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Multivalued and singlevalued fixed point results in partially ordered metric spaces

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Abstract Fixed point theory in partially ordered metric spaces has greatly developed in recent times. In this paper we prove certain fixed point theorems for multivalued and singlevalued mappings in such spaces. The mappings we consider here are assumed to satisfy certain metric inequalities in the case where the arguments of the functions are related by partial order. In one of our theorems we assume a weak contractive inequality. It is in the line with the research following the establishing of weak contraction principle in metric spaces [Rhoades BE. Some theorems on weakly contractive maps. Nonlinear Anal 2001;47(4):2683–93] and subsequently in partially ordered metric spaces [Harjani J, Sadarangani K. Fixed point theorems for weakly contractive mappings in partially ordered sets. Nonlinear Anal 2009;71:3403–10]. Two illustrative examples are also given.

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1. Introduction and mathematical preliminaries

Let (X, d) be a metric space. We denote the class of nonempty and bounded subsets of X by $B(X)$. For $A, B \in B(X)$, functions $D(A, B)$ and $\delta(A, B)$ are defined as follows:

$$D(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\},$$
$$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}.$$

If $A = \{a\}$, then we write $D(A, B) = D(a, B)$ and $\delta(A, B) = \delta(a, B)$. Also in addition, if $B = \{b\}$, then $D(A, B) = d(a, b)$ and $\delta(A, B) = d(a, b)$. Obviously, $D(A, B) \leqslant \delta(A, B)$. For all $A, B, C \in B(X)$, the definition of $\delta(A, B)$ yields the following:

$$\delta(A, B) = \delta(B, A),$$
$$\delta(A, B) \leqslant \delta(A, C) + \delta(C, B),$$
$$\delta(A, B) = 0 \quad \text{iff } A = B = \{a\},$$
$$\delta(A, A) = \operatorname{diam} A \quad (\text{Fisher, 1981; Fisher and Ise'ki, 1983}).$$

Fixed point theory of multivalued functions is a vast chapter of functional analysis. In particular, the function $\delta(A, B)$ has been used in many works in this area. Some of these works are noted in Choudhury (1996), Fisher (1981) and Fisher and Ise'ki (1983).

We will use the following relation between two nonempty subsets of a partially ordered set.

Definition 1.1 (Beg and Butt, 2010). Let A and B be two nonempty subsets of a partially ordered set $(X, \preceq)$. The relation between A and B is denoted and defined as follows: $A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

We will utilize the following control function which is also referred to as Altering distance function.

Definition 1.2 (Khan et al., 1984). A function $\psi : [0, \infty) \to [0, \infty)$ is called an Altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

The above control function has been utilized in a large number of works in metric fixed point theory. Some recent references are Choudhury (2005), Dorić (2009), Dutta and Choudhury (2008), Naidu (2003) and Sastry and Babu (1999). This control function has also been extended and applied to fixed point problems in probabilistic metric spaces (Choudhury and Das, 2009; Choudhury et al., 2009; Miheţ, 2009) and fuzzy metric spaces (Choudhury and Dutta, 2005).

The purpose of this paper is to establish the existence of fixed points of multivalued mappings in partially ordered metric spaces. The mappings are assumed to satisfy certain inequalities which involve the above mentioned control function. One of these inequalities, which has been used in Theorem 2.5, is a weak contractive type inequality. This type of inequality was considered by Alber and Guerre-Delabriere (1997) in Hilbert spaces where they established the weak contraction principle. Rhoades established that the weak contraction principle is also valid in an arbitrary complete metric space (Rhoades, 2001). The importance of weak contraction is that it is intermediate to a contraction and a non-expansive mapping. The former has a unique fixed point in a complete metric space whereas the latter need not have a fixed point. According to the result in Rhoades (2001) the weak contractions necessarily have fixed points in a complete metric space. Afterwards, weak contraction and functions satisfying weak contractive type inequalities have been considered in a large number of papers, some of which are noted in Chidume et al. (2002), Choudhury and Metiya (2010), Choudhury and Metiya (2010), Choudhury et al. (2011), Rouhani and Moradi (2010), Zhang and Song (2009). In particular, in partially ordered metric spaces, there are a number of such works (Altun and Simsek, 2010; Altun, 2011; Gnana Bhaskar and Lakshmikantham, 2006; Harjani and Sadarangani, 2009; Lakshmikantham and Ćirić, 2009; Nieto and Rodr guez-Lpez, 2005; Ran and Reurings, 2004; Samet, 2010; Zhang, 2010). Further we have established that in the corresponding single-valued cases a partial order condition of the metric space can be omitted if the function is continuous. Finally, we have concluded our paper with two illustrative examples.

2. Main results

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$,*
(ii) *for $x, y \in X, x \preceq y$ implies $Tx \prec_1 Ty$,*
(iii) *if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n,*
(iv) $\psi(\delta(Tx, Ty)) \leqslant \alpha\psi\left(\max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\right\}\right)$, for all comparable $x, y \in X$, where $0 < \alpha < 1$ and ψ is an altering distance function.

Then T has a fixed point.

Proof. By the assumption (i), there exists $x_1 \in Tx_0$ such that $x_0 \preceq x_1$. By the assumption (ii), $Tx_0 \prec_1 Tx_1$. Then there exists $x_2 \in Tx_1$ such that $x_1 \preceq x_2$. Continuing this process we construct a monotone increasing sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for all $n \geqslant 0$. Thus we have $x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$

If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \geqslant 0$.

Using the monotone property of ψ and the condition (iv), we have for all $n \geqslant 0$,

$$\begin{aligned}
&\psi(d(x_{n+1}, x_{n+2})) \leqslant \psi(\delta(Tx_n, Tx_{n+1}))\\
&\leqslant \alpha\psi\left(\max\left\{d(x_n, x_{n+1}), D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1}), \frac{D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)}{2}\right\}\right)\\
&\leqslant \alpha\psi\left(\max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\}\right).
\end{aligned}$$

Since $\frac{d(x_n,x_{n+2})}{2} \leqslant \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$\psi(d(x_{n+1}, x_{n+2})) \leqslant \alpha\psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{2.1}$$

Suppose that $d(x_n, x_{n+1}) \leqslant d(x_{n+1}, x_{n+2})$, for some positive integer n.

Then from (2.1), we have

$$\psi(d(x_{n+1}, x_{n+2})) \leqslant \alpha\psi(d(x_{n+1}, x_{n+2})),$$

which implies that $d(x_{n+1}, x_{n+2}) = 0$, or that $x_{n+1} = x_{n+2}$, contradicting our assumption that $x_n \neq x_{n+1}$, for each n.

Therefore, $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, for all $n \geqslant 0$ and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geqslant 0$ such that

$$d(x_n, x_{n+1}) \to r \quad \text{as } n \to \infty. \tag{2.2}$$

Taking the limit as $n \to \infty$ in (2.1) and using the continuity of ψ, we have

$$\psi(r) \leqslant \alpha\psi(r),$$

which is a contradiction unless $r = 0$.

$$\text{Hence} \quad \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.3}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers $k, n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geqslant \epsilon$.

Assuming that $n(k)$ is the smallest such positive integer, we get

$$\begin{aligned}
&n(k) > m(k) > k,\\
&d(x_{m(k)}, x_{n(k)}) \geqslant \epsilon \quad \text{and}\\
&d(x_{m(k)}, x_{n(k)-1}) < \epsilon.
\end{aligned}$$

Now,

$$\epsilon \leqslant d(x_{m(k)}, x_{n(k)}) \leqslant d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}),$$

that is, $\epsilon \leqslant d(x_{m(k)}, x_{n(k)}) < \epsilon + d(x_{n(k)-1}, x_{n(k)})$.

Taking the limit as $k \to \infty$ in the above inequality and using (2.3), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.4}$$

Again

$$d(x_{m(k)}, x_{n(k)}) \leqslant d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{m(k)+1}, x_{n(k)+1}) \leqslant d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}).$$

Taking the limit as $k \to \infty$ in the above inequalities and using (2.3) and (2.4), we have

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon. \tag{2.5}$$

Again,

$$d(x_{m(k)}, x_{n(k)}) \leqslant d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{m(k)}, x_{n(k)+1}) \leqslant d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}).$$

Letting $k \to \infty$ in the above inequalities and using (2.3) and (2.4), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon. \tag{2.6}$$

Similarly, we have that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \epsilon. \tag{2.7}$$

For each positive integer $k, x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ψ and the condition (iv), we have

$$\begin{aligned}
&\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leqslant \psi(\delta(Tx_{m(k)}, Tx_{n(k)}))\\
&\leqslant \alpha\psi\left(\max\left\{d(x_{m(k)}, x_{n(k)}), D(x_{m(k)}, Tx_{m(k)}), D(x_{n(k)}, Tx_{n(k)}), \frac{D(x_{m(k)}, Tx_{n(k)}) + D(x_{n(k)}, Tx_{m(k)})}{2}\right\}\right)\\
&\leqslant \alpha\psi\left(\max\left\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2}\right\}\right).
\end{aligned}$$

Letting $k \to \infty$ in above inequality, using (2.3)–(2.7) and using the continuity of ψ, we have

$$\psi(\epsilon) \leqslant \alpha\psi(\epsilon),$$

which is a contradiction by virtue of a property of ψ.

Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a $z \in X$ such that

$$x_n \to z \quad \text{as } n \to \infty. \tag{2.8}$$

By the assumption (iii), $x_n \preceq z$, for all n.

Then by the monotone property of ψ and the condition (iv), we have

$$\begin{aligned}\psi(\delta(x_{n+1}, Tz)) &\leqslant \psi(\delta(Tx_n, Tz))\\ &\leqslant \alpha\psi\left(\max\left\{d(x_n, z), D(x_n, Tx_n), D(z, Tz), \frac{D(x_n, Tz) + D(z, Tx_n)}{2}\right\}\right)\\ &\leqslant \alpha\psi\left(\max\left\{d(x_n, z), d(x_n, x_{n+1}), D(z, Tz), \frac{D(x_n, Tz) + d(z, x_{n+1})}{2}\right\}\right).\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, using (2.3) and (2.8) and the continuity of ψ, we have

$$\psi(\delta(z, Tz)) \leqslant \alpha\psi(D(z, Tz)) \leqslant \alpha\psi(\delta(z, Tz)),$$

which implies that $\delta(z, Tz) = 0$, or that $\{z\} = Tz$. Moreover, z is a fixed point of T. □

Taking ψ an identity function in Theorem 2.1, we have the following result.

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$,*
(ii) *for $x, y \in X$, $x \preceq y$ implies $Tx \prec_1 Ty$,*
(iii) *if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n,*
(iv) $\delta(Tx, Ty) \leqslant \alpha \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\right\}$, for all comparable $x, y \in X$, where $0 < \alpha < 1$.

Then T has a fixed point.

The following corollary is a special case of Theorem 2.1 when T is a singlevalued mapping.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *for $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$,*
(iii) *if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n,*

(iv) $\psi(d(Tx,Ty)) \leqslant \alpha\psi\Big(\max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\Big\}\Big)$, *for all comparable* $x, y \in X$, *where* $0 < \alpha < 1$ *and* ψ *is an altering distance function.*

Then T *has a fixed point.*

In the following theorem we replace condition (iii) of the above corollary by requiring T to be continuous.

Theorem 2.4. *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $T: X \to X$ *be a continuous mapping such that the following conditions are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$,
(ii) *for* $x, y \in X$, $x \preceq y$ *implies* $Tx \preceq Ty$,
(iii) $\psi(d(Tx,Ty)) \leqslant \alpha\psi\Big(\max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\Big\}\Big)$, *for all comparable* $x, y \in X$, *where* $0 < \alpha < 1$ *and* ψ *is an altering distance function.*

Then T *has a fixed point.*

Proof. We can treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. Arguing exactly as in the proof of Theorem 2.1, we have that $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = z$. Then, the continuity of T implies that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz$$

and this proves that z is a fixed point of T. $\square$

Theorem 2.5. *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $T: X \to B(X)$ *be a multivalued mapping such that the following conditions are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $\{x_0\} \prec_1 Tx_0$,
(ii) *for* $x, y \in X$, $x \preceq y$ *implies* $Tx \prec_1 Ty$,
(iii) *if* $x_n \to x$ *is a nondecreasing sequence in* X, *then* $x_n \preceq x$, *for all* n,
(iv) $\psi(\delta(Tx,Ty)) \leqslant \psi\Big(\max\Big\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\Big\}\Big) - \phi(\max\{d(x,y), \delta(y,Ty)\})$, *for all comparable* $x, y \in X$, *where* ψ *is an altering distance function and* $\phi : [0,\infty) \to [0,\infty)$ *is any continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$.

Then T *has a fixed point.*

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \geqslant 0$.

Using the monotone property of ψ and the condition (iv), we have for all $n \geqslant 0$,

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leqslant \psi(\delta(Tx_n, Tx_{n+1})) \\
&\leqslant \psi\left(\max\left\{d(x_n, x_{n+1}), D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1}), \frac{D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)}{2}\right\}\right) \\
&\quad - \phi(\max\{d(x_n, x_{n+1}), \delta(x_{n+1}, Tx_{n+1})\}) \\
&\leqslant \psi\left(\max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\}\right) \\
&\quad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}).
\end{aligned}$$

Since $\frac{d(x_n, x_{n+2})}{2} \leqslant \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leqslant \psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\
&\quad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}).
\end{aligned} \tag{2.9}$$

Suppose that $d(x_n, x_{n+1}) \leqslant d(x_{n+1}, x_{n+2})$, for some positive integer n.

Then from (2.9), we have

$$\psi(d(x_{n+1}, x_{n+2})) \leqslant \psi(d(x_{n+1}, x_{n+2})) - \phi(d(x_{n+1}, x_{n+2})),$$

that is, $\phi(d(x_{n+1}, x_{n+2})) \leqslant 0$, which implies that $d(x_{n+1}, x_{n+2}) = 0$, or that $x_{n+1} = x_{n+2}$, contradicting our assumption that $x_n \neq x_{n+1}$, for each n.

Therefore, $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, for all $n \geqslant 0$ and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists an $r \geqslant 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{2.10}$$

In view of the above facts, from (2.9) we have for all $n \geqslant 0$,

$$\psi(d(x_{n+1}, x_{n+2})) \leqslant \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})).$$

Taking the limit as $n \to \infty$ in the above inequality, using (2.10) and the continuities of ϕ and ψ, we have

$$\psi(r) \leqslant \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$.

Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then using an argument similar to that given in Theorem 2.1, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ for which

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon, \tag{2.12}$$

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon, \tag{2.13}$$

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon, \tag{2.14}$$

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \epsilon. \tag{2.15}$$

For each positive integer k, $x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ψ and the condition (iv), we have

$$\begin{aligned}
\psi(d(x_{m(k)+1}, x_{n(k)+1})) &\leqslant \psi(\delta(Tx_{m(k)}, Tx_{n(k)})) \\
&\leqslant \psi\Bigg(\max\Bigg\{d(x_{m(k)}, x_{n(k)}), D(x_{m(k)}, Tx_{m(k)}), D(x_{n(k)}, Tx_{n(k)}) \\
&\qquad \frac{D(x_{m(k)}, Tx_{n(k)}) + D(x_{n(k)}, Tx_{m(k)})}{2}\Bigg\}\Bigg) \\
&\qquad - \phi\big(\max\{d(x_{m(k)}, x_{n(k)}), \delta(x_{n(k)}, Tx_{n(k)})\}\big), \\
&\leqslant \psi(\max\Bigg\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \\
&\qquad \frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, Tx_{m(k)+1})}{2}\Bigg\}\Bigg) \\
&\qquad - \phi\big(\max\{d(x_{m(k)}, x_{n(k)}), d(x_{n(k)}, x_{n(k)+1})\}\big).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (2.11)–(2.15) and the continuities of ψ and ϕ, we have

$$\psi(\epsilon) \leqslant \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction by virtue of a property of ϕ.

Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a $z \in X$ such that

$$x_n \to z \quad \text{as } n \to \infty. \tag{2.16}$$

By the assumption (iii), $x_n \preceq z$, for all n.

Then by the monotone property of ψ and the condition (iv), we have

$$\begin{aligned}
\psi(\delta(x_{n+1}, Tz)) &\leqslant \psi(\delta(Tx_n, Tz)) \\
&\leqslant \psi\left(\max\left\{d(x_n, z), D(x_n, Tx_n), D(z, Tz), \frac{D(x_n, Tz) + D(z, Tx_n)}{2}\right\}\right) \\
&\quad - \phi(\max\{d(x_n, z), \delta(z, Tz)\}) \\
&\leqslant \psi\left(\max\left\{d(x_n, z), d(x_n, x_{n+1}), D(z, Tz), \frac{D(x_n, Tz) + d(z, x_{n+1})}{2}\right\}\right) \\
&\quad - \phi(\max\{d(x_n, z), \delta(z, Tz)\}).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, using (2.11) and (2.16) and the continuities of ψ and ϕ, we have

$$\psi(\delta(z, Tz)) \leqslant \psi(D(z, Tz)) - \phi(\delta(z, Tz)),$$

which implies that

$$\psi(\delta(z, Tz)) \leqslant \psi(\delta(z, Tz)) - \phi(\delta(z, Tz)),$$

which is a contradiction unless $\delta(z, Tz) = 0$, or that $\{z\} = Tz$; that is, z is a fixed point of T. $\square$

Taking ψ an identity function in Theorem 2.5, we have the following result.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$,*
(ii) *for $x, y \in X, x \preceq y$ implies $Tx \prec_1 Ty$,*
(iii) *if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n,*
(iv) $\delta(Tx, Ty) \leqslant \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\right\} - \phi(\max\{d(x, y), \delta(y, Ty)\})$, for all comparable $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$. *Then T has a fixed point.*

The following corollary is a special case of Theorem 2.5 when T is a singlevalued mapping.

Corollary 2.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *for $x, y \in X,\ x \preceq y$ implies $Tx \preceq Ty$,*
(iii) *if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n,*
(iv) $\psi(d(Tx, Ty)) \leqslant \psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right) - \phi(\max\{d(x,y), d(y, Ty)\})$*, for all comparable $x, y \in X$, where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$.*

Then T has a fixed point.

In the following theorem we replace condition (iii) of the above corollary by requiring T to be continuous.

Theorem 2.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *for $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$,*
(iii) $\psi(d(Tx, Ty)) \leqslant \psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)$
$-\phi(\max\{d(x,y), d(y,Ty)\})$, *for all comparable $x, y \in X$, where ψ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$.*

Then T has a fixed point.

Proof. We can treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.5. Arguing exactly as in the proof of Theorem 2.5, we have that $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = z$. Then, the continuity of T implies that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz$$

and this proves that z is a fixed point of T. $\square$

Remark 2.1. The Corollary 2.7 and Theorem 2.8 are the corresponding results in partially ordered metric spaces of Theorem 3.1 in Choudhury et al. (2011).

Example 1. Let $X = \left\{(0,0), (0,-\frac{1}{5}), \left(-\frac{1}{8}, 0\right)\right\}$ be a subset of R^2 with the order $\preceq$ defined as: for $(x_1, y_1), (x_2, y_2) \in X$, $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 \leqslant x_2, y_1 \leqslant y_2$. Let $d : X \times X \to \mathbb{R}$ be given as

$$d(x,y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}, \quad \text{for } x = (x_1, y_1),\ y = (x_2, y_2) \in X.$$

Then (X, d) is a complete metric space with the required properties of Theorems 2.1 and 2.5.

Let $T : X \to B(X)$ be defined as follows:

$$Tx = \begin{cases} \{(0,0)\}, & \text{if } x = (0,0), \\ \left\{(0,0), \left(-\frac{1}{8}, 0\right)\right\}, & \text{if } x = \left(0, -\frac{1}{5}\right), \\ \{(0,0)\}, & \text{if } x = \left(-\frac{1}{8}, 0\right). \end{cases}$$

Then T has the properties mentioned in Theorems 2.1 and 2.5.

Let $\psi : [0, \infty) \to [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$ be defined respectively as follows:

$$\psi(t) = t^2 \quad \text{and} \quad \phi(t) = t^3.$$

Then ψ and ϕ have the properties mentioned in Theorems 2.1 and 2.5.

Without loss of generality, we assume that $x \preceq y$ and discuss the following cases.

(i) If $x = (0, -\frac{1}{5})$ and $y = (0, 0)$, then $\delta(Tx, Ty) = \frac{1}{8}$,

$$\max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\right\} = \frac{1}{5} \quad \text{and}$$

$$\max\{d(x, y), \delta(y, Ty)\} = \frac{1}{5}.$$

(ii) If $x = \left(-\frac{1}{8}, 0\right)$ and $y = (0, 0)$, then $\delta(Tx, Ty) = 0$,

$$\max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\right\} = \frac{1}{8} \quad \text{and}$$

$$\max\{d(x, y), \delta(y, Ty)\} = \frac{1}{8}.$$

(iii) If $x = (0, 0)$ and $y = (0, 0)$, then $\delta(Tx, Ty) = 0$,

$$\max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\right\} = 0 \quad \text{and}$$

$$\max\{d(x, y), \delta(y, Ty)\} = 0.$$

(iv) If $x = (0, -\frac{1}{5})$ and $y = \left(0, -\frac{1}{5}\right)$, then $\delta(Tx, Ty) = \frac{1}{8}$,

$$\max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\right\} = \frac{1}{5} \quad \text{and}$$

$$\max\{d(x, y), \delta(y, Ty)\} = \frac{1}{5}.$$

(v) If $x = \left(-\frac{1}{8}, 0\right)$ and $y = \left(-\frac{1}{8}, 0\right)$, then $\delta(Tx, Ty) = 0$,

$$\max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\right\} = \frac{1}{8} \quad \text{and}$$

$$\max\{d(x, y), \delta(y, Ty)\} = \frac{1}{8}.$$

In above all cases, clearly,

$$\psi(\delta(Tx, Ty)) \leqslant \alpha\psi\left(\max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\right\}\right),$$
for $\alpha = \frac{1}{2}$,

and

$$\psi(\delta(Tx, Ty)) \leqslant \psi\left(\max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\right\}\right)$$
$$- \phi(\max\{d(x,y), \delta(y,Ty)\}).$$

Hence the conditions of Theorems 2.1 and 2.5 are satisfied and it is seen that $(0,0)$ is a fixed point of T.

Remark 2.2. The above example does not satisfy the Corollary 2.2 when $\alpha = \frac{1}{2}$. Hence Theorem 2.1 is a generalization of Corollary 2.2.

Remark 2.3. The condition (iv) of Corollary 2.7 or, alternately, the condition (iii) of Theorem 2.8 implies the condition (iv) of Corollary 2.3 and the condition (iii) of Theorem 2.4. For explanation, we start with the condition (iv) of Corollary 2.3

$$\psi(d(Tx, Ty)) \leqslant \alpha\psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)$$
$$= \psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)$$
$$- (1-\alpha)\psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)$$
$$\leqslant \psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)$$
$$- (1-\alpha)\psi(\max\{d(x,y), d(y,Ty)\}),$$

which is the special case of the condition (iv) of Corollary 2.7 or, alternately, the condition (iii) of Theorem 2.8 when ϕ defined by $\phi(t) = (1-\alpha)\psi(t)$. Hence the Corollary 2.6 and Theorem 2.8 are more general than Corollary 2.3 and Theorem 2.4, respectively. The following example demonstrate that the generalizations are actual.

Example 2. Let $X = \{0, 1, 2, 3, \ldots\}$ with the usual order $\leqslant$ be a partially ordered set. Let $d : X \times X \to \mathbb{R}$ be given as

$$d(x,y) = \begin{cases} x+y, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X, d) is a complete metric space with the required properties of Corollary 2.7 and Theorem 2.8.

Let $\psi : [0, \infty) \to [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$ be defined respectively as follows:

$$\psi(t) = t^2 \quad \text{and} \quad \phi(t) = \begin{cases} \frac{t^2}{2}, & \text{if } t \leqslant 1, \\ \frac{1}{2}, & \text{if } t > 1. \end{cases}$$

Then ψ and ϕ have the properties mentioned in Corollary 2.7 and Theorem 2.8.

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x - 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T has the properties mentioned in Corollary 2.7 and Theorem 2.8.

We discuss following cases for $x, y \in X$.

(i) If $x > y$ and $y \neq 0$, then

$$\psi(d(Tx, Ty)) = \psi(d(x-1, y-1)) = \psi(x + y - 2) = (x + y - 2)^2,$$

$$\begin{aligned} &\psi(\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}) \\ &= \psi\left(\max\left\{x + y, 2x - 1, 2y - 1, x + y - 1 \quad \text{or} \quad \frac{x + y - 1}{2}\right\}\right) \\ &= (2x - 1)^2 \end{aligned}$$

(since $\frac{1}{2}[d(x,Ty) + d(y,Tx)] = x + y - 1$ or $\frac{x+y-1}{2}$, according as $y \neq Tx$ or $y = Tx$) and

$$\phi(\max\{d(x,y), d(y,Ty)\}) = \phi(\max\{x + y, 2y - 1\}) = \frac{1}{2}.$$

(ii) If $y > x$ and $x \neq 0$, then

$$\psi(d(Tx, Ty)) = \psi(d(x-1, y-1)) = \psi(x + y - 2) = (x + y - 2)^2,$$

$$\begin{aligned} &\psi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\right\}\right) \\ &= \psi\left(\max\left\{x + y, 2x - 1, 2y - 1, x + y - 1 \text{ or } \frac{x + y - 1}{2}\right\}\right) \\ &= (2y - 1)^2 \end{aligned}$$

(Since $\frac{1}{2}[d(x,Ty) + d(y,Tx)] = x + y - 1$ or $\frac{x+y-1}{2}$, according as $x \neq Ty$ or $x = Ty$) and

$$\phi(\max\{d(x,y),d(y,Ty)\}) = \phi(\max\{x+y,2y-1\}) = \frac{1}{2}.$$

(iii) If $x > y$ and $y = 0$, then

$$\begin{aligned}\psi(d(Tx,Ty)) &= \psi(d(x-1,0)) = \psi(x-1)\\ &= (x-1)^2, \psi\left(\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\}\right)\\ &= \psi\left(\max\left\{d(x,0),d(x,Tx),d(0,0),\frac{1}{2}[d(x,0)+d(0,Tx)]\right\}\right)\\ &= \psi\left(\max\left\{x,2x-1,0,x-\frac{1}{2}\right\}\right)\\ &= (2x-1)^2 \quad \text{and} \quad \phi(\max\{(d(x,y),d(y,Ty))\})\\ &= \phi(\max\{d(x,0),d(0,0)\}) = \phi(x) = \frac{1}{2}.\end{aligned}$$

(iv) If $y > x$ and $x = 0$, then

$$\begin{aligned}\psi(d(Tx,Ty)) &= \psi(d(0,y-1)) = \psi(y-1)\\ &= (y-1)^2, \psi\left(\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\}\right)\\ &= \psi\left(\max\left\{d(0,y),d(0,0),d(y,Ty),\frac{1}{2}[d(0,Ty)+d(y,0)]\right\}\right)\\ &= \psi\left(\max\left\{y,0,2y-1,y-\frac{1}{2}\right\}\right) = (2y-1)^2 \quad \text{and} \quad \phi(\max\{d(x,y),d(y,Ty)\})\\ &= \phi(\max\{d(0,y),d(y,Ty)\}) = \phi(\max\{y,2y-1\}) = \frac{1}{2}.\end{aligned}$$

(v) $x = y$, then

$$\begin{aligned}\psi(d(Tx,Ty)) = 0, \psi(\max\{&d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)\\ &+ d(y,Tx)]\})\\ =& \ 0 \quad \text{or} \quad (2x-1)^2,\end{aligned}$$

according as $x = y = 0$ or $x = y \neq 0$ and $\phi(\max\{d(x,y),d(y,Ty)\}) = 0$ or $\frac{1}{2}$, according as $x = y = 0$ or $x = y \neq 0$.

In all the above cases, the condition (iv) of Corollary 2.7 or, alternately, the condition (iii) of Theorem 2.8 is satisfied. Hence required conditions of Corollary 2.7 and Theorem 2.8 are satisfied and it is seen that 0 is a fixed point of T.

Note: In the above example, we set $x = n + 1$ and $y = n$, where n is a positive integer. Then according to the case (i), $\psi(d(Tx,Ty)) = (2n-1)^2$ and

$\psi(\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\}) = (2n+1)^2$. Then, $\psi(d(Tx,Ty)) = \alpha_n \psi(\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\})$, where $\alpha_n = (\frac{2n-1}{2n+1})^2$.

Since $\alpha_n \to 1$ as $n \to \infty$, condition (iv) of Corollary 2.3 or, alternately, the condition (iii) of Theorem 2.4 does not hold for all comparable $x, y \in X$. This shows that Corollary 2.7 and Theorem 2.8 are more general than Corollary 2.3 and Theorem 2.4, respectively.

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