Arab Journal of Mathematical Sciences

Arab J Math Sci 22 (2016) 77–85

Little Hankel operators on the Bergman space

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Received 17 May 2014; received in revised form 5 May 2015; accepted 27 May 2015
Available online 6 June 2015

Abstract. In this paper we obtain a characterization of little Hankel operators defined on the Bergman space of the unit disk and then extend the result to vector valued Bergman spaces. We then derive from it certain asymptotic properties of little Hankel operators.

Keywords: Little Hankel operators; Toeplitz operators; Inner functions; Bergman space; Hardy space

2010 Mathematics Subject Classification: 47B38; 47B33; 47B35

1. Introduction

Let $dA(z)$ be the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. The Bergman space $A^2(\mathbb{D})$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in A^2(\mathbb{D})$ such that $f(z) = \langle f, K_z \rangle$ for every $f \in A^2(\mathbb{D})$. The normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\|\cdot\|_2$ and the inner product $\langle , \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. Then $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis [23] for $A^2(\mathbb{D})$. Let $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\overline{w})^2} = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$. Let $L^\infty(\mathbb{D})$ be the space of all essentially bounded Lebesgue measurable functions on $\mathbb{D}$ with the norm $\|f\|_\infty = ess \sup_{z\in\mathbb{D}} |f(z)|$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $A^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$. The Hankel operator $H_\phi : A^2(\mathbb{D}) \to (A^2(\mathbb{D}))^\perp$ with symbol $\phi \in L^\infty(\mathbb{D})$ is defined by $H_\phi f = (I - P)(\phi f)$. The little Hankel

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Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.05.002

operator $S_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ is defined by $S_\phi f = PJ(\phi f)$ where $J : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ is defined as $Jf(z) = f(\overline{z})$. There are also many equivalent ways of defining little Hankel operators on the Bergman space. For example, for $\phi \in L^\infty(\mathbb{D})$, define $\Gamma_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ as $\Gamma_\phi f = P(\phi Jf)$. It is not difficult to see that $\Gamma_\phi = S_{J\phi}$. Here we refer both the operators S_ϕ and Γ_ϕ as little Hankel operators on the Bergman space. Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \, o \, \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a(z)} = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_a f$ on $\mathbb{D}$ by $U_a f(z) = k_a(z) f(\phi_a(z))$. Notice that U_a is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$ for all $a \in \mathbb{D}$. Further, it can be verified that $U_a^2 = I$, the identity operator, $U_a^* = U_a, U_a(A^2(\mathbb{D})) \subset A^2(\mathbb{D})$ and $U_a((A^2(\mathbb{D}))^\perp) \subset (A^2(\mathbb{D}))^\perp$ for all $a \in \mathbb{D}$. Thus $U_a P = P U_a$ for all $a \in \mathbb{D}$.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $L^2(\mathbb{T})$ be the space of square integrable, measurable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure on $\mathbb{T}$. The sequence $\{\varepsilon_n(z)\}_{n=-\infty}^{\infty} = \{e^{in\theta}\}_{n=-\infty}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{T})$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are $c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, n \in \mathbb{Z},$ where $\mathbb{Z}$ is the set of all integers. The Hardy space $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ consisting of functions f with $c_n(f) = 0$ for all negative integers n. Since $c_n = c_n(.)$ is a bounded linear functional on $L^2(\mathbb{T})$ for any fixed n, and $H^2(\mathbb{T}) = \bigcap_{n<0} \ker \ c_n$, it follows that $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$ and therefore a Hilbert space.

Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded measurable functions on $\mathbb{T}$. For $\phi \in L^\infty(\mathbb{T})$, we define the multiplication operator M_ϕ from $L^2(\mathbb{T})$ into itself by $M_\phi f = \phi f$. The multiplication here is to be understood in the obvious sense, namely it is the pointwise one: $(\phi f)(e^{i\theta}) = \phi(e^{i\theta})f(e^{i\theta})$ for all $\theta \in [0, 2\pi]$. Let $\mathbb{P}$ be the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. For $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator $\mathcal{T}_\phi$ on $H^2(\mathbb{T})$ is defined by $\mathcal{T}_\phi f = \mathbb{P}(\phi f)$ for f in $H^2(\mathbb{T})$ and the Hankel operator $\mathbb{S}_\phi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ as $\mathbb{S}_\phi f = \mathbb{P}\mathbb{J}(\phi f)$ where $\mathbb{J}(e^{i\theta}) = e^{-i\theta}, 0 \le \theta \le 2\pi$. The function ϕ is called the symbol of the Hankel operator $\mathbb{S}_\phi$. Let $H^2(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ which are harmonic extensions of functions in $H^2(\mathbb{T})$. It is not very important [10] to distinguish $H^2(\mathbb{D})$ from $H^2(\mathbb{T})$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself and $\mathcal{LC}(H)$ be the set of all compact operators in $\mathcal{L}(H)$.

In 1964, Brown and Halmos [5] established that $T \in \mathcal{L}(H^2(\mathbb{T}))$ is a Toeplitz operator if and only if $\mathcal{T}_z^* T \mathcal{T}_z = T$. Further, they have shown that $T\mathcal{T}_z = \mathcal{T}_z T$ if and only if $T = \mathcal{T}_\phi, \phi \in H^\infty(\mathbb{T})$. In 1996, Cao [6] established that there is no nonzero bounded operator $T \in \mathcal{L}(A^2(\mathbb{D}))$ such that $T_z^* T T_z = T$. Englis [11] in 1988 proved that if $A, B \in \mathcal{L}(A^2(\mathbb{D}))$ are such that $AT_f B = T_f$ for all $f \in L^\infty(\mathbb{D})$ then both A and B are scalar multiples of the identity. Nehari [22] in 1957 gave a characterization of Hankel operators on the Hardy space. Nehari showed that if $H \in \mathcal{L}(H^2(\mathbb{T}))$ then $\mathcal{T}_z^* H = H\mathcal{T}_z$ if and only if there exists $\phi \in L^\infty(\mathbb{T})$ such that $H = \mathbb{S}_\phi$, a Hankel operator with symbol ϕ. In 1991, Faour [15] established that if $S \in \mathcal{L}(A^2(\mathbb{D}))$ then $T_z^* S = ST_z$ if and only if there exists a $\phi \in L^\infty(\mathbb{D})$ such that $S = S_\phi$, a little Hankel operator on the Bergman space.

M. Englis [12,14] showed that Toeplitz operators on $A^2(\mathbb{D})$ are dense in $\mathcal{L}(A^2(\mathbb{D}))$ in the weak operator topology and their norm closure contains all compact operators. Barria and Halmos [3], Feintuch [16] and Das [7] introduced the concept of asymptotic Toeplitz and asymptotic Hankel operators on the Hardy and Bergman spaces.

In this paper we obtain a new characterization of little Hankel operators defined on the Bergman space of the disk $A^2(\mathbb{D})$ involving inner functions and then extend the result for operators defined on vector valued Bergman spaces. We also derive from these certain asymptotic properties of little Hankel operators and Toeplitz operators. In Section 2, we obtain a characterization of little Hankel operators defined on $A^2(\mathbb{D})$ and vector valued Bergman spaces $A^2_{\mathbb{C}^n}(\mathbb{D}), n \geq 1$. In Section 3, we derive certain asymptotic properties of little Hankel operators.

2. Characterization of little Hankel operators on the Bergman space

In this section we obtain a characterization of little Hankel operators defined on the Bergman space $A^2(\mathbb{D})$. We then extend the result to obtain a characterization for little Hankel operators defined on vector valued Bergman spaces.

Let $a_1, a_2, \ldots$ be a sequence of complex numbers such that $0 < |a_1| \leq |a_2| \leq \cdots < 1$ and $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. Then the infinite product $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$ converges uniformly in each disk $|z| \leq R < 1$. Each a_n is a zero of $B(z)$, with multiplicity equal to the number of times it occurs in the sequence, and $B(z)$ has no other zeros in $|z| < 1$. Finally, $|B(z)| < 1$ in $|z| < 1$ and $|B(e^{i\theta})| = 1$ a.e.

A function of the form $B(z) = z^m \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$ is called a Blaschke product. Here m is a nonnegative integer and $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. If the set $\{a_n\}$ is empty, it is understood that $B(z) = z^m$.

It is well known [10] that every function $u \in H^\infty(\mathbb{D})$ can be extended almost everywhere on $\mathbb{T}$ via taking radial limits: $u(e^{it}) = \lim_{r \to 1} u(re^{it})$. These radial limits uniquely determine u. Recall that a function $u \in H^\infty(\mathbb{D})$ is called an inner function if $|u(e^{it})| = 1$ almost everywhere on $\mathbb{T}$. A finite Blaschke product is a function of the form $B(z) = \lambda \prod_{j=1}^{n} \frac{a_j - z}{1 - \overline{a_j} z}$ where $|a_j| < 1$ for all j and $|\lambda| = 1$.

In Theorem 1, we obtain a characterization of little Hankel operators on the Bergman space.

Theorem 1. *Let $S \in \mathcal{L}(A^2(\mathbb{D}))$. Then $T_{\bar{q}} S = ST_q$ for all inner functions $q \in H^\infty(\mathbb{D})$ if and only if S is a little Hankel operator on the Bergman space. That is, $S = S_\phi$ for some $\phi \in L^\infty(\mathbb{D})$.*

Proof. Let $T_{\bar{q}} S = ST_q$ for all inner functions $q \in H^\infty(\mathbb{D})$. Since z is inner, we obtain $T_{\bar{z}} S = ST_z$. By Theorem 2 of [15], there exists $\psi \in L^\infty(\mathbb{D})$ such that $S = S_\psi$, a little Hankel operator on $A^2(\mathbb{D})$ with symbol ψ.

Now to prove the converse, without loss of generality, let $S = S_{\phi o \phi_a}$ for some $\phi \in L^\infty(\mathbb{D}), a \in \mathbb{D}$. For if $S = S_\psi$ then $S = S_{(\psi o \phi_a) o \phi_a}$ for all $a \in \mathbb{D}$. Then $T^*_{\phi_a} S = U_a T_z^* U_a S_{\phi o \phi_a} = U_a T_z^* U_a U_a S_\phi U_a = U_a T_z^* S_\phi U_a$ as $U_a^2 = I$. Similarly, $ST_{\phi_a} = U_a S_\phi U_a U_a T_z U_a = U_a S_\phi T_z U_a$. It is shown in [15] that $T_z^* S_\phi = S_\phi T_z$. Thus $T^*_{\phi_a} S = ST_{\phi_a}$. Let $B_n =$

$\prod_{i=1}^{n} \frac{\overline{a_i}}{|a_i|} \frac{a_i - z}{1-\overline{a_i}z} = \prod_{i=1}^{n} \frac{\overline{a_i}}{|a_i|} \phi_{a_i}$ be a finite Blaschke product in $H^\infty(\mathbb{D})$. Let $\alpha_i = \frac{\overline{a_i}}{|a_i|}, i = 1, \ldots, n$. It is well known [1] that $T^*_{\psi_1} T_{\psi_2} = T_{\bar{\psi}_1 \psi_2}$ if either $\psi_1 \in H^\infty(\mathbb{D})$ or $\psi_2 \in H^\infty(\mathbb{D})$. Using this we obtain,

$$\begin{aligned} T^*_{B_n} S &= \left(T^*_{\prod\limits_{i=1}^{n} \alpha_i \phi_{a_i}} \right) S \\ &= T^*_{\alpha_1 \phi_{a_1}} T^*_{\alpha_2 \phi_{a_2}} \cdots T^*_{\alpha_n \phi_{a_n}} S \\ &= S T_{\alpha_1 \phi_{a_1}} T_{\alpha_2 \phi_{a_2}} \cdots T_{\alpha_n \phi_{a_n}} \\ &= S T_{\prod\limits_{i=1}^{n} \alpha_i \phi_{a_i}} = S T_{B_n}. \end{aligned}$$

Since every Blaschke product B is the uniform limit [17] of finite Blaschke products, we obtain $T^*_B S = S T_B$. It is shown in [20] that every inner function is the uniform limit of Blaschke products and hence it follows that $T^*_q S = S T_q$ for all inner functions $q \in H^\infty(\mathbb{D})$. $\square$

Let M_n be the set of all $n \times n$ matrices with complex entries. Let $A^2_{\mathbb{C}^n}(\mathbb{D}) = A^2(\mathbb{D}) \otimes \mathbb{C}^n$. For $\Phi \in L^\infty_{M_n}(\mathbb{D}) = L^\infty(\mathbb{D}) \otimes M_n$, define $S_\Phi : A^2_{\mathbb{C}^n}(\mathbb{D}) \to A^2_{\mathbb{C}^n}(\mathbb{D})$ as $S_\Phi f = \mathcal{P}(\mathfrak{J}(\Phi f))$ where $\mathcal{P}$ is the orthogonal projection from $L^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ onto $A^2_{\mathbb{C}^n}(\mathbb{D})$, $\mathfrak{J} : L^{2,\mathbb{C}^n}(\mathbb{D}, dA) \to L^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ is defined as $\mathfrak{J}F(z) = F(\overline{z})$ and $(\Phi F)(z) = \Phi(z)F(z), z \in \mathbb{D}$. Define T_Φ on $A^2_{\mathbb{C}^n}(\mathbb{D})$ as $T_\Phi F = \mathcal{P}(\Phi F)$. Let $I_{n\times n}$ be the $n \times n$ identity matrix.

Theorem 2. *Let $S \in \mathcal{L}(A^2_{\mathbb{C}^n}(\mathbb{D}))$. Then $T_{\bar{q} I_{n\times n}} S = S T_{q I_{n\times n}}$ for all inner functions $q \in H^\infty(\mathbb{D})$ if and only if S is a little Hankel operator on the Bergman space $A^2_{\mathbb{C}^n}(\mathbb{D})$. That is, $S = S_\Phi$ for some $\Phi \in L^\infty_{M_n}(\mathbb{D})$.*

Proof. Let $S \in \mathcal{L}(A^2_{\mathbb{C}^n}(\mathbb{D}))$. Since $A^2_{\mathbb{C}^n}(\mathbb{D}) = A^2(\mathbb{D}) \oplus A^2(\mathbb{D}) \oplus \cdots \oplus A^2(\mathbb{D})$, hence

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & \cdots & S_{2n} \\ \cdots & \cdots\cdots & \cdots & \cdots & \\ S_{n1} & S_{n2} & \cdots & \cdots & S_{nn} \end{bmatrix}$$

where $S_{ij} \in \mathcal{L}(A^2(\mathbb{D})), 1 \le i, j \le n$. Thus $T_{\bar{q} I_{n\times n}} S = S T_{q I_{n\times n}}$ for all inner functions $q \in H^\infty(\mathbb{D})$ if and only if $T_{\bar{q}} S_{ij} = S_{ij} T_q$ for all inner functions $q \in H^\infty(\mathbb{D}), 1 \le i, j \le n$. But from Theorem 1, it follows that $S_{ij} = S_{\varphi_{ij}} \in \mathcal{L}(A^2(\mathbb{D}))$, a little Hankel operator on $A^2(\mathbb{D})$ with symbol $\varphi_{ij} \in L^\infty(\mathbb{D}), 1 \le i, j \le n$. Hence

$$S = \begin{bmatrix} S_{\varphi_{11}} & S_{\varphi_{12}} & \cdots & \cdots & S_{\varphi_{1n}} \\ S_{\varphi_{21}} & S_{\varphi_{22}} & \cdots & \cdots & S_{\varphi_{2n}} \\ \cdots & \cdots\cdots & \cdots & \cdots & \\ S_{\varphi_{n1}} & S_{\varphi_{n2}} & \cdots & \cdots & S_{\varphi_{nn}} \end{bmatrix} = S_\Phi \in \mathcal{L}(A^2_{\mathbb{C}^n}(\mathbb{D})),$$

a little Hankel operator with symbol $\Phi \in L^\infty_{M_n}(\mathbb{D})$ and

$$\Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \cdots & \varphi_{2n} \\ \varphi_{n1} & \varphi_{n2} & \cdots & \cdots & \varphi_{nn} \end{bmatrix}.$$

Now suppose S_Φ is a little Hankel operator on $A^2_{\mathbb{C}^n}(\mathbb{D})$ with symbol

$$\Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \cdots & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \cdots & \varphi_{2n} \\ \varphi_{n1} & \varphi_{n2} & \cdots & \cdots & \varphi_{nn} \end{bmatrix} \in L^\infty_{M_n}(\mathbb{D}).$$

Since $A^2_{\mathbb{C}^n}(\mathbb{D}) = A^2(\mathbb{D}) \oplus \cdots \oplus A^2(\mathbb{D})$, hence

$$S_\Phi = \begin{bmatrix} S_{\varphi_{11}} & S_{\varphi_{12}} & \cdots & \cdots & S_{\varphi_{1n}} \\ S_{\varphi_{21}} & S_{\varphi_{22}} & \cdots & \cdots & S_{\varphi_{2n}} \\ S_{\varphi_{n1}} & S_{\varphi_{n2}} & \cdots & \cdots & S_{\varphi_{nn}} \end{bmatrix}$$

where $S_{\varphi_{ij}} \in \mathcal{L}(A^2(\mathbb{D}))$ is a little Hankel operator on $A^2(\mathbb{D})$ with symbol $\varphi_{ij} \in L^\infty(\mathbb{D})$. From Theorem 1, it follows that $T_{\bar{q}} S_{\varphi_{ij}} = S_{\varphi_{ij}} T_q, 1 \leq i,j \leq n$, for all inner functions $q \in H^\infty(\mathbb{D})$. Hence

$$T_{\bar{q}I_{n\times n}} S_\Phi = S_\Phi T_{qI_{n\times n}} \text{ for all inner functions } q \in H^\infty(\mathbb{D}). \quad \square$$

3. Asymptotic properties of little Hankel operators

In this section we derive certain asymptotic properties of Toeplitz and little Hankel operators defined on $A^2(\mathbb{D})$. Define a map W from $H^2(\mathbb{D})$ onto $A^2(\mathbb{D})$ as $Wz^n = \sqrt{n+1}z^n$, $n \geq 0, n \in \mathbb{N}$ and $z \in \mathbb{D}$. The map W is an isometry operator [13] from $H^2(\mathbb{D})$ onto $A^2(\mathbb{D})$, $WW^* = I_{A^2(\mathbb{D})}$, the identity operator from $A^2(\mathbb{D})$ into itself and $W^*W = I_{H^2(\mathbb{D})}$, the identity operator from $H^2(\mathbb{D})$ into itself.

Theorem 3 is just a superposition of results of the authors appeared in [8]. To make the presentation self-contained, we give a sketch of the proof here again.

Theorem 3. *Suppose $T \in \mathcal{L}(A^2(\mathbb{D}))$. The following hold:*

(i) *If $\mathcal{T}^*_{q^+} W^*TW = W^*TW\mathcal{T}_q$ for all inner functions $q \in H^\infty(\mathbb{D})$ then there exists a $\phi \in L^\infty(\mathbb{T})$ such that $T = W\mathbb{S}_\phi W^*$ where $\mathbb{S}_\phi$ is a Hankel operator on the Hardy space $H^2(\mathbb{D})$. Here $q^+(z) = \overline{q(\overline{z})}$.*

(ii) *Suppose $N^*(W^*TW)N = W^*TW$ where N is the unilateral shift on $H^2(\mathbb{D})$. Then there exists $\phi \in L^\infty(\mathbb{T})$ such that $W^*TW = \mathcal{T}_\phi$, a Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ and $(WJ^m_q W^*)T(W\mathcal{T}^m_{(zq)}W^*) \to L$ strongly where $L = W\mathbb{S}_\phi W^* \in \mathcal{L}(A^2(\mathbb{D}))$ as $m \to \infty$.*

(iii) *If $K \in \mathcal{LC}(A^2(\mathbb{D}))$ then $(W\mathcal{T}^{*^m}_{q^+}W^*)K \to 0$ in norm as $m \to \infty$, for all inner functions $q \in H^\infty(\mathbb{D})$.*

(iv) *If $\mathcal{T}^*_{q^+} W^*TW = W^*TW\mathcal{T}_q$ for all inner functions $q \in H^\infty(\mathbb{D})$ then $T(W\mathcal{T}^m_q W^*) \to 0$ strongly as $m \to \infty$.*

Proof. If $T \in \mathcal{L}(A^2(\mathbb{D}))$ then $W^*TW \in \mathcal{L}(H^2(\mathbb{D}))$ and since $\mathcal{T}^*_q(W^*TW) = (W^*TW)\mathcal{T}_q$ for all inner functions $q \in H^\infty(\mathbb{D})$, hence by [8], there exists a function $\phi \in L^\infty(\mathbb{T})$ such that $W^*TW = \mathbb{S}_\phi$ where $\mathbb{S}_\phi$ is a Hankel operator on the Hardy space $H^2(\mathbb{D})$. This implies (i).

We shall now prove (ii). The first part of (ii) follows from [5]. Suppose $T \in \mathcal{L}(A^2(\mathbb{D}))$. By [8], $J^m_q(W^*TW)\mathcal{T}^m_{(zq)} \to \mathbb{S}_\phi$ where $\mathbb{S}_\phi$ is the Hankel operator on the Hardy space with symbol $\phi \in L^\infty(\mathbb{T})$. Hence $(WJ^m_q W^*)T(W\mathcal{T}^m_{(zq)}W^*) \to L$ strongly as $m \to \infty$ where $L = W\mathbb{S}_\phi W^*$.

To prove (iii), let $K \in \mathcal{LC}(A^2(\mathbb{D}))$. Then $W^*KW \in \mathcal{LC}(H^2(\mathbb{D}))$. By [8], $\mathcal{T}_{q^+}^{*^m}(W^*KW) \to 0$ in norm as $m \to \infty$ for all inner functions $q \in H^\infty(\mathbb{D})$. This implies $(W\mathcal{T}_{q^+}^{*^m}W^*)K \to 0$ in norm as $m \to \infty$.

Now we shall prove (iv). Suppose $\mathcal{T}_{q^+}^*(W^*TW) = (W^*TW)\mathcal{T}_q$ for all inner functions $q \in H^\infty(\mathbb{D})$. Then by [8], there exists a function $\phi \in L^\infty(\mathbb{T})$ such that $W^*TW = \mathbb{S}_\phi$. Again by [8], it follows that $(W^*TW)\mathcal{T}_q^m = \mathbb{S}_\phi\mathcal{T}_q^m \to 0$ strongly as $m \to \infty$. □

Theorem 4. *Let $\phi \in L^\infty(\mathbb{D})$ be such that $\|\phi\|_\infty \le 1, \ker(T_{1-\overline{\phi}}) = \{0\}$. Then*

$$S_\Xi^* T^n_{\frac{1+\phi}{2}} \xrightarrow{w} 0 \quad \textit{for all } \Xi \in L^\infty(\mathbb{D}).$$

Proof. Assume that $\|\phi\|_\infty \le 1$. Then $\|T_{\frac{1+\phi}{2}}\| \le 1$. Hence the sequence $\{T^{*n}_{\frac{1+\phi}{2}}\}_{n=0}^\infty$ is bounded. So by [2] the sequence $\{T^{*^n}_{\frac{1+\phi}{2}}\}_{n=0}^\infty$ has a subsequence which converges to an operator $K \in \mathcal{L}(A^2(\mathbb{D}))$ in the weak operator topology. Without loss of generality, we shall assume the original sequence $\{T^{*n}_{\frac{1+\phi}{2}}\}_{n=0}^\infty$ converges to an operator $K \in \mathcal{L}(A^2(\mathbb{D}))$ in the weak operator topology. Hence $\left\langle \left(T^{*n+1}_{\frac{1+\phi}{2}} f - T^{*n}_{\frac{1+\phi}{2}} f\right), g\right\rangle \longrightarrow 0$ for every $f, g \in A^2(\mathbb{D})$ and $\left\{\langle T^{*n+1}_{\frac{1+\phi}{2}} f, g\rangle\right\}_{n=0}^\infty$ converges to $\langle Kf, g\rangle$ as n tends to ∞ for all $f, g \in A^2(\mathbb{D})$. This implies

$$\langle T^{*^n}_{\frac{1+\phi}{2}} f, T_{\frac{1+\phi}{2}} g\rangle \longrightarrow \langle Kf, g\rangle \quad \text{for all} f, g \in A^2(\mathbb{D}).$$

Thus $\langle Kf, T_{\frac{1+\phi}{2}} g\rangle = \langle Kf, g\rangle$ for all $f, g \in A^2(\mathbb{D})$ and therefore $T^*_{\frac{1+\phi}{2}} K = K$. Further since $\left\{\langle T^{*^n}_{\frac{1+\phi}{2}} T^*_{\frac{1+\phi}{2}} f, g\rangle\right\}_{n=0}^\infty$ converges to $\langle Kf, g\rangle$ for all $f, g \in A^2(\mathbb{D})$, we get

$$\left\langle KT^*_{\frac{1+\phi}{2}} f, g\right\rangle = \langle Kf, g\rangle \quad \text{for all } f, g \in A^2(\mathbb{D}).$$

Thus $KT^*_{\frac{1+\phi}{2}} = K$ and $T^{*^n}_{\frac{1+\phi}{2}} K = K$ for all $n \in \mathbb{Z}_+$. That is,

$$\langle T^{*^n}_{\frac{1+\phi}{2}} Kf, g\rangle = \langle Kf, g\rangle \quad \text{for all } f, g \in A^2(\mathbb{D}).$$

Taking limits on both sides, we obtain $K^2 = K$. This proves that the operator K is an idempotent. Moreover, $T^*_{\frac{1+\phi}{2}} K = K$ implies $T^*_\phi K = K$ and $KT^*_{\frac{1+\phi}{2}} = K$ implies $KT^*_\phi = K$. So $Ran\ (K) \subseteq \ker(T_{1-\overline{\phi}})$.

On the other hand, for $f \in \ker(T_{1-\overline{\phi}})$, we have $T^*_\phi f = f$, so $T^*_{\frac{1+\phi}{2}} f = f$. Hence $T^{*^n}_{\frac{1+\phi}{2}} f = f$ for all $n \in \mathbb{Z}_+$ and this implies $Kf = f$. Hence $Ran\ (K) = \ker(T_{1-\overline{\phi}})$.

To prove the inclusion $Ran\ (T_{1-\overline{\phi}}) \subseteq \ker(K)$, let $f \in A^2(\mathbb{D})$ be an arbitrary element and $g = f - T^*_\phi f$. Then we have $Kg = Kf - KT^*_\phi f = Kf - Kf = 0$. Hence $g \in \ker(K)$. Thus we have shown that, if $\|\phi\|_\infty \le 1$ then there exists an idempotent $K \in \mathcal{L}(A^2(\mathbb{D}))$ whose range is $\ker(T_{1-\overline{\phi}})$ and kernel contains $Ran\ (T_{1-\overline{\phi}})$.

Now suppose $\langle T^n_{\frac{1+\phi}{2}} f, g\rangle \longrightarrow 0$ for all $f, g \in A^2(\mathbb{D})$. Then

$$\langle f, T^{*^n}_{\frac{1+\phi}{2}} g\rangle \longrightarrow 0$$

for all $f, g \in A^2(\mathbb{D})$. Since $\|\phi\|_\infty \leq 1$, the sequence $\{T^{*^n}_{\frac{1+\phi}{2}}\}$ is bounded and we have already seen in the first part that

$$T^{*^n}_{\frac{1+\phi}{2}} \xrightarrow{w} K$$

in $\mathcal{L}(A^2(\mathbb{D}))$ and the operator K is an idempotent. Thus it follows that $K = 0$, the sequence $T^{*^n}_{\frac{1+\phi}{2}} \xrightarrow{w} 0$ and $Ran\ (K) = \ker(T_{1-\overline{\phi}}) = \{0\}$.

Conversely, assume that $\ker(T_{1-\overline{\phi}}) = \{0\}$. Since $\|\phi\|_\infty \leq 1$, the sequence $\{T^{*^n}_{\frac{1+\phi}{2}}\}$ is bounded and

$$\left\{T^{*^n}_{\frac{1+\phi}{2}}\right\} \xrightarrow{w} K \in \mathcal{L}(A^2(\mathbb{D})),$$

the operator K is an idempotent and $Ran\ K = \ker T_{1-\overline{\phi}} = \{0\}$. Thus it follows that $K = 0$ and $T^{*^n}_{\frac{1+\phi}{2}} \xrightarrow{w} 0$. The result follows since $S^*_{\Xi} \in \mathcal{L}(A^2(\mathbb{D}))$ for all $\Xi \in L^\infty(\mathbb{D})$. $\square$

If m is a nonnegative integer and $z \in \mathbb{D}$, the function $K^{(m)}_z(w) = \frac{1}{(1-\bar{z}w)^{2+m}}, w \in \mathbb{D}$ is the reproducing kernel at z in the weighted Bergman space $L^2_a(dA_m)$, where

$$dA_m(w) = (m+1)(1-|w|^2)^m dA(w).$$

The m-Berezin transform of an operator $S \in \mathcal{L}(A^2(\mathbb{D}))$ is defined as

$$(B_m S)(z) = (m+1)(1-|z|^2)^{2+m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \left\langle S(w^j K^{(m)}_z), w^j K^{(m)}_z \right\rangle.$$

It is clear that $B_m S \in L^\infty(\mathbb{D})$ for every $S \in \mathcal{L}(A^2(\mathbb{D}))$. Using the fact that

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j |w|^{2j} = (1-|w|^2)^m,$$

we see that if $S = T_\phi$ with $\phi \in L^\infty(\mathbb{D})$, then

$$\begin{aligned}(B_m\phi)(z) &= (B_m T_\phi)(z) \\ &= (m+1)(1-|z|^2)^{2+m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \int_{\mathbb{D}} \frac{\phi(w)|w|^{2j}}{|1-\bar{z}w|^{2(2+m)}} dA(w) \\ &= \int_{\mathbb{D}} \phi(w) \frac{(1-|z|^2)^{2+m}}{|1-\bar{z}w|^{2(2+m)}} (m+1)(1-|w|^2)^m dA(w) \\ &= \int_{\mathbb{D}} \phi(\phi_z(\rho))(m+1)(1-|\rho|^2)^m dA(\rho),\end{aligned}$$

where the last equality comes from the change of variables $w = \phi_z(\rho)$. Notice that $\|B_m(\phi)\|_\infty \leq \|\phi\|_\infty$ for all $\phi \in L^\infty(\mathbb{D})$. The 0-Berezin transform of an operator is the usual Berezin transform. The m-Berezin transforms of functions (not necessarily bounded) were introduced by Berezin in [4]. It is not difficult to verify that for $S \in \mathcal{L}(A^2(\mathbb{D}))$ and $m \geq 0$;

$$(m+2)(1-|z|^2) B_m \left(S - T_{\bar{w}} S T_w\right)(z) = (m+1) B_{m+1} \left(T_{1-\bar{w}z} S T_{1-w\bar{z}}\right)(z)$$

for every $z \in \mathbb{D}$ and $\|B_m S\|_\infty \leq (m+2)2^m \|S\|$.

Definition 1. A function $q \in A^2(\mathbb{D})$ is called an inner function in $A^2(\mathbb{D})$ if $\int_{\mathbb{D}}(|q(z)|^2 - 1)g(z)dA(z) = 0$ for all $g \in H^\infty(\mathbb{D})$.

For more details about Bergman space inner functions see [18].

Theorem 5. *Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic. Suppose there is $p > 3$ such that*

$$\sup_{z\in\mathbb{D}} \|T_{(B_m S_{J\phi})o\phi_z} 1\|_p < C \quad \textit{and} \quad \sup_{z\in\mathbb{D}} \|T^*_{(B_m S_{J\phi})o\phi_z} 1\|_p < C \tag{1}$$

where $C > 0$ is independent of m and q be a nonconstant inner function in $A^2(\mathbb{D})$. Then $T_{\overline{q}^m B_m S_{J\phi}} \longrightarrow 0$ as $m \to \infty$.

Proof. Assume that the operator $S_{J\phi} \in \mathcal{L}(A^2(\mathbb{D}))$ satisfies the condition (1). It follows from [21] that $T_{B_m S_{J\phi}} \longrightarrow S_{J\phi}$ in the operator norm as $m \longrightarrow \infty$. In particular $T_{B_m S_{J\phi}} \xrightarrow{w} S_{J\phi}$. Further, notice that $\|q\|_{A^2(\mathbb{D})} = 1$ and $|q(z)| < 1$ for all $z \in \mathbb{D}$. It is well known [23] that the reproducing kernels $\{K_\lambda\}_{\lambda\in\mathbb{D}}$ span $A^2(\mathbb{D})$. So, let $f = \sum_{i=1}^m b_i K_{\lambda_i}$. Then $T_{\overline{q}^m}\left(\sum_{i=1}^m b_i K_{\lambda_i}\right) = \sum_{i=1}^m b_i \overline{q(\lambda_i)}^m K_{\lambda_i}$. Hence

$$\left\|T_{\overline{q}^m}\left(\sum_{i=1}^m b_i K_{\lambda_i}\right)\right\| \le \sum_{i=1}^m |b_i||q(\lambda_i)|^m \|K_{\lambda_i}\| \to 0$$

as $m \to \infty$. Thus $T_{\overline{q}^m} \longrightarrow 0$ as $m \to \infty$. Hence, it follows from [9] that $T_{\overline{q}^m B_m S_{J\phi}} \longrightarrow 0$. $\square$

Theorem 6. *Let S_ϕ be a little Hankel operator on $A^2(\mathbb{D})$ with the symbol $\phi \in L^\infty(\mathbb{D})$ with $S_\phi = V|S_\phi|$ be the polar decomposition of S_ϕ and $\{z_n\}_{n=1}^\infty$ be a sequence in $\mathbb{D}$ with $|z_n| \to 1$. Then $U_{z_n}(S_\phi)_n \xrightarrow{w} 0$ as $n \to \infty$. Here $(S_\phi)_n = S_\phi(|S_\phi| + \frac{1}{n})^{-1}$.*

Proof. We shall first show that $U_{z_n} \xrightarrow{w} 0$. Since $\text{span}\{k_z : z \in \mathbb{D}\}$ is dense in $A^2(\mathbb{D})$, it suffices to show that for all $z, w \in \mathbb{D}$, we have $\lim_{n\to\infty}\langle U_{z_n}k_z, k_w\rangle = 0$. Fix $z, w \in \mathbb{D}$. For $n \ge 1$,

$$\langle U_{z_n}k_z, k_w\rangle = \frac{[(1-|w|^2)(1-|z|^2)(1-|z_n|^2)]}{[(1-\langle\phi_{z_n}(w), z\rangle)(1-\langle w, z_n\rangle)]^2}.$$

Since $|\langle\phi_{z_n}(w), z\rangle| \le |z|$ and $|\langle w, z_n\rangle| \le |w|$, we obtain

$$|\langle U_{z_n}k_z, k_w\rangle| \le \frac{(1-|w|^2)(1-|z|^2)(1-|z_n|^2)}{(1-|z|)^2(1-|w|)^2}.$$

It then follows that $\lim_{n\to\infty}\langle U_{z_n}k_z, k_w\rangle = 0$. For more details see [19].

Further, let $(S_\phi)_n = S_\phi(|S_\phi| + \frac{1}{n})^{-1}$. Next, we shall prove that $(S_\phi)_n \longrightarrow V$ strongly as $n \to \infty$. Let $\{E_\lambda\}$ be the spectral family for $|S_\phi|$. Then $(S_\phi)_n$ strongly converges to $I - E_0$ as $n \to \infty$. The reason is as follows: Notice that $|S_\phi| = \int_0^\infty \lambda dE_\lambda$ is the spectral decomposition of $|S_\phi|$. Let $\mathfrak{S}_n = |S_\phi|(|S_\phi| + \frac{1}{n})^{-1}$. Then $\mathfrak{S}_n E_0 f = (|S_\phi| + \frac{1}{n})^{-1}|S_\phi|E_0 f = 0$ for

$f \in A^2(\mathbb{D})$ and

$$\begin{aligned}\|\mathfrak{S}_n f - (I - E_0)f\|^2 &= \|(\mathfrak{S}_n - I)(I - E_0)f\|^2 \\ &= \int_0^\infty \left|\frac{\lambda}{\lambda + \frac{1}{n}} - 1\right|^2 d\|E_\lambda(I - E_0)f\|^2 \\ &= \int_0^\infty \left|\frac{\frac{1}{n}}{\lambda + \frac{1}{n}}\right|^2 d\|E_\lambda(I - E_0)f\|^2.\end{aligned}$$

From Lebesgue's dominated convergence theorem, it follows that $\mathfrak{S}_n$ strongly converges to $I - E_0$ as $n \to \infty$. Thus we have $(S_\phi)_n \to V(I - E_0)$ strongly as $n \to \infty$. Since E_0 is the projection onto the eigenspace $\{f \in A^2(\mathbb{D}) : S_\phi f = 0\}$, we get $VE_0 = 0$. Consequently, $(S_\phi)_n \longrightarrow V$ strongly as $n \to \infty$. Hence it follows from [9] that $U_{z_n}(S_\phi)_n \xrightarrow{w} 0$ as $n \to \infty$. $\square$

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