T[(m_1+m_2)\|u_0\|+k]}{1-\gamma}$, $k = \max_{t\in J}|N(u_0)|$.

Proof. From Theorem 3 and inequality (13) we have

$$\|u_p - u_q\| \leqslant \frac{\gamma^q}{1-\gamma}\|u_1 - u_0\|,$$

as $p \to \infty$ then $u_p \to u_{exact}$ and

$$\|u_1 - u_0\| = \max_{t\in J}|-\int_0^t [Lu_0 + R\,u_0 + N(u_0)]d\tau|$$
$$\leqslant \max_{t\in J}\int_0^t [|Lu_0| + |Ru_0| + |N(u_0)|]d\tau \leqslant T[(m_1+m_2)\|u_0\| + k],$$

so, the maximum absolute error in the interval J is

$$\|u_{exact} - u_p\| = \max_{t\in J}|u_{exact} - u_p| \leqslant \beta.$$

This completes the proof. For more details about the convergence of VIM, see, [28,32] $\square$

4. Numerical application

In this section, we discuss the numerical treatment of some problems of linear DDE (LDDE) and nonlinear DDE (NDDE) by using the proposed method.

Example 1. Consider the LDDE of first-order [5,25]

$$\frac{du(x)}{dx} = \frac{1}{2}e^{x/2}u\left(\frac{x}{2}\right) + \frac{1}{2}u(x), \quad 0 \leqslant x \leqslant 1, \quad u(0) = 1. \tag{15}$$

The exact solution of this example is $u(x) = e^x$. To solve Eq. (15) by means of VIM, we construct a correction functional which reads

$$u_{p+1}(x) = u_p(x) + \int_0^x \lambda(\tau)\left[u_{p\tau} - \frac{1}{2}e^{\tau/2}\tilde{u}_p\left(\frac{\tau}{2}\right) - \frac{1}{2}\tilde{u}_p(\tau)\right]d\tau, \quad p \geqslant 0. \tag{16}$$

Making the above correction functional stationary, and noticing that $\delta u(0) = 0$, we obtain

$$\delta u_{p+1}(x) = \delta u_p(x) + \delta \int_0^x \lambda(\tau)\left[u_{p\tau} - \frac{1}{2}e^{\tau/2}\tilde{u}_p\left(\frac{\tau}{2}\right) - \frac{1}{2}\tilde{u}_p(\tau)\right]d\tau$$
$$= \delta u_p + [\lambda(\tau)\delta u_p]_{\tau=x} - \int_0^x \dot{\lambda}(\tau)[\delta u_p]d\tau = 0,$$

where $\delta\tilde{u}_p$ is considered as a restricted variation, i.e., $\delta\tilde{u}_p = 0$, yields the following stationary conditions

$$\dot{\lambda}(\tau) = 0, \quad 1 + \lambda(\tau)|_{\tau=x} = 0. \tag{17}$$

The solution of this equation gives the Lagrange multiplier $\lambda(\tau) = -1$.

Now, the following variational iteration formula can be obtained

$$u_{p+1}(x) = u_p(x) - \int_0^x \left[u_{p\tau} - \frac{1}{2}e^{\tau/2}u_p\left(\frac{\tau}{2}\right) - \frac{1}{2}\ u_p(\tau)\right]d\tau, \quad p \geqslant 0. \tag{18}$$

We start with an initial approximation $u_0(x) = u(0)$, and by using the above iteration formula (18), we can obtain directly the other components as

$$u_0(x) = 1, \quad u_1(x) = e^{x/2} + \frac{1}{2}x, \ldots$$

In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the numerical solutions using $p = 5$ terms approximation. Fig. 1 shows the behavior of the error between the exact solution and the numerical solution in $[0, 1]$. We achieved a very good approximation with the exact solution of Eq. (15) by using five terms only of the iteration equation derived above, where in [11] obtained the solution after 13 iterations using Adomian decomposition method. It is evident that the overall errors can be made smaller by adding new terms of the iteration formula. The obtained numerical results justify the advantage of the proposed method, even in the few terms approximation is accurate.

Example 2. Consider the LDDE of second-order [6]

$$\frac{d^2u(x)}{dx^2} = \frac{3}{4}\ u(x) + u\left(\frac{x}{2}\right) - x^2 + 2, \quad 0 \leqslant x \leqslant 1, \quad u(0) = 1, \quad \frac{du(0)}{dx} = 0. \tag{19}$$

The exact solution of this example is $u(x) = x^2$. To solve Eq. (19) by means of VIM, we construct a correction functional which reads

$$u_{p+1}(x) = u_p(x) + \int_0^x \lambda(\tau)\left[u_{p\tau\tau} - \frac{3}{4}\ \tilde{u}_p(\tau) - \tilde{u}_p\left(\frac{\tau}{2}\right) + \tau^2 - 2\right]d\tau, \quad p \geqslant 0. \tag{20}$$

Making the above correction functional stationary, and noticing that $\delta u(0) = 0$, we obtain

$$\delta u_{p+1}(x) = \delta u_p(x) + \delta \int_0^x \lambda(\tau)\left[u_{p\tau\tau} - \frac{3}{4}\tilde{u}_p(\tau) - \tilde{u}_p\left(\frac{\tau}{2}\right) + \tau^2 - 2\right]d\tau$$
$$= \delta u_p + [\dot{\lambda}(\tau)\delta u_p - \lambda(\tau)\delta\dot{u}_p]_{\tau=x} + \int_0^x \ddot{\lambda}(\tau)[\delta u_p]d\tau = 0,$$

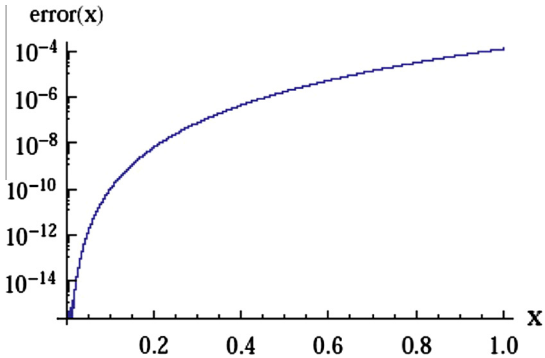

Fig. 1 Example 1.

where $\delta\tilde{u}_p$ is considered as a restricted variation, i.e., $\delta\tilde{u}_p = 0$, yields the following stationary conditions

$$\ddot{\lambda}(\tau) = 1, \quad \dot{\lambda}(\tau)|_{\tau=x} = 0, \quad \lambda(\tau)|_{\tau=x} = 0. \tag{21}$$

The solution of this equation gives the Lagrange multiplier $\lambda(\tau) = \tau - x$.

Now, the following variational iteration formula can be obtained

$$u_{p+1}(x) = u_p(x) + \int_0^x (\tau - x)\left[u_{p\tau\tau} - \frac{3}{4}u_p(\tau) - u_p\left(\frac{\tau}{2}\right) + \tau^2 - 2\right]d\tau, \quad p \geqslant 0. \tag{22}$$

We start with an initial approximation $u_0(x) = u(0) + x\frac{du(0)}{dx}$, and by using the above iteration formula (22), we can obtain directly the other components as

$$u_0(x) = 0, \quad u_1(x) = x^2 - \frac{1}{2}x^4, \quad u_2(x) = x^2 - 0.00225694x^6, \ldots$$

In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the numerical solutions using $p = 5$ terms approximation. Fig. 2 shows the behavior of the error between the exact solution and the numerical solution in $[0, 1]$. We achieved a very good approximation with the exact solution of Eq. (19) by using five terms only of the iteration equation derived above, wherein [11] obtained the solution after 8 iterations using Adomian decomposition method. This shows the advantage of the VIM.

Example 3. Consider the LDDE of third-order

$$\frac{d^3u(x)}{dx^3} = -u(x) - u(x - 0.3) + e^{-x+0.3}, \quad 0 \leqslant x \leqslant 1, \tag{23}$$

with the initial conditions

$$u(0) = 1, \quad \frac{du(0)}{dx} = -1, \quad \frac{d^2u(0)}{dx^2} = 1, \quad y(x) = e^{-x}, \quad x \leqslant 0.$$

The exact solution of this example is $u(x) = e^{-x}$.

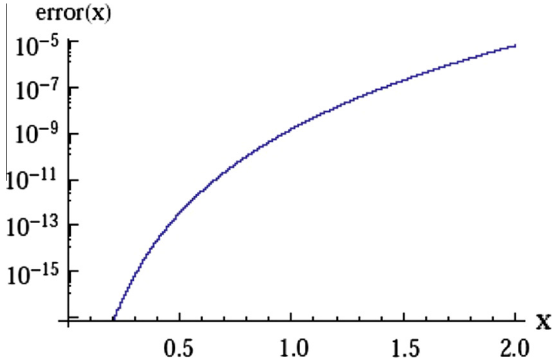

Fig. 2 Example 2.

To solve Eq. (23) by means of VIM, we can obtain the Lagrange multiplier λ, as follows:

$$\lambda(\tau) = -\frac{1}{2}(\tau - x)^2.$$

Now, the following variational iteration formula can be obtained

$$u_{p+1}(x) = u_p(x) + \int_0^x -\frac{1}{2}(\tau - x)^2[u_{p\tau\tau\tau} + u_p(\tau) + u_p(\tau - 0.3) - e^{-\tau+0.3}]\, d\tau, \quad p \geqslant 0. \tag{24}$$

We start with an initial approximation $u_0(x) = \frac{1}{2}x^2 - x + 1$, and by using the above iteration formula (24), we can obtain directly the components of the solution $u(x)$.

In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the numerical solutions using $p = 6$ terms approximation. Fig. 3 shows the behavior of the error between the exact solution and the numerical solution in $[0, 1]$. We achieved a very good approximation with the exact solution of Eq. (23) by using six terms only of the iteration equation derived above, wherein [11] obtained the solution after 6 iterations using Adomian decomposition method.

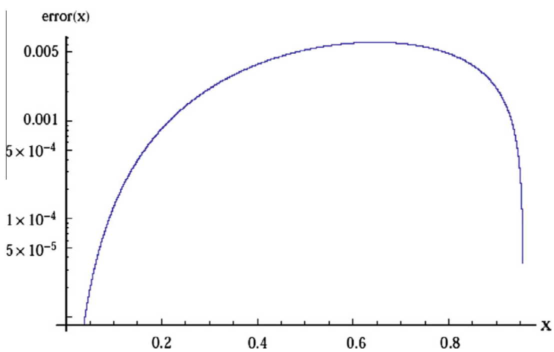

Fig. 3 Example 3.

Example 4. Consider the NDDE of first-order [7,17]

$$\frac{du(x)}{dx} = 1 - 2u^2\left(\frac{x}{2}\right), \quad u(0) = 0, \quad 0 \leqslant x \leqslant 1. \tag{25}$$

The exact solution of this example is $u(x) = \sin(x)$. To solve Eq. (25) by means of VIM, we can obtain the Lagrange multiplier λ, as follows $\lambda(\tau) = -1$.

Now, the following variational iteration formula can be obtained

$$u_{p+1}(x) = u_p(x) - \int_0^x \left[u_{p\tau} + 2u_p^2\left(\frac{\tau}{2}\right) - 1\right] d\tau, \quad p \geqslant 0. \tag{26}$$

We start with an initial approximation $u_0(x) = 0$, and by using the above iteration formula (26), we can obtain directly the other components as

$$u_0(x) = 0, \quad u_1(x) = x, \quad u_2(x) = x - \frac{x^3}{6}, \quad u_3(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{8064}, \ldots$$

From these components, we can note that it is the Taylor series of the exact solution $u(x) = \sin(x)$.

In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the numerical solutions using $p = 8$ terms approximation. Fig. 4 shows the behavior of the error between the exact solution and the numerical solution in $[0, 1]$. We achieved a very good approximation with the actual solution of Eq. (25) by using eight terms only of the iteration equation derived above, wherein [11] obtained the solution using Adomian decomposition method.

Example 5. Consider the NDDE of third-order

$$\frac{d^3u(x)}{dx^3} = -1 + 2u^2\left(\frac{x}{2}\right), \quad u(0) = 0, \quad \frac{du(0)}{dx} = 1, \quad \frac{d^2u(0)}{dx^2} = 0, \quad 0 \leqslant x \leqslant 1. \tag{27}$$

To solve Eq. (27) by means of VIM, we can obtain the Lagrange multiplier $\lambda(\tau) = -\frac{1}{2}(\tau - x)^2$.

Now, the following variational iteration formula can be obtained

$$u_{p+1}(x) = u_p(x) - \int_0^x \frac{1}{2}(\tau - x)^2\left[u_{p\tau\tau\tau} - 2u_p^2\left(\frac{\tau}{2}\right) + 1\right] d\tau, \quad p \geqslant 0. \tag{28}$$

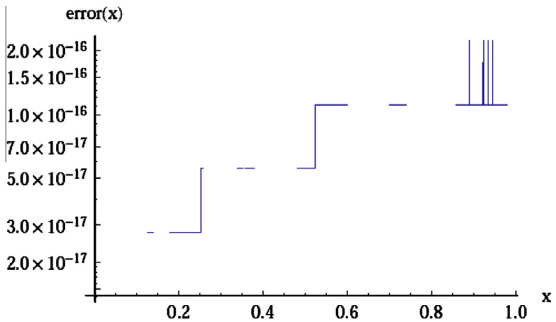

Fig. 4 Example 4.

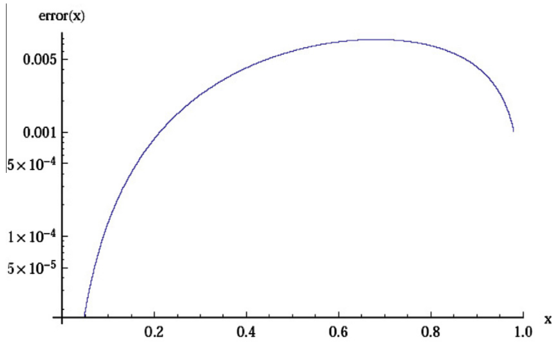

Fig. 5 Example 5.

We start with an initial approximation $u_0(x) = x$, and by using the above iteration formula (28), we can obtain the components of the solution $u(x)$. Consequently, the exact solution may be obtained by using

$$u = \lim_{p\to\infty} u_p = \sin(x). \tag{29}$$

Fig. 5 shows the behavior of the error between the exact solution and the numerical solution in $[0, 1]$. We achieved a very good approximation with the actual solution of Eq. (27) by using eight terms only of the iteration equation derived above, wherein [11] obtained the solution after 10 iterations using ADM (Fig. 6).

Example 6. This example is concerned with the implementation of VIM to obtain the numerical solution of the Logistic equation with delay of the form

$$\frac{du(t)}{dt} = \rho u(t)(1 - u(t - r)), \quad t > 0, \quad \rho > 0. \tag{30}$$

With the following initial condition $u(0) = u_0$, $u_0 > 0$.

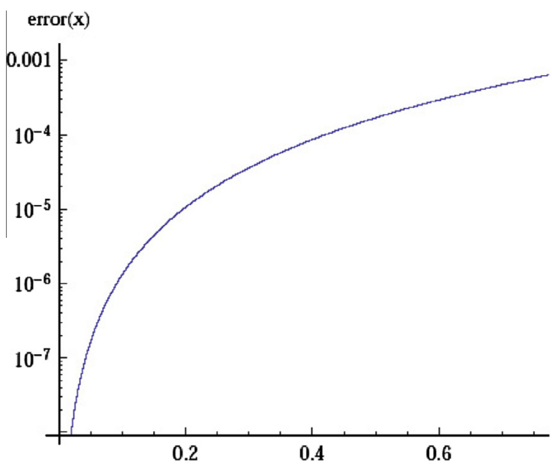

Fig. 6 Example 6.

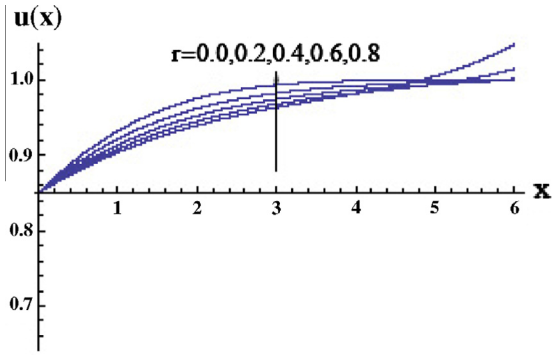

Fig. 7 The behavior of approximate solution of Example 6, at $r = 0.0, 0.2, 0.4, 0.6, 0.8$.

Applications of logistic equation.

A typical application of the logistic equation is a common model of population growth. Let $u(t)$ presents the population size and t represents time where the constant ρ defines the growth rate. Another application of logistic curve is in medicine, where the logistic differential equation is used to model the growth of tumors. This application can be considered an extension of the above mentioned use in the framework of ecology. Denoting with $u(t)$ the size of the tumor at time t.

To solve Eq. (30) by means of VIM, we can obtain the Lagrange multiplier $\lambda(\tau) = -1$. Now, the following variational iteration formula can be obtained as

$$u_{p+1}(t) = u_p(t) - \int_0^t [u_{p\tau}(\tau) - \rho u_p(\tau)(1 - u_p(\tau - r))]d\tau. \tag{31}$$

We start with an initial approximation $u_0(t) = 0.85$, and by using the above iteration formula (31), we can obtain the components of the solution $u(t)$ with $r = 0.0$

$$u_0(t) = 0.85,$$
$$u_1(t) = 0.85 + 0.06375t,$$
$$u_2(t) = 0.85 + 0.06375t - 0.0111563t^2 - 0.000677344t^3.$$

Therefore, the complete approximate solution can be readily obtained by the same iterative process. Consequently, the exact solution may be obtained by using:

$$u = \lim_{p\to\infty} u_p. \tag{32}$$

In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the numerical solutions using $p = 8$ terms approximation.

The behavior of the approximate solution and the exact solution in the interval $[0, 6]$ at $\rho = 0.5$ and using eight-iterations of the recurrence formula (31) using VIM is presented in Fig. 7 with $r = 0.0$, where in this case the exact solution is known and given by

$$u(t) = \frac{u_0}{u_0 + (1 - u_0)e^{-\rho t}}.$$

Also, Fig. 8 the behavior of the approximate solution with different values of the parameter of delay r is given. From this figure we can see that the approximate solution depends on this parameter.

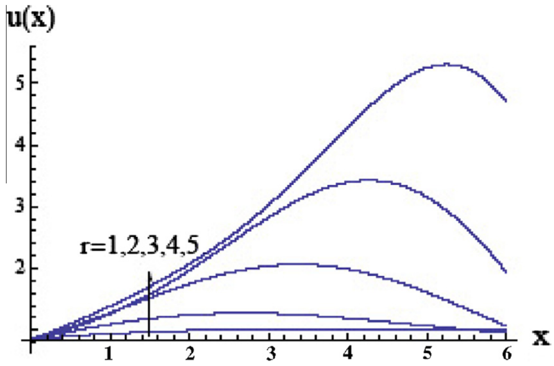

Fig. 8 The behavior of approximate solution of Example 6 at $r = 1, 2, 3, 4, 5$.

The numerical results we obtained justify the advantage of the proposed method, even in the few terms approximation is accurate and the solutions are very rapidly convergent. It is evident that the overall errors can be made smaller by adding new terms of the iteration formula. Also, it must be noted that VIM used here gives the possibility for obtaining an analytical satisfactory solution for which the other techniques of calculation are more laborious and the results contain a great complexity.

5. Conclusion

In this paper, the He's variational iteration method has been successfully applied to find the approximate solution of linear and nonlinear delay differential equations. The presented examples show that the results of the proposed method are in excellent agreement with those of Adomian decomposition method [11], but with less number of iterations. In our work, we use the Mathematica Package. An interesting point about VIM is that only few iterations or, even in some special cases, one iteration, leads to exact solutions or solutions with high accuracy. The main merits of VIM are, VIM can overcome the difficulties arising in the calculation of Adomian's polynomials in ADM, VIM does not require small parameters which are needed in perturbation method and no linearization is needed; the method is very promising for solving wide application in nonlinear differential equations.

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