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Multiples of repunits as sum of powers of ten

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Abstract. The sequence $P_{k,n} = 1 + 10^k + 10^{2k} + \cdots + 10^{(n-1)k}$ can be used to generate infinitely many Smith numbers with the help of a set of suitable multipliers. We prove the existence of such a set, consisting of constant multiples of repunits, that generalizes to any value of $k \geqslant 9$. This fact complements the earlier results which have been established for $k \leqslant 9$.

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1. Introduction

A natural number n is called a Smith number if n is a composite for which the digital sum $S(n)$ equals the p-digit sum $S_p(n)$, where $S_p(n)$ is given by the digital sum of all the prime factors of n, counting multiplicity. For example, based on the factorization $636 = 2^2 \times 3 \times 53$, we have $S_p(636) = 2 + 2 + 3 + 5 + 3 = 15$. Since $S(636) = 6 + 3 + 6 = 15$, then $S(636) = S_p(636)$ and therefore, 636 is a Smith number.

Smith numbers were first introduced in 1982 by Wilansky [2]. We already know that Smith numbers are infinitely many—a fact first proved in 1987 by McDaniel [1]. In a quite recent publication [3], an alternate method for constructing Smith numbers was introduced, involving the sequence $P_{k,n}$ defined by

$$P_{k,n} = \sum_{i=0}^{n-1} 10^{ki}.$$

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The established fact [3, Theorem 9] can be restated as follows.

Theorem 1.1. *Let $k \geqslant 2$ be fixed, and let M_k be a set of seven natural numbers with two conditions:*

1. *The set $\{S_p(t) \mid t \in M_k\}$ is a complete residue system modulo 7.*
2. *Every element $t \in M_k$ can be expressed as $t = \sum_{j=1}^{k} 10^{e_j}$, where the set $\{e_j \mid 1 \leqslant j \leqslant k\}$ is a complete residue system modulo k.*

Then there exist infinitely many values of $n \geqslant 1$ for which the product

$$9 \times P_{k,n} \times t_{k,n} \times 10^{f_{k,n}}$$

is a Smith number for some element $t_{k,n} \in M_k$ and exponent $f_{k,n} \geqslant 0$.

Following this result, the article continues with the construction of a set M_k which satisfies the hypothesis of Theorem 1.1, for each $k = 2, 3, \ldots, 9$.

This paper is a response to the challenge to continue with the search for such M_k for $k > 9$. Quite surprisingly we are able to give a relatively clean construction of M_k, which consists of seven constant multiples of the repunit $R_k = (10^k - 1)/9$, and which is valid for all $k \geqslant 9$ but not for lesser values of k.

2. Main results

Theorem 2.1. *Consider the repunit R_k and let $m = 9 \cdot 10^a + 9 \cdot 10^b + 1$, where* $k > a > b > 0$. Then we can write $mR_k = \sum_{j=1}^{k} 10^{e_j}$ *such that the set $\{e_j \mid 1 \leqslant j \leqslant k\}$ serves as a complete residue system modulo k.*

Proof. Since we will be dealing with strings of repeated digits, let us agree on the following notation. By (u_d), where $0 \leqslant u \leqslant 9$, we mean a string of u's of length d digits. In particular, when $d = 1$, we simply write (u) instead of (u_1). We also allow concatenation, e.g., the notation $(1_5,0,9_3,0_2,1)$ represents the number 111110999001.

Now let $A = 9 \cdot 10^a \cdot R_k$ and $B = 9 \cdot 10^b \cdot R_k$, hence $mR_k = A + B + R_k$. In order to help visualize how the addition $B + R_k$ is performed, we right-align the two strings and add columnwise, right to left, as follows.

$$\begin{aligned} R_k &= (1_k) = (1_{k-b}, 1_b), \\ B &= (9_k, 0_b) = (9_b, 9_{k-b}, 0_b), \\ B + R_k &= (1, 0_b, 1_{k-b-1}, 0, 1_b). \end{aligned}$$

Now with $A = (9_k,0_a)$, we prepare the addition operation for $mR_k = A + (B + R_k)$ in a similar way:

$$\begin{aligned} B + R_k &= (1, 0_b, 1_{k-a}, 1_{a-b-1}, 0, 1_b), \\ A &= (9_{a-b-1}, 9, 9_b, 9_{k-a}, 0_{a-b-1}, 0, 0_b), \\ mR_k &= (1, 0_{a-b-1}, 1, 0_b, 1_{k-a-1}, 0, 1_{a-b-1}, 0, 1_b). \end{aligned}$$

(Note that in the case $a - b - 1 = 0$, each string of length $a - b - 1$ appearing above is simply nonexistent, and similarly for $k - a - 1$ if equals 0.)

We see that the digits in mR_k are but zeros and ones, where the 1's (as read from right to left) precisely correspond to the powers 10^e, with

$$e \in \{0, \ldots, b-1,\ b+1, \ldots, a-1,\ a+1, \ldots, k-1,\ k+b,\ k+a\},$$

which is a complete residue system modulo k. $\square$

Theorem 2.2. *Let R_k represent the k-th repunit, and let the set M be given by*

$$M = \{1,\ 991,\ 90091,\ 99001,\ 900901,\ 9900001,\ 900090001\}.$$

Then the set $M_k = \{mR_k \mid m \in M\}$ satisfies the hypothesis of Theorem 1.1 *for all $k \geqslant 9$.*

Proof. We will first prove that the seven elements of M_k as stated in Theorem 2.2 have distinct p-digit sums modulo 7. It suffices to show that the set M has this same property, and this can be routinely checked from the prime factors of the elements $m \in M$ given in Table 1. Note that $m = 1$ is not included in the table, about which by convention we shall agree that $S_p(1) = 0$.

Table 1 The p-digit sums and the factorizations of the elements $m \in M$.

m	Factorization of m	$S_p(m)$	$S_p(m)$ mod 7
991	991	19	5
90,091	23×3917	25	4
99,001	$7 \times 14{,}143$	20	6
900,901	163×5527	29	1
9,900,001	$17 \times 449 \times 1297$	44	2
900,090,001	$421 \times 2{,}137{,}981$	38	3

To complete the proof of Theorem 2.2, we need to show that for each $m \in M$, the number mR_k can be expressed as the sum of k powers of 10 satisfying the condition described in Theorem 1.1. The case $m = 1$ is of course trivial, while the remaining six readily follow from Theorem 2.1. $\square$

As a further remark, we point out that the condition of being "a complete residue system modulo k" demanded by Theorem 1.1 is actually equivalent to the sum of the powers of 10 being a multiple of R_k. Although the necessity part is already claimed [3, Remark 8], we shall now write a complete proof for this fact.

Theorem 2.3. *Let $t = \sum_{j=1}^{k} 10^{e_j}$, where the exponents e_j are not assumed distinct. Then the set $C = \{e_j \mid 1 \leqslant j \leqslant k\}$ is a complete residue system modulo k if and only if t is a multiple of R_k.*

Proof. We note that $10^k \equiv 1 \pmod{R_k}$, so that $10^{e_j} \equiv 10^{e_j \bmod k} \pmod{R_k}$. Set $t' = \sum_{j=1}^{k} 10^{e_j \bmod k}$, and we have $t' \equiv t \pmod{R_k}$.

As a first case, suppose that among the numbers $e_1 \bmod k, \ldots, e_k \bmod k$, at most 9 of them can be identical. Then $t' \leqslant 9R_k$, and it follows that R_k divides t' if and only if $t' = hR_k$ for some h in the range $1 \leqslant h \leqslant 9$. The fact that $S(t') = k$ leads us to conclude that $t' = hR_k$ if and only if $h = 1$ and $\{e_1 \bmod k, \ldots, e_k \bmod k\} = \{0, 1, \ldots, k-1\}$. Thus R_k divides t if and only if C is a complete residue system modulo k.

Assume now that there exist more than 9 identical items among the numbers $e_1 \bmod k, \ldots, e_k \bmod k$. Clearly in this case, C is not a complete residue system modulo k. Observe that evaluating t' by adding the k summands $10^{e_j \bmod k}$ will involve carries and as a result, we have $S(t') < k$.

If it happens that $t' \leqslant 9R_k$, then as before, we see that R_k divides t' if and only if $t' = hR_k$ with $1 \leqslant h \leqslant 9$—this would be impossible since $S(t') < k$. Hence, neither does R_k divide t in this case.

We next consider the subcase $t' \geqslant 10^k$. For this let us write $t' = q \cdot 10^k + r$, where $r = t' \bmod 10^k$, and let $t'' = q + r$. Note that $t'' \equiv t' \pmod{R_k}$ and that $S(t'') \leqslant S(q) + S(r) = S(t') < k$. We will claim that $t'' \leqslant 9R_k$, so that once again we conclude that t'' is not a multiple of R_k, and neither is t.

The remainder of the proof is therefore showing that $t'' < 10^k$. By contradiction, suppose that $q + r \geqslant 10^k$. We note that $t' \leqslant k \cdot 10^{k-1}$, and so $q \leqslant \frac{k}{10}$. It follows that the number q is composed of at most $\lfloor \log k \rfloor$ digits. Since 10^k has $k+1$ digits, the inequality $q + r \geqslant 10^k$ implies that r has at least k digits and also that at least $k - \lfloor \log k \rfloor - 1$ left-most digits in r must be all 9's. With the fact that $S(r) = S(t') - S(q) < k$, we derive the inequality

$$9(k - \lfloor \log k \rfloor - 1) < k,$$

which is equivalent to

$$\frac{8}{9}k - \lfloor \log k \rfloor - 1 < 0.$$

This is a contradiction, because the quantity $\frac{8}{9}k - \lfloor \log k \rfloor - 1$ increases with k and is positive for all $k \geqslant 2$. □

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