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Notes on meromorphic functions sharing small function and its derivatives

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Abstract. In this paper we study the uniqueness theorems of meromorphic functions which share a small function with its derivatives, and give some results which are related to the results of P. Li.

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1. INTRODUCTION AND RESULTS

Let $\mathbb{C}$ be the complex plane. Throughout this paper f denotes a meromorphic function, i.e. a function that is holomorphic in $\mathbb{C}$ except for poles. It is assumed that the reader is familiar with the notations of Nevanlinna theory (see, for example, [4,11,10]). We denote by $S(r, f)$, as usual, any function satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r with finite Lebesgue measure. If a meromorphic function β satisfies $T(r, \beta) = S(r, f)$, then we call that β a small function of f. Let f and g be non-constant meromorphic functions, and let β be a meromorphic small function or constant in $\mathbb{C} \cup \{\infty\}$. We say that f and g share β CM (IM) if f and g have the same β-points with the same multiplicities (ignoring multiplicities). Let k be a positive integer, we denote by $N_{k)}(r, \frac{1}{f-\beta})$ ($N_{(k}(r, \frac{1}{f-\beta})$) the counting function of β-points of f with multiplicity $\leq k$ ($> k$). In the same way we can define $\bar{N}_{k)}(r, \frac{1}{f-\beta})$ and $\bar{N}_{(k}(r, \frac{1}{f-\beta})$ where in counting the β-points of f we ignore the multiplicities (see [11]).

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In 1979, Ruble and Yang [9] proved that if f is entire function and shares two finite values CM with f', then $f \equiv f'$. Mues and Steinmetz [6], and Gundersen [3] improved this result and proved the following:

Theorem 1.1. *Let f be a non-constant meromorphic function, a and b be two distinct values. If f and f' share the values a and b CM, then $f \equiv f'$.*

Frank and Weissenborn [2] improved Theorem 1.1 and proved the following result

Theorem 1.2. *Let f be a non-constant meromorphic function. If f and $f^{(k)}$ share two distinct values a and b CM, then $f \equiv f^{(k)}$.*

Yu [12], Lahiri–Sarkar [5], Zhang [13], Banerjee [1], Zhang–Lü [14], and many other authors have obtained results on the uniqueness problems of meromorphic functions that share one small function with their first or kth derivatives.

In 2003 P. Li [8] introduced the notation $S_1(r, f)$ which is defined to be any quantity such that for any positive number ϵ there exists a set $E(\epsilon)$ whose upper logarithmic density is less than ϵ and $S_1(r, f) = o(T(r, f))$ as $r \to \infty$, $r \notin E$. It is clear that every $S(r, f)$ is $S_1(r, f)$. In the same paper he improved Theorem 1.2 and proved the following:

Theorem 1.3 (*[8]*). *Let f be a non-constant meromorphic function, a_1 and a_2 $(a_j \neq \infty)$ $(j = 1, 2)$ be two distinct meromorphic functions satisfying $T(r, a_j) = S_1(r, f)$, $j = 1, 2$ and let $k > 1$ be a positive integer. If f and $f^{(k)}$ share a_1 and a_2 CM, then $f \equiv f^{(k)}$.*

Theorem 1.4 (*[8]*). *Let f be a non-constant meromorphic function, a_1 and a_2 $(a_j \neq \infty)$ $(j = 1, 2)$ be two distinct meromorphic functions satisfying $T(r, a_j) = S_1(r, f)$, $j = 1, 2$. If f and f' share a_1 and a_2 CM, and if $f \not\equiv f'$, then f can be expressed as $f = \alpha_2 + (\alpha_2 - \alpha_1)/(h - 1)$, where h is a transcendental meromorphic function satisfying $\bar{N}(r, h) + \bar{N}(r, \frac{1}{h}) = S_1(r, f)$, and $\alpha_j (\neq a_j)$, $j = 1, 2$ are two distinct meromorphic functions satisfying $\alpha_1' = a_2$, $\alpha_2' = a_1$, $a_1 - a_2 = \alpha_1 - \alpha_2$ and $T(r, \alpha_j) = S_1(r, f)$, $N(r, \frac{1}{f - \alpha_j}) = S_1(r, f)$, $j = 1, 2$.*

It is natural to ask whether the conditions of Theorems 1.3 and 1.4 remain true when f and $f^{(k)} (k \geq 1)$ share only one small function. In the present paper, we shall answer this question and prove the following theorems:

Theorem 1.5. *Let f be a non-constant meromorphic function and let β be a small meromorphic function of f such that $\beta \not\equiv 0, \infty$ and let $k \geq 1$ be an integer. If f and $f^{(k)}$ share β CM, and if $\bar{N}(r, \frac{1}{f}) = S(r, f)$, then either $f \equiv f^{(k)}$ or $k = 1$ and*

$$f(z) = \frac{\int_0^z \beta(t)dt + b}{1 + ce^{-z}} \tag{1.1}$$

where b and $c \neq 0$ are constants.

Theorem 1.6. *Let f be a non-constant meromorphic function and let β be a small meromorphic function of f such that $\beta \not\equiv 0, \infty$ and let $k \geq 1$ be an integer. If f and $f^{(k)}$ share β*

IM, and if $\bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f^{(k)}}) = S(r, f)$*, then either* $f \equiv f^{(k)}$ *or* $k = 1$*,* $\beta \equiv$ *constant and*

$$f(z) = \frac{2\beta}{1 + ce^{-2z}}. \tag{1.2}$$

where c *is a nonzero constant.*

2. Some lemmas

Lemma 2.1 ([4])**.** *Let* f *be a non-constant meromorphic function, and* a_1, a_2, a_3 *be distinct small functions of* f*. Then*

$$T(r, f) \leq \sum_{j=1}^{3} \bar{N}\Big(\frac{1}{f - a_j}\Big) + S(r, f).$$

Lemma 2.2. *Let* $k \geq 1$*,* f *be a non-constant meromorphic function and* $\nu \not\equiv \infty$ *be a small meromorphic function of* f*. Then either*

$$f^{(k)}(z) - \nu(z) = c(z + b)^{-(k+1)}, \tag{2.1}$$

where b *and* $c \neq 0$ *are constants, or*

$$(k-1)N_{1)}(r, f) \leq 2\bar{N}\Big(r, \frac{1}{f^{(k)} - \nu}\Big) + 2\bar{N}_{(2}(r, f) + S(r, f). \tag{2.2}$$

Proof. We consider the following meromorphic function:

$$W = \Big(\frac{f^{(k+1)} - \nu'}{f^{(k)} - \nu}\Big)^2 - (k+1)\Big(\frac{f^{(k+1)} - \nu'}{f^{(k)} - \nu}\Big)'. \tag{2.3}$$

From Nevanlinna's fundamental estimate of logarithmic derivative we obtain

$$m(r, W) = S(r, f). \tag{2.4}$$

Let z_∞ be a simple pole of f and $\nu(z_\infty) \neq 0, \infty$. By a simple calculation on the local expansion we see that

$$W(z) = O\Big((z - z_\infty)^{k-1}\Big). \tag{2.5}$$

In the following we shall treat two cases $W \equiv 0$ and $W \not\equiv 0$ separately.

Case i. $W \equiv 0$. We rewrite (2.3) in the form

$$\Big(\frac{f^{(k+1)} - \nu'}{f^{(k)} - \nu}\Big)^{-2}\Big(\frac{f^{(k+1)} - \nu'}{f^{(k)} - \nu}\Big)' = \frac{1}{k+1}.$$

Integrating twice, we get (2.1).

Case ii. $W \not\equiv 0$. Then we deduce from (2.5), (2.4) and (2.3) that

$$\begin{aligned}(k-1)N_{1)}(r,f) &\le N\Big(r,\frac{1}{W}\Big)+S(r,f)\le T(r,W)+S(r,f)\\ &\le N(r,W)+m(r,W)+S(r,f)\le N(r,W)+S(r,f)\\ &\le 2\bar{N}\Big(r,\frac{1}{f^{(k)}-\nu}\Big)+2\bar{N}_{(2}(r,f)+S(r,f).\end{aligned}$$

This is (2.2). □

Lemma 2.3. *Let $k \ge 1$, f be a non-constant meromorphic function and $\nu \not\equiv 0,\infty$ be a meromorphic small function of f. If f and $f^{(k)}$ share ν IM, then only (2.2) holds.*

Proof. If (2.1) holds, then $\nu \equiv constant$. Integrating (2.1) k times we deduce that

$$f(z)-\nu=\frac{(-1)^k c+(z+b)[(z^k-k!)\nu+k!P_{k-1}]}{k!(z+b)},$$

where P_{k-1} is a polynomial of degree at most $k-1$. Since f and $f^{(k)}$ share ν IM, we must have $(z^k-k!)\nu+k!P_{k-1}\equiv 0$. This implies that $\nu=0$, which contradicts with assumption of Lemma 2.3. Thus from Lemma 2.2 we find (2.2) holds. □

Lemma 2.4 ([7]). *Let f be a meromorphic function and $\Psi = a_n f^n + a_{n-1}f^{n-1}+\cdots + a_1 f + a_0$, where $a_n \not\equiv 0, a_{n-1},\dots,a_1,a_0$ be meromorphic small functions of f. If $\bar{N}(r,\frac{1}{\Psi})=S(r,f)$, then three cases are possible*

(i) $\Psi = a_n\Big(f+\frac{a_{n-1}}{na_n}\Big)^n$;

(ii) *There exist a meromorphic small function $\alpha_0 \not\equiv 0$ and an integer μ such that $n=2\mu$ and*

$$\Psi = a_n\Big(f^2+2\frac{a_{n-1}}{na_n}f+\Big(\frac{a_{n-1}}{na_n}\Big)^2+\alpha_0\Big)^{\mu};$$

(iii) *There exist a meromorphic small function $\alpha_0 \not\equiv 0$, positive integers μ_1 and μ_2, and distinct complex numbers λ_1 and λ_2 such that $\mu_1+\mu_2=n$, $\mu_1\lambda_1+\mu_2\lambda_2=0$, and*

$$\Psi = a_n\Big(f+\frac{a_{n-1}}{na_n}-\lambda_1\alpha_0\Big)^{\mu_1}\Big(f+\frac{a_{n-1}}{na_n}-\lambda_2\alpha_0\Big)^{\mu_2}.$$

3. PROOF OF THEOREM 1.5

If $f \equiv f^{(k)}$, there is nothing to prove, so we assume that $f \not\equiv f^{(k)}$. We distinguish three cases below.

Case 1. $\bar{N}(r,f)=S(r,f)$. From this, $\bar{N}(r,\frac{1}{f})=S(r,f)$ and Lemma 2.1 we obtain

$$\begin{aligned}T(r,f) &\le \bar{N}\Big(r,\frac{1}{f}\Big)+\bar{N}\Big(r,\frac{1}{f-\beta}\Big)+\bar{N}(r,f)+S(r,f)\\ &= \bar{N}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f). \qquad (3.1)\end{aligned}$$

Since f and $f^{(k)}$ share β CM, it follows that

$$\begin{aligned}\bar{N}\Big(r,\frac{1}{f-\beta}\Big) &\le N\Big(r,\frac{1}{(f^{(k)}/f)-1}\Big) \le T\Big(r,\frac{f^{(k)}}{f}\Big)+O(1)\\ &= N\Big(r,\frac{f^{(k)}}{f}\Big)+S(r,f)\\ &\le k\Big[\bar{N}\Big(r,\frac{1}{f}\Big)+\bar{N}(r,f)\Big]+S(r,f)=S(r,f).\end{aligned}$$

Together with (3.1) we have $T(r,f)=S(r,f)$ which is a contradiction.

Case 2. $\bar{N}(r,\frac{1}{f-\beta})=S(r,f)$. Again by Lemma 2.1 we find that $T(r,f)=\bar{N}(r,f)+S(r,f)$ which implies

$$N_{(2}(r,f)+m(r,f)=S(r,f). \tag{3.2}$$

Hence, employing Lemma 2.3, we find that $T(r,f)=S(r,f)$. This is impossible unless $k=1$. Set

$$\begin{aligned}\Gamma &= \frac{1}{f}\Big[\frac{(f'/\beta)'}{(f'/\beta)-1}-2\frac{(f/\beta)'}{(f/\beta)-1}\Big] && (3.3)\\ &= \frac{1}{\beta}\Big[\frac{f'}{f}\Big(\frac{(f'/\beta)'}{(f'/\beta)-1}-\frac{(f'/\beta)'}{f'/\beta}\Big)-2\Big(\frac{(f/\beta)'}{(f/\beta)-1}-\frac{(f/\beta)'}{f/\beta}\Big)\Big]. && (3.4)\end{aligned}$$

Then from Nevanlinna's fundamental estimate of the logarithmic derivative we have

$$m(r,\Gamma)=S(r,f). \tag{3.5}$$

It follows from (3.3) that if z_∞ is a pole of f of order $p\ge 1$, then

$$\Gamma(z)=\begin{cases}O((z-z_\infty)) & \text{if } p=1\\ O\Big((z-z_\infty)\Big)^{p-1} & \text{if } p\ge 2.\end{cases} \tag{3.6}$$

From the hypotheses of Theorem 1.5, (3.4) and (3.6) we deduce that

$$N(r,\Gamma)\le 2\bar{N}\Big(r,\frac{1}{f}\Big)+\bar{N}\Big(r,\frac{1}{f-\beta}\Big)+S(r,f)=S(r,f). \tag{3.7}$$

If $\Gamma\equiv 0$, then from (3.3) we obtain by integrating once,

$$(f-\beta)^2=c\beta(f'-\beta), \tag{3.8}$$

where c is a nonzero constant. We have thus derived the result

$$2N\Big(r,\frac{1}{f-\beta}\Big)=N\Big(r,\frac{1}{f'-\beta}\Big)+S(r,f).$$

Because of f and f' share β CM,

$$N\Big(r,\frac{1}{f-\beta}\Big)=S(r,f). \tag{3.9}$$

(3.8) can be rewritten as

$$\frac{\beta' - \beta}{f - \beta} = \frac{1}{c\beta}(f - \beta) - \frac{(f - \beta)'}{f - \beta}.$$

If $\beta \not\equiv \beta'$, from (3.2) we see that

$$m\Big(r, \frac{1}{f - \beta}\Big) \leq m(r, f) + S(r, f) = S(r, f).$$

Combining with (3.9) we get $T(r, f) = S(r, f)$ a contradiction. Therefore $\beta \equiv \beta'$ and so $\beta = be^z$ for some nonzero constant b. Thus (3.8) becomes

$$(f - \beta)^{-2}(f - \beta)' = \frac{1}{cb}e^{-z}.$$

By integration once,

$$(f - \beta)^{-1} = \frac{1}{c\beta} + d,$$

where d is a constant. This gives the contradiction $T(r, f) = S(r, f)$. If $\Gamma \not\equiv 0$, then from (3.6), (3.5) and (3.7) we have

$$\begin{aligned}\bar{N}(r, f) &\leq N\Big(r, \frac{1}{\Gamma}\Big) + S(r, f) \leq T(r, \Gamma) + S(r, f)\\ &= N(r, \Gamma) + m(r, \Gamma) + S(r, f) = S(r, f).\end{aligned}$$

This is impossible.

Case 3. $\bar{N}(r, f) \neq S(r, f)$ and $\bar{N}(r, \frac{1}{f-\beta}) \neq S(r, f)$, Let Λ be the function defined by

$$\Lambda = \frac{1}{f}\Big[\frac{(f^{(k)}/\beta)'}{(f^{(k)}/\beta) - 1} - \frac{(f/\beta)'}{(f/\beta) - 1}\Big]. \tag{3.10}$$

Similarly as the formula (3.3) we obtain

$$m(r, \Lambda) = S(r, f). \tag{3.11}$$

From (3.10) it can be seen that if z_∞ is a pole of f of order $p \geq 1$, then z_∞ is possible a zero of Λ of order $p - 1$. i.e.

$$\Lambda(z) = O\Big((z - z_\infty)^{p-1}\Big). \tag{3.12}$$

Let z_0 be a zero of $f - \beta$ and $\beta(z_0) \neq 0, \infty$. In view of f and $f^{(k)}$ share β CM, from (3.10)

$$\Lambda(z_0) = O(1). \tag{3.13}$$

We can also conclude from (3.10) that if z_1 is a zero of f of order $n \geq 1$, then z_1 is a zero of Λ of order at most $n + 1 + s$. i.e.

$$\Lambda(z) = O\Big((z - z_1)^{-k-1+s}\Big), \tag{3.14}$$

where $\beta(z) = O((z - z_1)^s)$ and s is an integer number. Thus, from (3.12)–(3.14) and $\bar{N}(r, \frac{1}{f}) = S(r, f)$ we deduce that

$$N(r, \Lambda) \leq (k+1)\bar{N}\Big(r, \frac{1}{f}\Big) + N(r, \beta) + N\Big(r, \frac{1}{\beta}\Big) = S(r, f).$$

Combining with (3.11) we get

$$T(r, \Lambda) = S(r, f). \tag{3.15}$$

If $\Lambda \equiv 0$, then from integration of (3.10) we find $f - \beta = c(f^{(k)} - \beta)$. Hence $\bar{N}(r, f) = S(r, f)$ which is impossible. Therefore, we must have $\Lambda \not\equiv 0$. Writing (3.10) as

$$f = \frac{1}{\Lambda}\Big[\frac{(f^{(k)}/\beta)'}{(f^{(k)}/\beta) - 1} - \frac{(f/\beta)'}{(f/\beta) - 1}\Big].$$

Consequently, from (3.15),

$$m(r, f) \leq m\Big(r, \frac{1}{\Lambda}\Big) + S(r, f) \leq T(r, \Lambda) + S(r, f) = S(r, f). \tag{3.16}$$

Further, it follows from (3.12) and (3.15) that

$$\begin{aligned} N_{(2}(r, f) - \bar{N}_{(2}(r, f) &\leq N\Big(r, \frac{1}{\Lambda}\Big) + S(r, f) \\ &\leq T(r, \Lambda) + S(r, f) = S(r, f), \end{aligned} \tag{3.17}$$

and we may therefore conclude that

$$N_{(2}(r, f) = S(r, f). \tag{3.18}$$

We next define

$$\Omega = \frac{(f^{(k)}/\beta)'}{(f^{(k)}/\beta) - 1} - \frac{(f/\beta)'}{(f/\beta) - 1} - k\frac{f'}{f}. \tag{3.19}$$

Then

$$m(r, \Omega) = S(r, f). \tag{3.20}$$

If z_∞ is a simple pole of f, then from (3.19) we find that Ω is holomorphic at z_∞. Thus from this, the assumptions of Theorem 1.5 and (3.18) we conclude

$$N(r, \Omega) \leq \bar{N}\Big(r, \frac{1}{f}\Big) + \bar{N}_{(2}(r, f) + S(r, f) = S(r, f).$$

Together with (3.20) we have

$$T(r, \Omega) = S(r, f). \tag{3.21}$$

Eliminating $\frac{(f^{(k)}/\beta)'}{(f^{(k)}/\beta)-1} - \frac{(f/\beta)'}{(f/\beta)-1}$ between (3.19) and (3.10) leads to

$$kf' = \Lambda f^2 - \Omega f. \tag{3.22}$$

Suppose that z_∞ is a simple pole of f and $\beta(z_\infty) \neq 0, \infty$. In the neighborhood of z_∞, the function f has the following Laurent expansion

$$f(z) = \frac{a_{-1}}{z - z_\infty} + a_0 + O((z - z_\infty)),$$

where $a_{-1} \neq 0$ is the residue of f at z_∞. By a simple computing, we find that Λ and Ω have the following expansions:

$$\Lambda(z) = \frac{-k}{a_{-1}} + \frac{(k-1)a_0 + \beta}{a_{-1}^2}(z - z_\infty) + O\Big((z - z_\infty)^2\Big) \tag{3.23}$$

and

$$\Omega(z) = \frac{\beta - (k+1)a_0}{a_{-1}} + O((z - z_\infty)). \tag{3.24}$$

Differentiating (3.23) once,

$$\Lambda'(z) = \frac{(k-1)a_0 + \beta}{a_{-1}^2} + O((z - z_\infty)). \tag{3.25}$$

If we now eliminate a_0 and a_{-1} among (3.23)–(3.25) we arrive at

$$\Omega(z) = \frac{-2}{k-1}\Lambda(z)\beta(z) + \frac{k(k+1)}{k-1}\frac{\Lambda'(z)}{\Lambda(z)} + O((z - z_\infty)), \tag{3.26}$$

provided that $k > 1$. If $\Omega \not\equiv \frac{-2}{k-1}\Lambda\beta + \frac{k(k+1)}{k-1}\frac{\Lambda'}{\Lambda}$, then from (3.18), (3.26), (3.21) and (3.15) we see

$$\begin{aligned}\bar{N}(r, f) = N_{1)}(r, f) + S(r, f) &\leq N\Big(r, \frac{1}{\Omega + \frac{2}{k-1}\Lambda\beta - \frac{k(k+1)}{k-1}\frac{\Lambda'}{\Lambda}}\Big) + S(r, f) \\ &\leq T(r, \Omega) + 3T(r, \Lambda) + S(r, f) = S(r, f),\end{aligned}$$

a contradiction. Therefore

$$\Omega \equiv \frac{-2}{k-1}\Lambda\beta + \frac{k(k+1)}{k-1}\frac{\Lambda'}{\Lambda}, \tag{3.27}$$

provided that $k > 1$. If we next eliminate Ω between (3.27) and (3.22) gives

$$kf' = \Lambda f^2 + \Big(\frac{2}{k-1}\Lambda\beta - \frac{k(k+1)}{k-1}\frac{\Lambda'}{\Lambda}\Big)f. \tag{3.28}$$

Since $\bar{N}(r, \frac{1}{f^{(k)} - \beta}) = \bar{N}(r, \frac{1}{f - \beta})$, we may obtain from Lemma 2.3 and (3.18),

$$(k-1)N_{1)}(r, f) \leq 2\bar{N}\Big(r, \frac{1}{f - \beta}\Big) + S(r, f).$$

That is $(k-1)T(r, f) \leq 2T(r, f) + S(r, f)$, so that $k \leq 3$. Let

$$F = \frac{f^{(k)} - \beta}{f - \beta}, \tag{3.29}$$

which, in view of f and $f^{(k)}$ share β CM, leads to

$$N\Big(r, \frac{1}{F}\Big) \equiv 0. \tag{3.30}$$

In the following we shall treat three cases only $k = 3$, $k = 2$ and $k = 1$ respectively.

Case 3.1. $k = 3$. Differentiating (3.28) three times we arrive at

$$f''' = (2/9)\Lambda^3 f^4 + ((4/9)\Lambda^3\beta - 2\Lambda\Lambda')f^3 + \alpha_1 f^2 + \alpha_2 f, \tag{3.31}$$

where α_1 and α_2 are small functions of f. Because of f and f''' share β CM, it follows from (3.31) that

$$(2/9)\Lambda^3\beta^3 + ((4/9)\Lambda^3\beta - 2\Lambda\Lambda')\beta^2 + \alpha_1\beta + \alpha_2 \equiv 1. \tag{3.32}$$

Substituting (3.31) into (3.29) and then using (3.32), we arrive at

$$F = (2/9)\Lambda^3 f^3 + ((2/3)\Lambda^3\beta - 2\Lambda\Lambda')f^2 + \alpha_3 f + 1, \tag{3.33}$$

where α_3 is a small function of f. Applying Lemma 2.4 to (3.33) we shall have the following three cases:

Case 3.1.1. F can be expressed as

$$F = (2/9)\Lambda^3\Big(f + \beta - 3\frac{\Lambda'}{\Lambda^2}\Big)^3. \tag{3.34}$$

From this and (3.33) we see that $(\Lambda\beta - 3\frac{\Lambda'}{\Lambda})^3 = 9/2$. This implies that

$$\Lambda\beta - 3\frac{\Lambda'}{\Lambda} = A, \tag{3.35}$$

where A is a constant and $A^3 = \frac{9}{2}$. If we next eliminate $\Lambda\beta$ between (3.35) and (3.27) (when $k = 3$) we obtain $\Omega = 3\frac{\Lambda'}{\Lambda} - A$. Integration of each members of this and (3.19) yields the following $F = c\Lambda^3 e^{-Az} f^3$, where c is a nonzero constant. By using (3.33), a contradiction occurs.

Case 3.1.2. Since the power of f is three in (3.33) which contradicts with $3 = 2\mu$ in Lemma 2.4(ii).

Case 3.1.3. F can be expressed as

$$F = (2/9)\Lambda^3(f + \theta_1)^{\mu_1}(f + \theta_2)^{\mu_2}, \tag{3.36}$$

where $\theta_1 = \beta - 3\frac{\Lambda'}{\Lambda^2} - \lambda_1\alpha_0$, $\theta_2 = \beta - 3\frac{\Lambda'}{\Lambda^2} - \lambda_2\alpha_0$ and μ_1, μ_2, λ_1, λ_2, α_0 have the same meaning as in Lemma 2.4 from which, (3.36) and (3.33) it follows readily that $\theta_1 \not\equiv \theta_2$, $\theta_1 \not\equiv 0$, $\theta_2 \not\equiv 0$ and

$$\bar{N}\Big(r, \frac{1}{\theta_1}\Big) + \bar{N}\Big(r, \frac{1}{\theta_2}\Big) = S(r, f).$$

Combining this with $\bar{N}(r, \frac{1}{f}) = S(r, f)$ and Lemma 2.1 we get $T(r, f) = S(r, f)$ a contradiction.

Case 3.2. $k = 2$. Differentiating (3.28) (when $k = 2$) twice, we obtain

$$f'' = (1/2)\Lambda^2 f^3 + (1/2)(3\Lambda^2\beta - 8\Lambda')f^2 + \alpha_4 f,$$

where $\alpha_4 = 2\Lambda\beta - 6\frac{\Lambda'}{\Lambda}$. Similarly as Case 3.1, we arrive at the conclusion

$$F = (1/2)\Lambda^2 f^2 + 2(\Lambda^2\beta - 2\Lambda')f + 1. \tag{3.37}$$

That is

$$Ff = (1/2)\Lambda^2 f^3 + 2(\Lambda^2\beta - 2\Lambda')f^2 + f. \tag{3.38}$$

Obviously,

$$\bar{N}\Big(r, \frac{1}{Ff}\Big) \le \bar{N}\Big(r, \frac{1}{F}\Big) + \bar{N}\Big(r, \frac{1}{f}\Big) = S(r, f).$$

By Lemma 2.4, only three cases are possible.

Case 3.2.1. $Ff = (1/2)\Lambda^2(f + (4/3)(\beta - 2\frac{\Lambda'}{\Lambda^2}))^3$ which contradicts with (3.38).

Case 3.2.3. Similarly as Case 3.1.2, we will arrive at the same contradiction.

Case 3.2.3. Ff can be expressed as

$$\begin{aligned} Ff = {} & (1/2)\Lambda^2\Big(f + (4/3)\Big(\beta - 2\frac{\Lambda'}{\Lambda^2}\Big) - \lambda_1\alpha_0\Big)^{\mu_1} \\ & \times\Big(f + (4/3)\Big(\beta - 2\frac{\Lambda'}{\Lambda^2}\Big) - \lambda_2\beta_0\Big)^{\mu_2} \end{aligned} \tag{3.39}$$

where μ_1, μ_2, λ_1, λ_2, α_0 have the same meaning as in Lemma 2.4. Without loss of generality, we can assume that $\mu_1 = 1$ and $\mu_2 = 2$. It can be obtained from (3.39) and (3.38) that $(4/3)(\beta - 2\frac{\Lambda'}{\Lambda^2}) - \lambda_1\alpha_0 \equiv 0$. From this, (3.39) and $\lambda_1 + 2\lambda_2 = 0$ we deduce that $F = (1/2)\Lambda^2(f + 2(\beta - 2\frac{\Lambda'}{\Lambda^2}))^2$. This and (3.37) imply that $(\Lambda\beta - 2\frac{\Lambda'}{\Lambda^2})^2 \equiv 1/2$. Using an argument similar to that in the proof of Case 3.1.1, we have $F = c\Lambda^2 e^{-2bz} f^2$, where b and $c \ne 0$ are constants and $b^2 = 1$. By (3.37) this is a contradiction again.

Case 3.3. $k = 1$. Since f and f' share the β CM, we conclude from (3.15), (3.21) and (3.22) that

$$\begin{aligned} \bar{N}\Big(r, \frac{1}{f-\beta}\Big) &\le N\Big(r, \frac{1}{\beta\Lambda - \Omega - 1}\Big) + S(r, f) \\ &\le T(r, \Lambda) + T(r, \Omega) + T(r, \beta) + S(r, f) = S(r, f). \end{aligned}$$

Thus, we have a contradiction and it follows that $\beta\Lambda - \Omega \equiv 1$. From this, (3.10), (3.19) and (3.29) we can show that $F'/(F-1) - F'/F = 1$. Integration of each member of this yields $F = \frac{1}{1-ce^z}$, where c is a nonzero constant. Together with (3.29) we find that $[f(\frac{1-ce^z}{e^z})]' = -\beta c$. By integration we get (1.1). This proves Theorem 1.5. □

4. Proof of Theorem 1.6

Consider the following function

$$H = \frac{(f^{(k)}/\beta)'[(f/\beta) - 1]}{(f^{(k)}/\beta)[(f^{(k)}/\beta) - 1]} = \Big[\frac{(f^{(k)}/\beta)'}{(f^{(k)}/\beta) - 1} - \frac{(f^{(k)}/\beta)'}{f^{(k)}/\beta}\Big][(f/\beta) - 1]. \tag{4.1}$$

By lemma of logarithmic derivative, we get

$$m(r, H) \le m(r, f) + S(r, f). \tag{4.2}$$

Similar to Case 1 and Case 2 in the proof of Theorem 1.5, we can prove that $\bar{N}(r, f) = S(r, f)$ is impossible, and if $\bar{N}(r, \frac{1}{f-\beta}) = S(r, f)$, then we have only (3.8), which may also be written

$$c\beta f' = [f + \beta(i\sqrt{c} - 1)][f - \beta(i\sqrt{c} + 1)].$$

From this, $\bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f'}) = S(r, f)$ and Lemma 2.1 we find $T(r, f) = S(r, f)$ which is a contradiction. Therefore in the following, we assume that $\bar{N}(r, \frac{1}{f-\beta}) \neq S(r, f)$ and $\bar{N}(r, f) \neq S(r, f)$. It follows from (4.1) that if z_∞ is a pole of f of order $p \geq 1$ and $\beta(z_\infty) \neq 0, \infty$, then

$$H(z) = O\Big((z - z_\infty)^{k-1}\Big). \tag{4.3}$$

Since f and $f^{(k)}$ share β IM, we deduce from (4.1) that if z_0 is a zero of $f - \beta$ of order $q \geq 1$ and $\beta(z_\infty) \neq 0, \infty$, then

$$H(z) = O\Big((z - z_0)^{q-1}\Big). \tag{4.4}$$

Thus from (4.1), (4.3), (4.4) and $\bar{N}(r, \frac{1}{f^{(k)}}) = S(r, f)$, we find

$$N(r, H) \leq \bar{N}\Big(r, \frac{1}{f^{(k)}}\Big) + S(r, f) = S(r, f).$$

Together with (4.2) we have

$$T(r, H) \leq m(r, f) + S(r, f). \tag{4.5}$$

Obviously, $H \not\equiv 0$. By (4.3)–(4.5) we see that

$$\begin{aligned}(k-1)\bar{N}(r, f) + \bar{N}_{(2}\Big(r, \frac{1}{f-\beta}\Big) &\leq N\Big(r, \frac{1}{H}\Big) + S(r, f) \\ &\leq T(r, H) + S(r, f) \\ &\leq m(r, f) + S(r, f).\end{aligned} \tag{4.6}$$

By using the same methods as those in the proof of Theorem 1.5,

$$T(r, \Lambda) \leq \bar{N}_{(2}\Big(r, \frac{1}{f-\beta}\Big) + \bar{N}_{(2}\Big(r, \frac{1}{f^{(k)}-\beta}\Big) + S(r, f)$$

and

$$m(r, f) + N_{(2}(r, f) - \bar{N}_{(2}(r, f) \leq N\Big(r, \frac{1}{\Lambda}\Big) + m\Big(r, \frac{1}{\Lambda}\Big) + S(r, f).$$

Combining these two inequalities, (4.6) and $\bar{N}(r, \frac{1}{f^{(k)}}) = S(r, f)$ yields

$$\begin{aligned}(k-1)\bar{N}(r, f) + N_{(2}(r, f) &\leq \bar{N}_{(2}\Big(r, \frac{1}{f^{(k)}-\beta}\Big) + \bar{N}_{(2}(r, f) + S(r, f) \\ &\leq N_{(2}\Big(r, \frac{1}{f^{(k+1)}/f^{(k)} - \beta'/\beta}\Big) + \bar{N}_{(2}(r, f) + S(r, f) \\ &\leq T\Big(r, \frac{f^{(k+1)}}{f^{(k)}}\Big) + \bar{N}_{(2}(r, f) + S(r, f)\end{aligned}$$

$$\leq \bar{N}\Big(r, \frac{1}{f^{(k)}}\Big) + \bar{N}(r, f) + \bar{N}_{(2}(r, f) + S(r, f)$$
$$= \bar{N}(r, f) + \bar{N}_{(2}(r, f) + S(r, f).$$

Therefore

$$(k-2)\bar{N}(r, f) + N_{(2}(r, f) \leq \bar{N}_{(2}(r, f) + S(r, f). \tag{4.7}$$

This is impossible unless $k \leq 2$. If $k = 2$, from (4.7) we have $N_{(2}(r, f) = S(r, f)$. This, $\bar{N}(r, \frac{1}{f''}) = S(r, f)$ and Lemma 2.2 (with $\nu \equiv 0$) give a contradiction. Hence, $k = 1$. In view of (3.3) in the proof of Theorem 1.5 we can consider two cases.

Case I. $\Gamma \not\equiv 0$. Denote by $\bar{N}_{(1,2)}(r, \frac{1}{f-\beta})$ is the counting function of those zeros of $f - \beta$ of order one and zeros of $f' - \beta$ of order two, each zero in this counting function is counted only once. From (3.3) and (3.6) it is easy to conclude that

$$\begin{aligned} N(r, f) - \bar{N}_{(2}(r, f) &\leq N\Big(r, \frac{1}{\Gamma}\Big) + S(r, f) \leq T(r, \Gamma) - m\Big(r, \frac{1}{\Gamma}\Big) + S(r, f) \\ &\leq \bar{N}\Big(r, \frac{1}{f-\beta}\Big) - \bar{N}_{(1,2)}\Big(r, \frac{1}{f-\beta}\Big) - m\Big(r, \frac{1}{\Gamma}\Big) \\ &\quad + S(r, f). \end{aligned} \tag{4.8}$$

Writing (3.3) as

$$f = \frac{1}{\Gamma}\left[\frac{(f'/\beta)'}{(f'/\beta) - 1} - 2\frac{(f/\beta)'}{(f/\beta) - 1}\right].$$

Hence

$$m(r, f) \leq m\Big(r, \frac{1}{\Gamma}\Big) + S(r, f).$$

Combining with (4.8) we obtain

$$m(r, f) + N(r, f) + \bar{N}_{(1,2)}\Big(r, \frac{1}{f-\beta}\Big) \leq \bar{N}\Big(r, \frac{1}{f-\beta}\Big) + \bar{N}_{(2}(r, f) + S(r, f).$$

Because of f and f' share β IM and $\bar{N}(r, \frac{1}{f}) = S(r, f)$,

$$\begin{aligned} \bar{N}\Big(r, \frac{1}{f-\beta}\Big) &\leq N\Big(r, \frac{1}{(f'/f) - 1}\Big) + S(r, f) \leq T\Big(r, \frac{f'}{f}\Big) + S(r, f) \\ &\leq \bar{N}\Big(r, \frac{1}{f}\Big) + \bar{N}(r, f) + S(r, f) = \bar{N}(r, f) + S(r, f). \end{aligned}$$

Combining these two inequalities, we find

$$m(r, f) + N(r, f) + \bar{N}_{(1,2)}\Big(r, \frac{1}{f-\beta}\Big) \leq \bar{N}_{(2}(r, f) + \bar{N}(r, f) + S(r, f).$$

Hence

$$m(r, f) + N_{(3}(r, f) + \bar{N}_{(1,2)}\Big(r, \frac{1}{f-\beta}\Big) = S(r, f). \tag{4.9}$$

From (4.1) we deduce that if z_∞ is a pole of f of order $p \ge 1$ and $\beta(z_\infty) \ne 0, \infty$,

$$H(z_\infty) = 1 + \frac{1}{p}. \tag{4.10}$$

If $p = 1$ and $H \not\equiv 2$, then from (4.10), (4.5) and (4.9) we see

$$N_{1)}(r, f) \le N\left(r, \frac{1}{H-2}\right) + S(r, f) \le T(r, H) + S(r, f) = S(r, f). \tag{4.11}$$

If $p = 2$ and $H \not\equiv 3/2$, then again from (4.10), (4.5) and (4.9) we get

$$\bar{N}_{2)}(r, f) - N_{1)}(r, f) \le N\left(r, \frac{1}{H-3/2}\right) + S(r, f) = S(r, f). \tag{4.12}$$

Then from (4.9), (4.11) and (4.12) we have a contradiction $\bar{N}(r, f) = S(r, f)$. Therefore either $H \equiv 2$ or $H \equiv 3/2$. If $H \equiv 2$, from this, (4.12) and (4.1) we obtain

$$\bar{N}_{(2}(r, f) + N_{(2}\left(r, \frac{1}{f-\beta}\right) = S(r, f). \tag{4.13}$$

From (2.3) we obtain by putting $k = 1$ and $\nu \equiv 0$ the function

$$W_1 = \left(\frac{f''}{f'}\right)^2 - 2\left(\frac{f''}{f'}\right)'. \tag{4.14}$$

By using the same methods as those in the proof of Lemma 2.2, we get

$$T(r, W_1) \le 2\bar{N}\left(r, \frac{1}{f'}\right) + 2\bar{N}_{(2}(r, f) + S(r, f).$$

Combining with $\bar{N}(r, \frac{1}{f'}) = S(r, f)$ and (4.13) we find

$$T(r, W_1) = S(r, f). \tag{4.15}$$

If z_0 is a zero of $f' - \beta$ of order $p \ge 3$ and $\beta(z_0) \ne 0, \infty$, then

$$\frac{f''}{f'} = \frac{\beta'}{\beta} + O\left((z - z_0)^{p-1}\right). \tag{4.16}$$

Substituting (4.16) into (4.14), W_1 is changed to

$$W_1 = \left(\frac{\beta'}{\beta}\right)^2 - 2\left(\frac{\beta'}{\beta}\right)' + O\left((z - z_0)^{p-2}\right). \tag{4.17}$$

If $W_1 \equiv (\beta'/\beta)^2 - 2(\beta'/\beta)'$, then (4.14) becomes

$$\left(\frac{f''}{f'} - \frac{\beta'}{\beta}\right)\left(\frac{f''}{f'} + \frac{\beta'}{\beta}\right) = 2\left(\frac{f''}{f'} - \frac{\beta'}{\beta}\right)'.$$

Hence $(f''/f') - (\beta'/\beta) = O(1)$, which contradicts with (4.16). Therefore $W_1 \not\equiv (\beta'/\beta)^2 - 2(\beta'/\beta)'$, and so, from (4.17) and (4.15) we see

$$\begin{aligned} N_{(3}\left(r, \frac{1}{f'-\beta}\right) &\le 3N\left(r, \frac{1}{W_1 - (\beta'/\beta)^2 + 2(\beta'/\beta)'}\right) \\ &\le 3T(r, W_1) + S(r, f) = S(r, f). \end{aligned}$$

Together with (4.9) and (4.13) we have $N_{(2}(r, \frac{1}{f'-\beta}) = S(r, f)$. Thus by this and (4.13),

$$N\Big(r, \frac{1}{f-\beta}\Big) = N_{1)}\Big(r, \frac{1}{f-\beta}\Big) + S(r, f) = N_{1)}\Big(r, \frac{1}{f'-\beta}\Big) + S(r, f)$$
$$= N\Big(r, \frac{1}{f'-\beta}\Big) + S(r, f),$$

which, in view of f and f' share the value IM, leads to f and f' share β CM "at most". Using an argument similar to that in the proof of Theorem 1.5, we arrive at the conclusion (1.1). From this it is easy to see that $\bar{N}(r, \frac{1}{f'}) \neq S(r, f)$, a contradiction. If $H \equiv 3/2$, from this and (4.11) we find

$$\bar{N}_{1)}(r, f) + N_{(2}\left(r, \frac{1}{f-\beta}\right) = S(r, f). \tag{4.18}$$

We set

$$\Phi = 2\frac{f''}{f'} - 3\frac{f'}{f}. \tag{4.19}$$

Then it is clear that $m(r, \Phi) = S(r, f)$ and if z_∞ is a pole of f of order 2, from (4.19) we see that Φ is holomorphic at z_∞. Thus, from (4.18), (4.9) and $\bar{N}(r, \frac{1}{f'}) + \bar{N}(r, \frac{1}{f}) = S(r, f)$,

$$T(r, \Phi) \le N_{1)}(r, f) + \bar{N}_{(3}(r, f) + \bar{N}\Big(r, \frac{1}{f'}\Big) + \bar{N}\Big(r, \frac{1}{f}\Big) = S(r, f). \tag{4.20}$$

Similarly according to the above discussion, we arrive at the result either $\Phi \not\equiv 2\frac{\beta'}{\beta} - 3$, and so

$$\bar{N}_{(2}\Big(r, \frac{1}{f'-\beta}\Big) \le \bar{N}\Big(r, \frac{1}{\Phi - 2\frac{\beta'}{\beta} + 3}\Big) + S(r, f)$$
$$\le T(r, \Phi) + S(r, f) = S(r, f).$$

From this, (4.18), (3.12) and (3.15) we reach the contradiction $\bar{N}(r, f) = S(r, f)$. Or $\Phi \equiv 2\frac{\beta'}{\beta} - 3$. Combining with (4.19) we obtain $2(\frac{f''}{f'} - \frac{\beta'}{\beta}) \equiv 3\frac{f'}{f} - 3$. Hence, by direct integration, we have $f'^2 = c\beta^2 e^{-3z} f^3$, where c is a nonzero constant. Because of f and f' share β IM and $\bar{N}(r, \frac{1}{f-\beta}) \neq S(r, f)$, the last equation becomes $f^{-3/2} f' = \beta^{-1/2}$, where $\beta = \frac{e^z}{\sqrt[3]{c}}$. Then by integration, we conclude the contradiction $T(r, f) = S(r, f)$.

Case II. $\Gamma \equiv 0$. From (3.8) we know that $2T(r, f) = T(r, f') + S(r, f)$. From this it is easy to see that $m(r, f) + N_{(2}(r, f) = S(r, f)$. It follows from this and (4.11) that $H \equiv 2$. We write (4.1) in the form $2 \equiv (\frac{f''}{f'} - \frac{\beta'}{\beta})(\frac{f-\beta}{f'-\beta})$, and eliminating $f' - \beta$ between this and (3.8) gives

$$2(f - \beta)f' = c(f''\beta - \beta' f'). \tag{4.21}$$

Differentiating (3.8) and then using (4.21), we arrive at $\beta' f \equiv 0$, which results in $\beta' \equiv 0$, so that β is a constant and rewrite (3.8) as

$$f' = \frac{1}{c\beta}[f + \beta(i\sqrt{c} - 1)][f - \beta(i\sqrt{c} + 1)], \tag{4.22}$$

and so, from $\bar{N}(r, \frac{1}{f'}) = S(r, f)$,

$$\bar{N}\Big(r, \frac{1}{f+\beta(i\sqrt{c}-1)}\Big) + \bar{N}\Big(r, \frac{1}{f-\beta(i\sqrt{c}+1)}\Big) = S(r, f).$$

Hence, employing Lemma 2.1 and $\bar{N}(r, \frac{1}{f}) = S(r, f)$ we find $T(r, f) = S(r, f)$. This is impossible unless $c = -1$. Thus (4.22) reads $\frac{f'}{f-2\beta} - \frac{f'}{f} = -2$. By integration once, we conclude (1.2) and the required result is proved. □

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