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Multiplicity of solutions for a general $p(x)$-Laplacian Dirichlet problem

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Abstract. We establish some results on the existence of multiple nontrivial solutions for a class of $p(x)$-Laplacian elliptic equations. Our approach relies on the variable exponent theory of generalized Lebesgue–Sobolev spaces, combined with adequate variational methods and a variant of the Mountain Pass lemma.

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1. INTRODUCTION

Consider the $p(x)$-Laplacian Dirichlet problem

$$(\mathcal{P})\begin{cases} -\Delta_{p(x)}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geqslant 1)$, $p \in C(\overline{\Omega})$ with

$$1 < p^- = \inf_{x\in\Omega} p(x) \leqslant p^+ = \sup_{x\in\Omega} p(x) < \infty,$$

$\Delta_{p(x)} = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, which becomes p-Laplacian when $p(x) \equiv p$ (a constant) and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the subcritical growth condition

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(F_0)

$$|f(x,t)| \leqslant c(1+|t|^{q(x)-1}), \quad \forall t \in \mathbb{R}, \quad \text{a.e.} \quad x \in \Omega,$$

for some $c > 0, q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x), \forall x \in \overline{\Omega}$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geqslant N. \end{cases}$$

The study of differential and partial differential involving variable exponent conditions is a new and an interesting topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics etc... These physical problems were facilitated by the development of Lebesgue and Sobolev spaces with variable exponent.

Recall that the weak solutions of ($\mathcal{P}$) are the critical points of the associated energy functional Φ, given by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - \int_\Omega F(x,u) \, dx,$$

acting on the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$, where $F(x,t) = \int_0^t f(x,s) \, ds$. It is well known that under (F_0), Φ is well defined and is a C^1 functional with derivative given by

$$\langle \Phi'(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \int_\Omega f(x,u) v \, dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

If $f(x,0) = 0$ for a.e. $x \in \Omega$, the constant function $u = 0$ is a trivial solution of problem ($\mathcal{P}$). In this case, the key point is proving the existence of nontrivial solutions for ($\mathcal{P}$). For this purpose, we need to introduce a condition that gives us information about the behaviors of the perturbed function $f(x,t)$ or its primitive $F(x,t)$ near infinity and near zero.

The existence and multiplicity of solutions of $p(x)$-Laplacian problems have been studied by several authors, see for example [2,3,7–10,12–15] and the references therein.

More recently, in [10], the authors investigated the eigenvalues of the $p(x)$-Laplacian Dirichlet problem. They showed that Λ, the set of eigenvalues, is a nonempty infinite set such that $\sup\Lambda = +\infty$. Moreover, they proved that if there is a vector $l \in \mathbb{R}^N \setminus \{0\}$ such that for any $x \in \Omega, p(x + tl)$ is monotone for $t \in I_x = \{t | x + tl \in \Omega\}$, then

$$\lambda_* = \inf \Lambda > 0. \tag{1.1}$$

In [13], the authors studied the existence and multiplicity of solutions of the $p(x)$-Laplacian operator in the particular case $f(x,t) = \lambda(t^{\gamma-1} - t^{\beta-1})$ with $1 < \beta < \gamma < \inf_{x\in\overline{\Omega}} p(x)$ and $t \geqslant 0$. They proved the existence of at least two distinct non-negative, nontrivial weak solutions, provided that $\lambda > 0$ is large enough.

In this paper, we start by proving the existence of at least one nontrivial solution under the following conditions.

(F_1)

$$\lim_{|t|\to\infty}\left(F(x,t)-\frac{(p^-)^2}{(p^+)^3}\lambda_*|t|^{p^-}\right)=-\infty \text{ uniformly for a.e. } x\in\Omega,$$

where λ_* is given in (1.1).
(F_2) There exist $\mu\in[1,p^-)$ and $\gamma>0$, such that

$$0<\mu F(x,t)\leqslant tf(x,t), \text{ for a.e. } x\in\Omega, \quad 0<|t|\leqslant\gamma.$$

The main result reads as follows.

Theorem 1.1. *Assume that (F_0), (F_1) and (F_2) are satisfied, then the problem$(\mathcal{P})$ has at least one nontrivial solution.*

The second purpose of this paper is to show the existence of at least two nontrivial solutions of problem $(\mathcal{P})$ under the following assumptions.

(F_3) There exist $\theta>p^+$ and $M>0$ such that

$$|t|\geqslant M\Rightarrow 0<\theta F(x,t)\leqslant tf(x,t)$$

for a.e. $x\in\Omega$ and each $t\in\mathbb{R}$.
(F_4) $f(x,t)=o(|t|^{p^+-1})$ as $t\to 0$ and uniformly for $x\in\Omega$, with $q^->p^+$.

We can state the following result.

Theorem 1.2. *Suppose (F_0), (F_3), (F_4) and $f(x,0)=0$ for a.e. $x\in\Omega$. Then the problem$(\mathcal{P})$ has at least two nontrivial solutions, in which one is non-negative and one is non-positive.*

Remark 1.3. By Theorem 4.3 in [9], problem $(\mathcal{P})$ has at least a weak solution. However, the proof in [9] does not state the fact that solution is nontrivial in the case when $f(x,0)=0$.

We point out that our results are inspired by [1], where a related property is proved in the case of the p-Laplace operators. We point out that the extension from p-Laplace operator to $p(x)$-Laplace operator is not trivial, since the $p(x)$-Laplacian has a more complicated structure than the p- Laplace operator, for example, it is inhomogeneous.

This paper contains three sections. We will first introduce some basic preliminary results and lemmas in Section 2. In Section 3, we will give the proofs of our main results.

2. Preliminary results

We recall in this section some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$.

Throughout this paper, we assume that $p(x)>1, p(x)\in C^{0,\alpha}(\overline{\Omega})$ with $\alpha\in(0,1)$.

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \min_{x\in\overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leqslant 1\right\}.$$

Variable exponent Lebesgue space resembles classical Lebesgue space in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leqslant p^+ < \infty$ and continuous functions are dense, if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leqslant p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{q(x)}(\Omega)$the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)},$$

holds true. For more details, we can refer to [11].

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the Modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $J : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If (u_n),$u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leqslant J(u) \leqslant |u|_{p(x)}^{p^+}, \tag{2.1}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leqslant J(u) \leqslant |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$|u_n - u|_{p(x)} \to 0 \Longleftrightarrow J(u_n - u) \to 0, \tag{2.3}$$

$$|u|_{p(x)} < 1(\text{resp.} = 1; > 1) \Longleftrightarrow J(u) < 1(\text{resp.} = 1; > 1). \tag{2.4}$$

Spaces with $p^+ = \infty$ have been studied by [6].

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}. \tag{2.5}$$

The Space $(W_0^{1,p(x)}(\Omega), \|.\|)$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$

is compact and continuous. We refer to [11,4,5] for further properties of variable exponent Lebesgue–Sobolev spaces.

Now, we consider the eigenvalues of the $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

For any $u \in W_0^{1,p(x)}(\Omega)$, define $F, G : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$F(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx; \quad G(u) = \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx.$$

For any $t > 0$, define

$$M_t = G^{-1}(t) = \{u \in W_0^{1,p(x)}(\Omega) : G(u) = t\},$$

then M_t is a C^1 submanifold of $W_0^{1,p(x)}(\Omega)$ since t is a regular value of G. Put,

$$\sum_{t,n} = \{H \subset M_t : H = -H, \gamma(H) \geqslant n\},$$

where $\gamma(H)$ is the genus of H.

Define

$$c_{(n,t)} = \inf_{H\in\sum_{t,n}} \sup_{u\in H} F(u), \quad n = 1, 2, \ldots$$

By [10] , the Dirichlet problem (2.6) has infinitely many eigenpair sequences $\{(u_{(n,t)},\lambda_{(n,t)})\}$ such that

$$G(\pm u_{(n,t)}) = t, F(\pm u_{(n,t)}) = c_{(n,t)},$$
$$\lambda_{(n,t)} = \frac{\langle F'(u_{(n,t)}), u_{(n,t)}\rangle}{\langle G'(u_{(n,t)}), u_{(n,t)}\rangle} \to \infty, \text{ as } n \to \infty.$$

Define

$$\bar{\mu}_* = \inf_{W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx},$$
$$\mu_* = \inf_{W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx},$$

$\lambda_* = \inf \Lambda$ where $\Lambda = \{\lambda \in \mathbb{R} : \lambda \text{is an eigenvalue of}(2.6)\}$.

So we obtain the following lemma.

Lemma 2.1.

$$\left(\frac{p^-}{p^+}\right)^2 \bar{\mu}_* \leqslant \lambda_* \leqslant \left(\frac{p^+}{p^-}\right)^2 \bar{\mu}_*.$$

Proof. First, we prove the following claim.

Claim: For all $t > 0$, let u_0 be an eigenfunction associated with $\lambda_{(1,t)}$ of the problem (2.6). Then

$$F(u_0) = c_{(1,t)} = \inf\{F(u) : u \in M_t\}.$$

Indeed, let $b_t = \inf\{F(u){:}u \in M_t\}$. Obviously $b_t \leqslant c_{(1,t)}$. Since the functional $F : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is coercive and weakly lower semi-continuous and M_t is weakly closed subset of $W_0^{1,p(x)}(\Omega)$, there exists $u_* \in M_t$ such that $F(\pm u_*) = b_t$. Let $H = \{\pm u_*\}$, then $\gamma(H) = 1$ and $c_{(1,t)} \leqslant b_t$. Thus the claim follows.

Second, we have

$$\frac{p^-}{p^+}\frac{\int_\Omega |\nabla u_0|^{p(x)}\ dx}{\int_\Omega |u_0|^{p(x)}\ dx} \leqslant \frac{F(u_0)}{G(u_0)} \leqslant \inf\left\{\frac{F(u)}{G(u)}|u \in M_t\right\}$$

and from this it follows that $\frac{p^-}{p^+}\lambda_{(1,t)} \leqslant \inf\{\frac{F(u)}{G(u)}|u \in M_t\}$.

Then,

$$\frac{p^-}{p^+}\lambda_* \leqslant \inf\left\{\frac{F(u)}{G(u)}|u \in M_t\right\} \quad \forall t > 0.$$

So, $\frac{p^-}{p^+}\lambda_* \leqslant \mu_*$ and consequently $\lambda_* \leqslant \frac{p^+}{p^-}\mu_*$.

Now, let $u_{(n,t)}$ be the eigenfunction associated with $\lambda_{(n,t)}$, we have$\lambda_{(n,t)} \geqslant \frac{p^-}{p^+}\frac{F(u_{(n,t)})}{G(u_{(n,t)})}$ for every $n \in \mathbb{N}$ and $t > 0$. It result that $\lambda_{(n,t)} \geqslant \frac{p^-}{p^+}\mu_*$ and hence, $\lambda_* \geqslant \frac{p^-}{p^+}\mu_*$. Finally, $\frac{p^-}{p^+}\mu_* \leqslant \lambda_* \leqslant \frac{p^+}{p^-}\mu_*$.

On the other hand, it is easy to see that $\frac{p^-}{p^+}\bar{\mu}_* \leqslant \mu_* \leqslant \frac{p^+}{p^-}\bar{\mu}_*$. Thus the lemma follows. □

Now, we consider the truncated problem

$$(\mathcal{P}_\pm)\begin{cases} -\Delta_{p(x)}u = f_\pm(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where

$$f_\pm(x,t) = \begin{cases} f(x,t) & \text{if} \pm t \geqslant 0,\\ 0 & \text{otherwise.}\end{cases}$$

We denote by $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$ the positive and negative parts of u.

We need the following lemmas.

Lemma 2.2.

(i) *If* $u \in W_0^{1,p(x)}(\Omega)$ *then* $u^+, u^- \in W_0^{1,p(x)}(\Omega)$ *and*

$$\nabla u^+ = \begin{cases} \nabla u, & \text{if } [u > 0],\\ 0, & \text{if } [u \leqslant 0],\end{cases} \qquad \nabla u^- = \begin{cases} 0, & \text{if } [u \geqslant 0],\\ \nabla u, & \text{if } [u < 0].\end{cases}$$

(ii) *The mappings* $u \mapsto u^\pm$ *are continuous on* $W_0^{1,p(x)}(\Omega)$.

Proof. The first assertion has been proved in [13, Lemma 3.3].

Now, we will show the second assertion. Indeed, since $u^{\pm} = \frac{1}{2}(|u| \pm u)$, it suffices to prove that the mapping $u \mapsto |u|$ is continuous on $W_0^{1,p(x)}(\Omega)$ i.e. $u_n \to u$ implies $|u_n| \to |u|$. We have $|u_n| \to |u|$ in $L^{p(x)}(\Omega)$ and $|u_n|$ is bounded in $W_0^{1,p(x)}(\Omega)$. Thus, from reflexivity of $W_0^{1,p(x)}(\Omega)$, $|u_n| \rightharpoonup z$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence. Hence $z = |u|$ and $|u_n| \rightharpoonup |u|$ for the whole sequence. On the other hand, we have $\nabla|u| = \text{sgn}(u)\nabla u$ a.e. and $\||u|\| = \|u\|$. By uniform convexity of $W_0^{1,p(x)}(\Omega)$ it follows that $|u_n| \to |u|$ in $W_0^{1,p(x)}(\Omega)$. Thus, the lemma follows. □

Lemma 2.3. *All solutions of* $(\mathcal{P}_+)$ *(resp.* $(\mathcal{P}_-)$*) are non-negative (resp. non-positive) solutions of* $(\mathcal{P})$.

Proof. Define $\Phi_\pm : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$,

$$\begin{aligned}\Phi_\pm(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\ dx - \int_\Omega F_\pm(x,u)\ dx\\ &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\ dx - \int_\Omega F(x,u^\pm)\ dx,\end{aligned}$$

where $F_\pm(x,s) = \int_0^s f_\pm(x,t)\ dt$. It is well known that from Lemma 2.2 and the condition (F_0),$\Phi_\pm$ is well defined on $W_0^{1,p(x)}(\Omega)$, weakly lower semi-continuous and C^1-functionals.

Let u be a solution of $(\mathcal{P}_+)$, or equivalently, u be a critical point of Φ_+. Taking $v = u^-$ in

$$\langle \Phi'_+(u), v\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v\ dx - f_+(x,u)v)\ dx = 0,$$

shows that $J(\nabla u^-) = \int_\Omega (|\nabla u^-|^{p(x)}\ dx = 0$. In view of (2.1) and (2.2) we have $\|u^-\| = 0$, so $u^- = 0$ and $u = u^+$ is also a critical point of Φ with critical value $\Phi(u) = \Phi_+(u)$.

Similarly, nontrivial critical points of Φ_- are non-positive solutions of $(\mathcal{P})$. □

3. Proof of main results

3.1. Proof of Theorem 1.1

Before to present the proof of Theorem 1.1, we start with the following auxiliary result.

Lemma 3.1. Φ *is coercive on* $W_0^{1,p(x)}(\Omega)$.

Proof. Put

$$G(x,t) = F(x,t) - \frac{(p^-)^2}{(p^+)^3}\lambda_*|t|^{p^-}.$$

Then, from (F_1) we conclude that, for every $M > 0$, there is $R_M > 0$ such that

$$G(x,t) \leqslant -M, \forall |t| \geqslant R_M, \quad \text{a.e. } x \in \Omega. \tag{3.1}$$

By contradiction, let $K \in \mathbb{R}$ and $(u_n) \subset W_0^{1,p(x)}(\Omega)$ be such that

$$\|u_n\| \to \infty \quad \text{and} \quad \Phi(u_n) \leqslant K.$$

Putting $v_n = \frac{u_n}{\|u_n\|}$, one has $\|v_n\| = 1$. For a subsequence, we may assume that for some $v_0 \in W_0^{1,p(x)}(\Omega)$, we have $v_n \rightharpoonup v_0$ weakly in $W_0^{1,p(x)}(\Omega)$, $v_n \to v_0$ strongly in $L^{p(x)}(\Omega)$, $v_n(x) \to v_0(x)$ a.e. in Ω.

Now, using (3.1) and (2.1) it follows that

$$\begin{aligned} K \geqslant \Phi(u_n) &= \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\, dx - \int_\Omega F(x,u_n)\, dx \\ &\geqslant \frac{1}{p^+} \int_\Omega |\nabla u_n|^{p(x)}\, dx - \frac{(p^-)^2}{(p^+)^3} \lambda_* \int_\Omega |u_n|^{p^-}\, dx - \int_\Omega G(x,u_n)\, dx \\ &\geqslant \frac{1}{p^+} \left(\|u_n\|^{p^-} - \left(\frac{p^-}{p^+}\right)^2 \lambda_* \int_\Omega |u_n|^{p^-}\, dx \right) + M_1, \end{aligned} \tag{3.2}$$

where $M_1 \in \mathbb{R}$.

Dividing (3.2) by $\|u_n\|^{p^-}$ and passing to the limit, we conclude

$$\frac{1}{p^+} \left(1 - \left(\frac{p^-}{p^+}\right)^2 \lambda_* \int_\Omega |v_0|^{p^-}\, dx \right) \leqslant 0,$$

Consequently, $v_0 \neq 0$. Let $\Omega_0 = \{x \in \Omega : v_0(x) \neq 0\}$, via the result above we have $|\Omega_0| > 0$ and

$$|u_n(x)| \to +\infty, \quad \text{a.e. } x \in \Omega_0.$$

Thus, from (F_1) and Lemma 2.1 we deduce that

$$\begin{aligned} K \geqslant \Phi(u_n) &\geqslant \frac{1}{p^+} \int_\Omega \left(|\nabla u_n|^{p(x)} - \left(\frac{p^-}{p^+}\right)^2 \lambda_* |u_n|^{p^-} \right) dx - \int_\Omega G(x,u_n)\, dx \\ &\geqslant \frac{1}{p^+} \int_\Omega \left(|\nabla u_n|^{p(x)} - \left(\frac{p^-}{p^+}\right)^2 \lambda_* |u_n|^{p(x)} \right) dx - \int_\Omega G(x,u_n)\, dx \\ &\geqslant - \int_\Omega G(x,u_n)\, dx \to +\infty. \end{aligned}$$

This is a contradiction. Hence Φ is coercive on $W_0^{1,p(x)}(\Omega)$. □

Lemma 3.2. *Under (F_0) and (F_2), zero is local maximum for the functional $\Phi(su), s \in \mathbb{R}$, for $u \neq 0$.*

Proof. From the condition (F_2), there exists a constant $c_0 > 0$ such that

$$F(x,t) \geqslant c_0 |t|^\mu, \quad \text{for } x \in \Omega, |t| \leqslant \gamma. \tag{3.3}$$

From (F_0) and $|t| > \gamma$, there exists $c_1 > 0$ such that

$$|F(x,t)| \leqslant c_1 |t|^{q(x)}, \quad x \in \Omega, |t| > \gamma.$$

Using the preceding inequality and (3.3), we get

$$F(x,t) \geqslant c_0 |t|^{\mu} - c_1 |t|^{q(x)}, \quad x \in \Omega, \quad t \in \mathbb{R} \tag{3.4}$$

for some $q^- = \inf_{x\in\Omega} q(x) > p^+$ and $c_1 > 0$.

Then, for $u \in W_0^{1,p(x)}(\Omega), u \neq 0$ and $s > 0$, we have

$$\begin{aligned} \Phi(su) &= \int_\Omega \frac{1}{p(x)} |s\nabla u|^{p(x)}\, dx - \int_\Omega F(x, su)\, dx \\ &\leqslant \frac{s^{p^+}}{p^-} \int_\Omega |\nabla u|^{p(x)}\, dx - \int_\Omega (c_0 |su|^{\mu} - c_1 |su|^{q(x)})\, dx \\ &\leqslant \frac{s^{p^+}}{p^-} \int_\Omega |\nabla u|^{p(x)}\, dx - c_0 s^{\mu} \|u\|_{L^\mu}^{\mu} + c_1 s^{q^+} \int_\Omega |u|^{q(x)}\, dx. \end{aligned}$$

Since $\mu < p^+ < q^+$, there exists a $s_0 = s_0(u) > 0$ such that

$$\Phi(su) < 0, \quad \text{for all } 0 < s < s_0. \qquad \square \tag{3.5}$$

Proof of Theorem 1.1. By Lemma 3.1, Φ is coercive on $W_0^{1,p(x)}(\Omega)$. Since Φ is weakly lower semi-continuous, Φ has a global minimizer u_0 on $W_0^{1,p(x)}(\Omega)$. Because $\Phi(0) = 0$, then, in order to prove $u_0 \neq 0$, it is sufficient to show that $\Phi(u_0) < 0$. Hence, the Theorem 1.1 follows from Lemma 3.2. $\square$

3.2. Proof of Theorem 1.2

To apply the mountain pass theorem, we will do separate studies of the compactness of $\Phi_\pm$ and its geometry.

Lemma 3.3. *Under (F_0) and (F_3), the functional Φ_+ satisfies the (PS) condition.*

Proof. Let $(u_n)_n$ be a (PS) sequence for the functional Φ_+: $\Phi_+(u_n)$ bounded and $\Phi'_+(u_n) \to 0$. Let us show that $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega)$. Using the hypothesis (F_3), since $\Phi_+(u_n)$ is bounded, we have

$$\begin{aligned} C_1 &\geqslant \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\, dx - \int_\Omega F(x, u_n^+)\, dx \\ &\geqslant \frac{1}{p^+} J(\nabla u_n) - \int_\Omega \frac{u_n^+}{\theta} f(x, u_n^+)\, dx + C_2, \end{aligned}$$

where C_1 and C_2 are two constants. Note that

$$\langle \Phi'_+(u_n), u_n\rangle = \int_\Omega |\nabla u_n|^{p(x)}\ dx - \int_\Omega f\big(x, u_n^+\big) u_n\ dx$$
$$= \int_\Omega |\nabla u_n|^{p(x)}\ dx - \int_\Omega f\big(x, u_n^+\big) u_n^+\ dx,$$

which implies

$$C_1 \geqslant \left(\frac{1}{p^+} - \frac{1}{\theta}\right) J(\nabla u_n) + \frac{1}{\theta}\langle \Phi'_+(u_n), u_n\rangle + C_2. \tag{3.6}$$

Suppose, by contradiction that $(u_n)_n$ unbounded in $W_0^{1,p(x)}(\Omega)$, so $\|u_n\| \geqslant 1$ for rather large values of n and it results that

$$\|u_n\|^{p^-} \leqslant J(\nabla u_n) \leqslant \|u_n\|^{p^+}$$

for rather large values of n. Furthermore, $\Phi'_+(u_n) \to 0$ assure that there exists $C_3 > 0$ such that

$$-C_3\|u_n\| \leqslant \langle \Phi'_+(u_n), u_n\rangle \leqslant C_3\|u_n\|$$

for rather large values of n. Consequently,

$$C_1 \geqslant \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|^{p^-} - \frac{C_3}{\theta}\|u_n\| + C_2.$$

Since $p^- > 1$ and $\left(\frac{1}{p^+} - \frac{1}{\theta}\right) > 0$, we have

$$\left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|^{p^-} - \frac{C_3}{\theta}\|u_n\| + C_2 \to +\infty \text{ as } n \to +\infty,$$

what is a contradiction. So $(u_n)_n$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$. The proof of Lemma 3.3 is complete. □

Lemma 3.4. *There exist $r > 0$ and $\alpha > 0$ such that $\Phi_+(u) \geqslant \alpha$, for all$u \in W_0^{1,p(x)}(\Omega)$-with $\|u\| = r$.*

Proof. The conditions (F_0) and (F_4) assure that

$$|F(x,t)| \leqslant \varepsilon |t|^{p^+} + C(\varepsilon)|t|^{q(x)} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

For $\|u\|$ small enough, we have

$$\Phi_+(u) \geqslant \frac{1}{p^+} J(\nabla u) - \int_\Omega F(x, u^+)\ dx$$
$$\geqslant \frac{1}{p^+}\|u\|^{p^+} - \varepsilon\int_\Omega |u^+|^{p^+}\ dx - C(\varepsilon)\int_\Omega |u^+|^{q(x)}\ dx$$
$$\geqslant \frac{1}{p^+}\|u\|^{p^+} - \varepsilon\int_\Omega |u|^{p^+}\ dx - C(\varepsilon)\int_\Omega |u|^{q(x)}\ dx. \tag{3.7}$$

By the condition (F_0), it follows

$$p^- \leqslant p \leqslant p^+ < q^- \leqslant q(x) < p^*$$

then

$$W_0^{1,p(x)}(\Omega) \subset L^{p^+}(\Omega) \quad \text{and} \quad W_0^{1,p(x)}(\Omega) \subset L^{q(x)}(\Omega),$$

with a continuous and compact embedding, what implies the existence of $C_4, C_5 > 0$ such that

$$\|u\|_{L^{p^+}} \leqslant C_4\|u\| \quad \text{and} \quad |u|_{q(x)} \leqslant C_5\|u\|$$

for all $u \in W_0^{1,p(x)}(\Omega)$. Since $\|u\|$ is small enough, we deduce

$$\int_\Omega |u|^{q(x)} \leqslant |u|_{q(x)}^{q^-} \leqslant C_6\|u\|^{q^-}.$$

Replacing in (3.7), it results that

$$\Phi_+(u) \geqslant \frac{1}{p^+}\|u\|^{p^+} - \varepsilon C_4^{p^+}\|u\|^{p^+} - C_7\|u\|^{q^-},$$

with C_i are positives constants. Let us choose $\varepsilon > 0$ such that $\varepsilon C_4^{p^+} \leqslant \frac{1}{2p^+}$, we obtain

$$\Phi_+(u) \geqslant \frac{1}{2p^+}\|u\|^{p^+} - C_7\|u\|^{q^-} \geqslant \|u\|^{p^+}\left(\frac{1}{2p^+} - C_7\|u\|^{q^- - p^+}\right). \tag{3.8}$$

Since $p^+ < q^-$, the function $t \to \left(\frac{1}{2p^+} - C_7 t^{q^- - p^+}\right)$ is strictly positive in a neighborhood of zero. It follows that there exist $r > 0$ and $\alpha > 0$ such that

$$\Phi_+(u) \geqslant \alpha \ \forall u \in W_0^{1,p(x)}(\Omega) : \|u\| = r. \qquad \square$$

Proof of Theorem 1.2. In order to apply the Mountain Pass Theorem, we must prove that

$$\Phi_+(su) \to -\infty \text{ as } s \to +\infty,$$

for a certain $u \in W_0^{1,p(x)}(\Omega)$. From the condition (F_3), we obtain

$$F(x,t) \geqslant c|t|^\theta \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Let $u \in W_0^{1,p(x)}(\Omega)$ and $s > 1$ we have

$$\begin{aligned} \Phi_+(su) &= \int_\Omega \frac{s^{p(x)}}{p(x)}|\nabla u|^{p(x)}\, dx - \int_\Omega F(x,(su)^+)\, dx, \\ &\leqslant s^{p^+}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - cs^\theta \int_\Omega |u^+|^\theta\, dx. \end{aligned}$$

The fact $\theta > p^+$, gives that

$$\Phi_+(su) \to -\infty \text{ as } s \to +\infty.$$

It follows that there exists $e \in W_0^{1,p(x)}(\Omega)$ such that $\|e\| > r$ and $\Phi_+(e) < 0$.

According to the Mountain Pass Theorem, Φ_+ admits a critical value $\mu \geqslant \alpha$ which is characterized by

$$\mu = \inf_{h\in\Lambda}\sup_{t\in[0,1]}\Phi_+(h(t))$$

where

$$\Lambda = \{h \in C\Big([0,1], W_0^{1,p(x)}(\Omega)\Big) : h(0) = 0 \text{ and } h(1) = e\}.$$

Then, the functional Φ_+ has a critical point u^+ with $\Phi_+(u^+) \geqslant \alpha$. But, $\Phi_+(0) = 0$, that is, $u^+ \neq 0$. Therefore, the problem $(\mathcal{P}_+)$ has a nontrivial solution which, by Lemma 2.3, is a non-negative solution of the problem $(\mathcal{P})$.

Similarly, using Φ_-, we show that there exists furthermore a non-positive solution. The proof of Theorem 1.2 is now complete. □

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