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Nonlinear anisotropic parabolic equations in L^m

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Abstract. In this paper, we give a result of regularity of weak solutions for a class of nonlinear anisotropic parabolic equations with lower-order term when the right-hand side is an L^m function, with m being ''small''. This work generalizes some results given in [2] and [3].

Mathematics Subject Classification: 34A60; 34B18; 34B15

Keywords: Nonlinear parabolic equations; Anisotropic equations; L^m data

1. INTRODUCTION

In this work we study the regularity of the solution of the following nonlinear anisotropic parabolic equation

$$(P)\qquad \begin{cases} \partial_t u - \operatorname{div}(\hat{a}(t,x,u,Du)) + F(t,x,u) = f & \text{in } Q, \\ u(0,x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

where Ω is a smooth bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$, $T > 0$ a real number, $Q = (0,T)\times\Omega$, and $\hat{a} : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, a.e. (t,x) in Q and $\forall u \in \mathbb{R}$ and $\forall \xi, \xi' \in \mathbb{R}^N$, the following:

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$$\hat{a}(t,x,u,\xi)\xi \geqslant \alpha\sum_{i=1}^{N}|\xi_i|^{p_i}, \quad \hat{a}(\cdot) = (a_1(\cdot),\ldots,a_N(\cdot)), \tag{1}$$

$$|a_i(t,x,u,\xi)| \leqslant \beta\left(b + |u|^{\bar{p}} + \sum_{j=1}^{N}|\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}}, \quad i = 1,\ldots,N, \tag{2}$$

$$(\hat{a}(t,x,u,\xi) - \hat{a}(t,x,u,\xi'))(\xi - \xi') > 0, \quad \xi \neq \xi', \tag{3}$$

where α, β are strictly positive real numbers, b is a given positive function in $L^1(Q)$, and $p_i > 1$, $i = 1,\ldots,N$.

Let $F(t,x,u) : Q \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following conditions:

$$\sup_{|\sigma|\leqslant\lambda}|F(t,x,\sigma)| \in L^1(Q), \quad \forall\lambda > 0, \tag{4}$$

$$F(t,x,u)\,\mathrm{sign}(u) \geqslant 0, \quad \text{a.e.}(t,x) \in Q, \ \forall u \in \mathbb{R}. \tag{5}$$

We assume that $f \in L^m(Q)$ when

$$1 < m < \frac{(N+2)\bar{p}}{(N+2)\bar{p} - N}, \quad \frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}, \quad m' = \frac{m}{m-1}, \tag{6}$$

where the exponents $p_1,\ldots,p_N$ are restricted as follows:

$$\begin{cases} 1 + \frac{N}{N+1} < p_i < \frac{\bar{p}(2-m+N)}{(1-m)(2+N)\bar{p}+mN}, \quad i = 1,\ldots,N \\ \bar{p} < \frac{N(N+2)}{m(N+1)}. \end{cases} \tag{7}$$

As a prototype example, we consider the model problem

$$\begin{cases} \partial_t u - \sum_{i=1}^{N}(D_i(|D_i u|^{p_i-2}D_i u) - |u|^{p_i-1}u) = f & \text{in } Q, \\ u(0,x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

with $f \in L^m(Q)$, m as in (6), and $p_1, p_2,\ldots,p_N$ are restricted as in (7).

Anisotropic operators are involved today in various branches of applied science. In some cases, they provide realistic models for the study of natural phenomena in biology and fluid mechanics (see references of [6]). In the isotropic case ($p_1 = p_2 = ,\ldots, = p_N = p$), it has been proved in [2] that there exists a weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ to nonlinear parabolic equations with $p > 1 + \frac{N}{N+1}$ and $1 < m < \frac{(N+2)p}{(N+2)p-N}$. Existence and regularity of solutions to nonlinear anisotropic elliptic equations, with $F = 0, 1 < m < \frac{N\bar{p}}{N\bar{p}-N+\bar{p}}$, and $2 - \frac{1}{N} < p_i < \frac{(N-1)\bar{p}}{N-\bar{p}}, \bar{p} < N$, has been proved in [3]. Recently in [7,6], the author has discussed the existence of solutions for anisotropic parabolic equations with measures (or $L^1(0,T;L^1\log L^1(\Omega))$) data under the assumptions

$$\begin{cases} 1 + \frac{N}{N+1} < p_i < \frac{\bar{p}(N+1)}{N}, \quad i = 1,\ldots,N, \\ \bar{p} \leqslant \frac{N(N+2)}{N+1}. \end{cases}$$

In this paper, we treat the anisotropic parabolic case with $f \in L^m(Q), 1 < m < (N+2)\bar{p}/[(N+2)\bar{p} - N]$, and p_i for $i = 1, \dots, N$ are restricted as in (7).

Before giving our results, it seems of interest to make some remarks.

Remark 1.1. We note that the assumption $p_i > 1 + \frac{N}{N+1}$ for $i = 1, \dots, N$ is usual when we search for weak solutions to problems, as (P), with L^m-data ($m > 1$ small). It corresponds to the well-known isotropic condition $p > 1 + \frac{N}{N+1}$.

Remarks 1.1. Let $p_i > 1 + \frac{N}{N+1}, i = 1, \dots, N$.

1. We have

$$\frac{\bar{p}(N+1)}{N} < \frac{\bar{p}(2 - m + N)}{(1-m)(2+N)\bar{p} + mN}, \quad \forall m > 1.$$

2. In the isotropic case ($p_i = \bar{p} = p$), the assumption

$$p < \frac{p(2 - m + N)}{(1-m)(2+N)p + mN}$$

is true for all $m > 1$.

3. If $\bar{p} \leqslant N$, we can write

$$\frac{(N+2)\bar{p}}{(N+2)\bar{p} - N} - \frac{N(N+2)}{\bar{p}(N+1)} = \frac{(N+2)((N+1)\bar{p} - N)(\bar{p} - N)}{\bar{p}(N+1)((N+2)\bar{p} - N)} \leqslant 0.$$

Then, the assumption $\bar{p} < \frac{N(N+2)}{m(N+1)}$ is true for all $m \in \left(1, \frac{(N+2)\bar{p}}{(N+2)\bar{p} - N}\right)$.

2. On some anisotropic Sobolev spaces

Let Ω be a bounded smooth open subset of $\mathbb{R}^N$ and $\vec{p} = (p_1, \dots, p_N) \in \mathbb{R}^N$ with $p_i \geqslant 1$ for $i = 1, \dots, N$. We define the anisotropic space $W_0^{1,p_i}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_i = \|u\|_{L^{p_i}(\Omega)} + \|D_i u\|_{L^{p_i}(\Omega)}, \quad D_i u = \frac{\partial u}{\partial x_i}.$$

Its dual is denoted by $\left(W_0^{1,p_i}(\Omega)\right)'$. Next, we put

$$X_0^{1,\vec{p}}(\Omega) = \bigcap_{i=1}^{N} W_0^{1,p_i}(\Omega) \quad \text{and} \quad \mathbb{L}^{\vec{p}}\left(0, T; X_0^{1,\vec{p}}(\Omega)\right) = \bigcap_{i=1}^{N} L^{p_i}\left(0, T; W_0^{1,p_i}(\Omega)\right)$$

with

$$\|u\|_{X_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^{N} \|D_i u\|_{L^{p_i}(\Omega)}, \quad \|u\|_{\mathbb{L}^{\vec{p}}\left(0,T;X_0^{1,\vec{p}}(\Omega)\right)} = \sum_{i=1}^{N} \|u\|_{L^{p_i}\left(0,T;W_0^{1,p_i}(\Omega)\right)}.$$

Then one has the following Lemma.

Lemma 2.1 (*Anisotropic Sobolev inequality, [10]*). *Let $\vec{p} = (p_1, p_2, \ldots, p_N)$ with $p_i \geqslant 1$ and $u \in X_0^{1,\vec{p}}(\Omega)$. Then*

$$\|u\|_{L^r(\Omega)} \leqslant C\left(\prod_{i=1}^{N} \|D_i u\|_{L^{p_i}(\Omega)}\right)^{\frac{1}{N}}, \tag{8}$$

where $r = \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$ if $\bar{p} < N$ with $\bar{p}$ given by $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$. The constant C depends on N and p_i, $i = 1, \ldots, N$. Furthermore, if $\bar{p} \geqslant N$, the inequality (8) is true for all $r \geqslant 1$ and C depends on r and $|\Omega|$.

Possible references on the theory of anisotropic Sobolev spaces are [10,1].

3. Statements of results

Definition 3.1. A function u is a weak solution of problem (P) if:

$$u \in L^1(0, T; W_0^{1,1}(\Omega)), \hat{a}(t, x, u, Du) \in L^1(Q)^N F(t, x, u) \in L^1(Q),$$

and

$$-\int_Q u\partial_t\varphi \; dxdt + \int_Q \hat{a}(t, x, u, Du)D\varphi \; dxdt + \int_Q F(t, x, u)\varphi \; dxdt = \int_Q \varphi f \; dxdt,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ which is zero in a neighborhood of $(0, T) \times \partial\Omega$ and $\{T\} \times \Omega$.

The main result of this paper is the following.

Theorem 3.1. *Let p_i, $i = 1, \ldots, N$ are restricted as in (7), m as in (6), and $f \in L^m(Q)$. Then under the assumptions (1)–(5),* the *problem (P) has at least one weak solution $u \in \mathbb{L}^{\vec{q}}\left(0, T; X_0^{1,\vec{q}}(\Omega)\right)$ where*

$$q_i = \frac{((N+1)\bar{p} - N)mp_i}{(N+2-m)\bar{p}}, \quad i = 1, \ldots, N.$$

Proof. The proof needs three steps.

Step1: Approximation.

Let $(f_n) \subset C_0^\infty(Q)$ such that

$$f_n \to f \quad \text{strongly in} \quad L^m(Q), \text{ as } n \to +\infty,$$

and

$$\|f_n\|_{L^m(Q)} \leqslant \|f\|_{L^m(Q)}, \quad \forall n \geqslant 1.$$

We define the problems (P_n) by

$$(P_n)\quad \begin{cases} \partial_t u_n - \operatorname{div}(\hat{a}(t,x,u_n,Du_n)) + F(t,x,u_n) = f_n & \text{in } Q, \\ u_n(0,x) = 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

The existence of a solution u_n in $\mathbb{L}^{\vec{p}}(0,T;X_0^{1,\vec{p}}(\Omega)) \cap C([0,T];L^2(\Omega))$ of the problem (P_n) is classical; see for instance [5].

Remark 3.1.

Under the assumption $f \in L^m(Q)$ in Theorem 3.1, we can deduce that f is never in the dual space $\left(\mathbb{L}^{\vec{p}}\left(0,T;X_0^{1,\vec{p}}(\Omega)\right)\right)'$, so that the result of this paper deals with irregular data as in [6,7] . If m tends to be 1, then q_i tends to be $\frac{p_i}{\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right)$, which is bound on q_i obtained in [4,7].

In the remainder of this paper, we denote by C or $C_j, j \in \mathbb{N}^*$, various positive constants depending only on the structure of $\hat{a}, F, |\Omega|, |Q|, \|f\|_{L^m(Q)}$, and T, never on n.

Step2: Uniform estimates on (u_n).

Lemma 3.1. *Let m as in (6). Assume (1)–(5) hold and the corresponding exponents $p_i, i = 1, \ldots, N$ are restricted as in (7). Then there exists a constant $C_1 > 0$ such that for all $n \geqslant 1$:*

$$\|u_n\|_{\mathbb{L}^{\vec{q}}\left(0,T;X_0^{1,\vec{q}}(\Omega)\right)} \leqslant C_1, \quad q_i = \frac{((N+1)\bar{p} - N)mp_i}{(N+2-m)\bar{p}}.$$

and

$$\|u_n\|_{L^{\bar{q}}(Q)} \leqslant C_1, \quad \frac{1}{\bar{q}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i}. \tag{9}$$

Proof of Lemma 3.1. By the definition of q_i and $\bar{q}$, we set $q_i = \theta p_i, \theta = \frac{\bar{q}}{\bar{p}}$. Thanks to (6), we have $\theta \in (0,1)$.

Let $\delta \in (0,1)$, $\tau \in (0,T)$, choosing

$$\varphi(u) = ((1+|u|)^{1-\delta} - 1)\operatorname{sgn}(u)\chi_{(0,\tau)}$$

as test function in (P_n), by (1), we obtain

$$\int_\Omega dx \int_0^{u_n(\tau,x)} \varphi(\sigma)d\sigma + (1-\delta)\alpha\sum_{i=1}^{N}\int_0^\tau \int_\Omega \frac{|D_i u_n|^{p_i}}{(1+|u_n|)^\delta}dxdt + \int_0^\tau \int_\Omega F(t,x,u_n)\varphi(u_n)dxdt \leqslant \int_0^\tau \int_\Omega |f_n||\varphi(u_n)|dxdt. \tag{10}$$

Observing that there exist two positive constants C_2 and C_3 such that

$$\forall\sigma \in \mathbb{R}, \quad \int_0^{\sigma} \varphi(t)dt \geqslant C_2|\sigma|^{2-\delta} - C_3,$$

by (10) and (5), we can write

$$C_4\|u_n\|^{2-\delta}_{L^\infty(0,T;L^{2-\delta}(\Omega))} + \alpha(1-\delta)\sum_{i=1}^{N}\int_Q \frac{|D_i u_n|^{p_i}}{(1+|u_n|)^{\delta}}dxdt$$
$$\leqslant C + C\|f_n\|_{L^m(Q)}\left(\int_Q (1+|u_n|)^{(1-\delta)m'}dxdt\right)^{1/m'}. \tag{11}$$

So that

$$\|u_n\|_{L^\infty(0,T;L^{2-\delta}(\Omega))} \leqslant C + C\left(\int_Q (1+|u_n|)^{(1-\delta)m'}dxdt\right)^{1/(2-\delta)m'}, \tag{12}$$

and

$$\sum_{i=1}^{N}\int_Q \frac{|D_i u_n|^{p_i}}{(1+|u_n|)^{\delta}}dxdt \leqslant C + C\left(\int_Q (1+|u_n|)^{(1-\delta)m'}dxdt\right)^{1/m'}. \tag{13}$$

Next, writing

$$y_{ni} = \int_Q |D_i u_n|^{q_i}dxdt = \int_Q \frac{|D_i u_n|^{q_i}}{(1+|u_n|)^{\delta\theta}}(1+|u_n|)^{\delta\theta},$$

and using Hölder inequality, the inequality (13) gives

$$y_{ni} \leqslant \left(C + C\left(\int_Q (1+|u_n|)^{(1-\delta)m'}dxdt\right)^{1/m'}\right)^{\theta}\left(\int_Q (1+|u_n|)^{\delta\frac{\theta}{1-\theta}}\right)^{1-\theta}, \tag{14}$$

for all $i = 1, \ldots, N$. Now we choose δ, such that

$$(1-\delta)m' = \frac{\delta\theta}{1-\theta} = \frac{N+2-\delta}{N}\bar{q} \tag{15}$$

so that

$$\delta = \frac{(N+2)(\bar{p}-\bar{q})}{N+\bar{p}-\bar{q}}. \tag{16}$$

Putting $d = \frac{N+2-\delta}{N}\bar{q}$, we obtain

$$d = \frac{((N+1)\bar{p}-N)m}{N+\bar{p}-m\bar{p}}. \tag{17}$$

Now, by (14) and (15), we can write

$$y_{ni}^{\frac{\bar{q}}{Nq_i}} \leqslant C\left(1 + \int_Q (1+|u_n|)^{d}dxdt\right)^{(1-\frac{\theta}{m})\frac{\bar{q}}{Nq_i}},$$

this implies that

$$\prod_{i=1}^{N} y_{ni}^{\frac{\bar{q}}{Nq_i}} \leqslant C^N \left(1 + \int_Q (1+|u_n|)^d dxdt\right)^{(1-\frac{\theta}{m})}. \tag{18}$$

Now, observe that

$$\bar{p} > 1 + \frac{N}{N+1} \Rightarrow \bar{p} > \frac{2N}{m+N} \Rightarrow 2-\delta < d < \bar{q}^*.$$

Using the interpolation inequality, we get

$$\|u_n(t,.)\|_{L^d(\Omega)} \leqslant \|u_n(t,.)\|_{L^{2-\delta}(\Omega)}^{1-\tau} \|u_n(t,.)\|_{L^{\bar{q}^*}(\Omega)}^{\tau}, \quad \tau = \frac{(2-\delta-d)\bar{q}^*}{(2-\delta-\bar{q}^*)d}. \tag{19}$$

By (16), (17), and (19), we see that

$$\tau = \frac{N}{N+2-\delta} \quad \text{and} \quad d\tau = \bar{q}.$$

We combine (12) and (19), we can write

$$\begin{aligned}
\int_0^T \|u_n\|_{L^d(\Omega)}^d dt &\leqslant \|u_n\|_{L^\infty(0,T;L^{2-\delta}(\Omega))}^{(1-\tau)d} \int_0^T \|u_n\|_{L^{\bar{q}^*}(\Omega)}^{d\tau} dt \\
&\leqslant C\left(1 + \int_Q (1+|u_n|)^d dxdt\right)^{\frac{(1-\tau)d}{(2-\delta)m'}} \int_0^T \|u_n\|_{L^{\bar{q}^*}(\Omega)}^{\bar{q}} dt \\
&= C\left(1 + \int_Q (1+|u_n|)^d dxdt\right)^{\frac{\bar{q}}{Nm'}} \int_0^T \|u_n\|_{L^{\bar{q}^*}(\Omega)}^{\bar{q}} dt
\end{aligned} \tag{20}$$

Let us apply Lemma 2.1, we get

$$\int_0^T \|u_n\|_{L^{\bar{q}^*}(\Omega)}^{\bar{q}} dt \leqslant C \int_0^T \prod_{i=1}^{N} \left(\int_\Omega |D_i u_n|^{q_i} dx\right)^{\frac{\bar{q}}{Nq_i}} dt.$$

The fact that $\sum_{i=1}^N \frac{\bar{q}}{Nq_i} = 1$ and the generalized Hölder inequality, lead to

$$\int_0^T \|u_n\|_{L^{\bar{q}^*}(\Omega)}^{\bar{q}} dt \leqslant C \prod_{i=1}^{N} \left(\int_Q |D_i u_n|^{q_i} dxdt\right)^{\frac{\bar{q}}{Nq_i}}. \tag{21}$$

From (18), (20), and (21) we can write

$$\begin{aligned}
\int_Q |u_n|^d dxdt &\leqslant C\left(1 + \int_Q |u_n|^d dxdt\right)^{1+\frac{\bar{q}}{Nm'}-\frac{\theta}{m}} \\
&= C\left(1 + \int_Q |u_n|^d dxdt\right)^{1+\frac{\bar{q}}{N}-\frac{\bar{q}}{Nm}-\frac{\bar{q}}{m\bar{p}}}.
\end{aligned} \tag{22}$$

Because

$$1 + \frac{\bar{q}}{N} - \frac{\bar{q}}{Nm} - \frac{\bar{q}}{m\bar{p}} < 1 \Longleftrightarrow m < \frac{N}{\bar{p}} + 1,$$

and

$$\frac{(N+2)\bar{p}}{(N+2)\bar{p}-N} < \frac{N}{\bar{p}} + 1,$$

thanks to (6), the inequality (22) implies that the sequence (u_n) is bounded on $L^d(Q)$. Which then yields, by (14), a bound on the norm of $(D_i u_n)$ in $L^{q_i}(Q)$ where q_i is given by (16). Thus (9) follows from (21). This finished the proof of Lemma 3.1. □

Remark 3.2. The condition $\bar{p} < \frac{N(N+2)}{m(N+1)}$ guarantees that $\bar{q} < N$.

Lemma 3.2. *There exists a constant C_4 such that*

$$\|F(.,.,u_n)\|_{L^1(Q)} \leqslant C_4, \quad \forall n \geqslant 1.$$

Proof of Lemma 3.2. By (10), (15), and (22), we can write

$$\int_Q F(t,x,u_n)\varphi(u_n)dxdt \leqslant C + C\|f_n\|_{L^m(Q)}\left(\int_Q (1+|u_n|)^{(1-\delta)m'}dxdt\right)^{1/m'} \leqslant C.$$

Using the above estimate and assumptions (4) and (5), we get

$$\begin{aligned}\int_Q |F(t,x,u_n)|dxdt &\leqslant \int_{Q\cap\{|u_n|\leqslant 1\}} |F(t,x,u_n)|dxdt + \frac{1}{\varphi(1)}\int_Q F(t,x,u_n)\varphi(u_n)dxdt \\ &\leqslant C. \qquad \square\end{aligned}$$

Lemma 3.3. *Assume that $p_i, i = 1, \ldots, N$ are restricted as in (7). Then*

$$r_i = \frac{m(N+1)p_i}{(2-m+N)(p_i-1)\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right) > 1, \quad i = 1, \ldots, N.$$

The sequence $(u'_n) = (\partial_t u_n)$ remains in a bounded set of $L^{r_-}(0,T;(X_0^{1,\vec{r}'}(\Omega))') + L^1(Q), \vec{r}' = (r'_1, \ldots, r'_N), r'_i$ is the conjugate r_i.

Proof of Lemma 3.3. Since $p_i < \frac{\bar{p}(2-m+N)}{(1-m)(2+N)\bar{p}+mN}$, we have $r_i > 1$. Knowing that $(f_n - F(.,.,u_n))$ is in the bounded set of $L^1(Q)$, we have to show that

$$v_n = \text{div } (\hat{a}(t,x,u_n,Du_n))$$

belongs to a bounded set of $L^{r_-}\left(0,T;\left(X_0^{1,\vec{r}'}(\Omega)\right)'\right)$. By using the inequality (11) of [7], we have

$$\|v_n\|^{r_-}_{(X_0^{1,\vec{r}'}(\Omega))'} \leqslant C\sum_{i=1}^N\left(\int_\Omega (b+|u_n|^{\bar{p}} + \sum_{j=1}^N |D_j u_n|^{p_j})^{(1-\frac{1}{p_i})r_i}dx\right)^{r_-/r_i}.$$

Let

$$s = \left(1-\frac{1}{p_i}\right)r_i = \frac{m(N+1)}{(2-m+N)\bar{p}}\left(\bar{p} - \frac{N}{N+1}\right) = \frac{\bar{q}}{\bar{p}} < 1.$$

Then

$$\|v_n\|^{r_-}_{(X_0^{1,\vec{r}}(\Omega))'} \leqslant C\sum_{i=1}^{N}\left(\int_\Omega \left(b^s + |u_n|^{\bar{q}} + \sum_{j=1}^{N}|D_j u_n|^{q_j}\right)dx\right)^{r_-/r_i}.$$

Using this inequality, (9), and Lemma 3.1, we have

$$\int_0^T \|v_n\|^{r_-}_{(X_0^{1,\vec{r}}(\Omega))'} \leqslant C, \quad \forall n \geqslant 1.$$

Step3: Passage to the limit.

Let $r_- = \min\{r_i, \quad i = 1, \ldots, N\}$ (r_i given as in Lemma 3.3). Then, thanks to Lemma 3.3, we have that (u_n') remains in a bounded set of the space

$$L^{r_-}\left(0, T; \left(X_0^{1,\vec{r}}(\Omega)\right)'\right) + L^1(Q) \subset L^{r_-}(0, T; W^{-1,r_-}(\Omega)) + L^1(Q).$$

By Lemma 3.1, (u_n) remains in a bounded set of

$$\mathbb{L}^{\vec{q}}\left(0, T; X_0^{1,\vec{q}}(\Omega)\right) \subset L^{q_-}\left(0, T; W_0^{1,q_-}(\Omega)\right), \quad q_i = \frac{((N+1)\bar{p} - N)mp_i}{(N+2-m)\bar{p}}.$$

Using Corollary 4 in [9], we obtain that

$$u_n \to u \quad \text{strongly in} \quad L^1(Q) \quad \text{and a.e. in} \quad Q. \tag{23}$$

Now, adopting the approach of [7], there exists a subsequence (still denoted (u_n)) such that

$$Du_n \to Du \quad \text{a.e. on} \quad Q. \tag{24}$$

Since $\hat{a}$ is a Carathéodory (23), (24), (2), and Lemma 3.1, we get for all $i = 1, \ldots, N$

$$a_i(t, x, u_n, Du_n) \to a_i(t, x, u, Du) \quad \text{weakly in } L^{r_i}(Q), \tag{25}$$

where r_i given as in Lemma 3.3. Using the fact that F is a Carathéodory function, (4), (5), (23), and (24), we work in exactly the same way as that of Lemma 4.5 in [8], we obtain that

$$F(t, x, u_n) \to F(t, x, u) \quad \text{strongly in} \quad L^1(Q). \tag{26}$$

By (23), (25), (26) we can pass to the limit for $n \to +\infty$ in the weak formulation of (P_n) and we obtain that u is a weak solution for (P). □

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