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Linear maps preserving *G*-unitary operators in Hilbert space

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space and $\mathscr{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We give the concrete forms of surjective continuous unital linear maps from $\mathscr{B}(\mathcal{H})$ onto itself that preserve $\boldsymbol{G}$-unitary operators.

Keywords: Linear preserver; Jordan homomorphisms; Operators on spaces with an indefinite metric; Unitary operators

Mathematics Subject Classifications: Primary 15A86; 15A04; 47B50; Secondary 47L30; 16W20

1. INTRODUCTION

Linear preserver problems are an active research area in Matrix, Operator Theory and Banach Algebras. It has attracted the attention of many mathematicians in the last few decades [3–5,7,8,11,12]. A linear preserver is a linear map of an algebra $\mathscr{A}$ into itself which, roughly speaking, preserves certain properties on some elements in $\mathscr{A}$. Linear preserver problems concern the characterization of such maps. Automorphisms and anti-automorphisms certainly preserve various properties of the elements. Therefore, it is not surprising that these two types of maps often appear in the conclusions of the results. In this paper, we shall concentrate on the case when $\mathscr{A} = \mathscr{B}(\mathcal{H})$, the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We should point out that a great deal of work has been devoted to the case when $\mathcal{H}$ is finite dimensional, it is the case when $\mathscr{A}$ is a matrix algebra (see survey articles [9,8]). The first papers concerning this case date back to the previous century [5].

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The aim of this paper is to characterize surjective continuous linear maps $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ preserving G-unitary-operators. As a consequence, we describe surjective continuous linear maps from $\mathscr{B}(\mathcal{H})$ onto itself that preserves G-quasi-unitary operators.

2. Preliminaries

Let $(\mathcal{H}, [.,.])$ be an indefinite inner product space (cf [1,2,13]). It is well known that $(\mathcal{H}, [.,.])$ is a complete indefinite inner product if and only if $\mathcal{H}$ is a Hilbert space with some inner product $\langle .,. \rangle$ and there exists an invertible self-adjoint operator $G \in \mathscr{B}(\mathcal{H})$ such that

$$[x, y] = [x, y]_G = \langle Gx, y \rangle$$

for all $x, y \in \mathcal{H}$, and the set of all bounded linear operators on $\mathcal{H}$ with respect to the indefinite inner product $[.,.]$ is the same as $\mathscr{B}(\mathcal{H}, \langle .,. \rangle)$. Thus, we may always assume that $\mathcal{H}$ is a Hilbert space with inner product $\langle .,. \rangle$. For an invertible self-adjoint operator G and $A \in \mathscr{B}(\mathcal{H})$, denote $A^\sharp$ the indefinite adjoint of A with respect to G, i.e., the G-adjoint of A, which is determined by

$$[x, A^\sharp y]_G = [Ax, y]_G$$

for all $x, y \in \mathcal{H}$. Clearly, $A^\sharp = G^{-1}A^*G$, where A^* is the adjoint of A with respect to the inner product $\langle .,. \rangle$.

We continue by some definitions.

Definition 2.1. An operator p on $\mathscr{B}(\mathcal{H})$ is called G-projection if and only if

$$p^2 = p = p^\sharp.$$

Definition 2.2. An operator T on $\mathscr{B}(\mathcal{H})$ is called G-unitary if and only if

$$T^\sharp T = TT^\sharp = I.$$

In 1977 Phadke et al. in [10] have introduced the notion of a quasi-unitary operator on a Hilbert space as follows.

Definition 2.3. An operator T on Hilbert space $\mathcal{H}$ is called quasi-unitary if

$$TT^* = T^*T = T + T^*.$$

It is easy to see that the following Proposition holds true.

Proposition 2.4. *An operator T on Hilbert space is quasi-unitary if and only if $I - T$ is a unitary operator.*

By combining definitions of G-unitary and quasi-unitary operator, we define G-quasi-unitary operator as follows.

Definition 2.5. An operator T on Hilbert space $\mathcal{H}$ is called G-quasi-unitary operator if

$$T^{\sharp}T = TT^{\sharp} = T^{\sharp} + T.$$

For every pair of vectors $x, y \in \mathcal{H}$, the symbol $x \otimes y$ stands for the rank-1 linear operator on $\mathcal{H}$ defined by $(x \otimes y)z = \langle z, y\rangle x$ for any $z \in \mathcal{H}$. Let $\mathscr{F}(\mathcal{H})$ and $\mathscr{K}(\mathcal{H})$ denote the ideals of all finite-rank and the compact operators, respectively. Afterward, a linear map ϕ from algebra $\mathscr{A}$ into an algebra $\mathscr{B}$ is called a Jordan homomorphism if $\phi(x^2) = \phi(x)^2$ for every $x \in \mathscr{A}$. A well known result of Herstein ([6], Theorem 3.1) shows that a Jordan homomorphism on prime algebra is either a homomorphism or an anti-homomorphism.

3. Linear map preserving G-unitary operators

We begin by mentioning that when studying linear maps from $\mathscr{B}(\mathcal{H})$ onto itself preserving G-unitary operators, there is no loss of generality in assuming that the map is unital. Indeed, if $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ preserves G-unitary operators, then $\phi(I)$ is G-unitary operator. We can instead work with the linear map ψ defined by $\psi(A) = \phi(I)^{\sharp}\phi(A)$ for all $A \in \mathscr{B}(\mathcal{H})$. This map clearly preserves G-unitary operators and is unital.

Let us start with a Lemma which will be useful in the proof of the main Theorems.

Lemma 3.1. *Operator* $\exp(itS)$ *is G-unitary operator for every $t \in \mathbb{R}$ and every G-self-adjoint $S \in \mathscr{B}(\mathcal{H})$.*

Proof. We have $(S^{\sharp})^k = G^{-1}(S^*)^k G$. Since S is a G-self-adjoint so $G^{-1}S^*G = S$ and so $G^{-1}(S^*)^k G = S^k$. It follows that

$$\begin{aligned}(\exp(itS))^{\sharp} &= \sum_{k=0}^{\infty} \frac{(-it)^k}{k!}(S^{\sharp})^k \\ &= \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} G^{-1}(S^*)^k G \\ &= \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} S^k \\ &= \exp(-itS).\end{aligned}$$

Which implies that

$$(\exp(itS))^{\sharp}(\exp(itS)) = (\exp(itS))(\exp(itS))^{\sharp} = I.$$

Consequently, the desired result follows. □

Lemma 3.2. *Let $\mathcal{H}$ be a separable complex Hilbert space. If $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ is a continuous and surjective homomorphism or anti-homomorphism, then ϕ is injective.*

Proof. We consider only the case where ϕ is a continuous and surjective homomorphism.

Let $A \in \mathscr{B}(\mathcal{H})$ such that $\phi(A) = 0$. If $A \neq 0$ then there exists a vector $z \in \mathcal{H}$ such that $Az \neq 0$. We have

$$\begin{aligned}
\phi(x \otimes y) &= \phi\left(\frac{1}{||Az||^2}(x \otimes Az)(Az \otimes y)\right) \\
&= \frac{1}{||Az||^2}\phi((x \otimes Az)(Az \otimes y)) \\
&= \frac{1}{||Az||^2}\phi(x \otimes Az)\phi(A)\phi(z \otimes y) \\
&= 0.
\end{aligned}$$

So $\phi(x \otimes y) = 0$ for all $x, y \in \mathcal{H}$. Consequently, ϕ sends all rank-one operators to zero. Moreover, every finite-rank operator can be written as a linear combination of finitely many rank-1 operators. It follows from the linear property of ϕ that $\phi(\mathscr{F}(\mathcal{H})) = 0$. Since $\mathscr{F}(\mathcal{H})$ is dense in $\mathscr{K}(\mathcal{H})$ and ϕ is continuous then $\phi(\mathscr{K}(\mathcal{H})) = 0$. Therefore, $\mathscr{K}(\mathcal{H}) \subset Ker(\phi)$. since ϕ is a continuous and surjective homomorphism, so $Ker(\phi)$ is a closed ideal of $\mathscr{B}(\mathcal{H})$, this implies that $Ker(\phi) = \mathscr{K}(\mathcal{H})$ and after surjectivity of ϕ we get that ϕ is a continuous isomorphism from $\mathscr{B}(\mathcal{H})/\mathscr{K}(\mathcal{H})$ to $\mathscr{B}(\mathcal{H})$, which contradicts the simplicity of $\mathscr{B}(\mathcal{H})/\mathscr{K}(\mathcal{H})$ because $\mathscr{B}(\mathcal{H})$ is not simple. Thus $A = 0$, so ϕ is injective. □

Theorem 3.3. *Let $\mathcal{H}$ be a separable complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a unital linear continuous surjective map. If ϕ preserves G-unitary operators in one direction, then there exist $\lambda = \pm 1$ and invertible operator $U \in \mathscr{B}(\mathcal{H})$ satisfying $UU^\sharp = U^\sharp U = \lambda I$ such that*

$$\phi(A) = \lambda U A U^\sharp,$$

or

$$\phi(A) = \lambda U A^t U^\sharp$$

for all $A \in \mathscr{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of $\mathcal{H}$.

Proof. Pick a G-self-adjoint operator $S \in \mathscr{B}(\mathcal{H})$. By Lemma 3.1 $\exp(itS)$ is a G-unitary operator for every $t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
I = \phi(\exp(itS))^\sharp\phi(\exp(itS)) &= \phi\left(I + itS + \frac{(it)^2}{2!}S^2 + \cdots\right)^\sharp \phi\left(I + itS + \frac{(it)^2}{2!}S^2 + \cdots\right) \\
&= I + it(\phi(S) - \phi(S)^\sharp) - t^2\frac{\phi(S^2)^\sharp + \phi(S^2)}{2} + t^2\phi(S)^\sharp\phi(S)\ldots.
\end{aligned}$$

Hence

$$\phi(S) = \phi(S)^{\sharp} \tag{3.1}$$

and

$$-\frac{\phi(S^2)^{\sharp} + \phi(S^2)}{2} + \phi(S)^{\sharp}\phi(S) = 0 \tag{3.2}$$

for every G-self-adjoint operator S. Using the fact that every $A \in \mathscr{B}(\mathcal{H})$ can be written as $A = S + iT$ with S, T G-self-adjoint. We get from (3.1) and the linearity of ϕ that

$$\begin{aligned}\phi(A^{\sharp}) &= \phi(S - iT)\\ &= \phi(S) - i\phi(T)\\ &= \phi(S)^{\sharp} - i\phi(T)^{\sharp}\\ &= \phi(S + iT)^{\sharp}\\ &= \phi(A)^{\sharp}.\end{aligned}$$

It follows that for every G-self-adjoint operator S, we have $\phi(S)^{\sharp} = \phi(S)$ and by (3.2) $\phi(S^2) = \phi(S)^2$. Operator $S + T$ is G-self-adjoint, hence

$$\phi((S + T)^2) = (\phi(S) + \phi(T))^2,$$

and so

$$\phi(ST + TS) = \phi(S)\phi(T) + \phi(T)\phi(S),$$

Consequently,

$$\begin{aligned}\phi(A^2) &= \phi((S + iT)^2)\\ &= \phi(S^2 - T^2 + i(ST + TS))\\ &= \phi(S^2) - \phi(T^2) + i\phi(ST + TS)\\ &= \phi(S)^2 - \phi(T)^2 + i\phi(S)\phi(T) + \phi(T)\phi(S)\\ &= \phi(S + iT)^2\\ &= \phi(A)^2.\end{aligned}$$

It follows that $\phi(A^2) = \phi(A)^2$ for all operators A in $\mathscr{B}(\mathcal{H})$, which implies that ϕ is a Jordan homomorphism. It is known that every Jordan homomorphism in prime algebra is a homomorphism or an anti-homomorphism. Since $\mathscr{B}(\mathcal{H})$ is a prime algebra, then ϕ is a homomorphism or an anti-homomorphism. By Lemma 3.2 ϕ is injective, so ϕ is an automorphism or an anti-automorphism. Note that an automorphism or an anti-automorphism on $\mathscr{B}(\mathcal{H})$ is of the form $\phi(A) = UAU^{-1}$ for every A in $\mathscr{B}(\mathcal{H})$ or $\phi(A) = UA^tU^{-1}$ for every A in $\mathscr{B}(\mathcal{H})$, where U is an invertible operator. We only consider the first form of ϕ, the proof of the second form is similar to the first. Moreover,

$$\phi(A^{\sharp}) = \phi(A)^{\sharp},$$

so

$$UA^\sharp U^{-1} = (U^{-1})^\sharp A^\sharp U^\sharp,$$

and so

$$UG^{-1}A^*GU^{-1} = G^{-1}(U^{-1})^* A^* U^* G$$

for all A in $\mathscr{B}(\mathcal{H})$. It follows that

$$U^*GUG^{-1}A^* = A^*U^*GUG^{-1},$$

for all A in $\mathscr{B}(\mathcal{H})$. Since the center of $\mathscr{B}(\mathcal{H})$ is the set of scalar operators, $U^*GUG^{-1} = \lambda I$. Then $U^\sharp U = \lambda I$, since U is a is an invertible operator, then also $UU^\sharp = \lambda I$. It is known that $U^\sharp U$ is a G-self-adjoint operator, so λ is a real scalar. In order to complete the proof can be reduced to $\lambda = \pm 1$. The proof is complete. □

When the Hilbert spaces is not necessarily separable, we have the following result.

Theorem 3.4. *Let $\mathcal{H}$ be a complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a unital linear continuous surjective map. If ϕ preserves G-unitary operators in both directions, then there exist $\lambda = \pm 1$ and invertible operator $U \in \mathscr{B}(\mathcal{H})$ satisfying $UU^\sharp = U^\sharp U = \lambda I$ such that*

$$\phi(A) = \lambda U A U^\sharp$$

or

$$\phi(A) = \lambda U A^t U^\sharp$$

for all $A \in \mathscr{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of $\mathcal{H}$.

For the proof of this Theorem we will also need the following simple Lemma.

Lemma 3.5. *ϕ is injective.*

Proof. Let $T \in \mathscr{B}(\mathcal{H})$ such that $\phi(T) = 0$, then

$$\phi(T + I) = I,$$
$$\phi(T - I) = -I$$

and

$$\phi(T + iI) = iI.$$

Since ϕ preserves G-unitary operators, $T + I$, $T - I$ and $T + iI$ are G-unitary operators. It follows that

$$T^\sharp T + T^\sharp + T = 0,$$
$$T^\sharp T - T^\sharp - T = 0.$$

and

$$T^\sharp T - iT^\sharp + iT = 0.$$

These equalities imply that $T^\sharp T = 0$, so $T + T^\sharp = 0$ and $T - T^\sharp = 0$. Thus $T = 0$, which implies that ϕ is injective. □

We will now prove Theorem 3.4.

Proof. By Lemma 3.5 ϕ is injective so ϕ is a continuous unital bijective linear map that preserves G-unitary operators in both directions. The rest of proof is similar to the proof of Theorem 3.3. □

4. Linear maps preserving G-quasi-unitary operators

Lemma 4.1. *An operator U is a G-quasi-unitary operator on a Hilbert space if and only if $I - U$ is a G-unitary operator.*

Proof. The claim easily follows from $(I - U)^\sharp (I - U) = I - U^\sharp - U + U^\sharp U$ and $(I - U)^\sharp (I - U)^\sharp = I - U^\sharp - U + UU^\sharp$.

Since U is G-quasi-unitary, then

$$(I - U)^\sharp (I - U) = (I - U)(I - U)^\sharp = I.$$

For the second implication, we know that

$$(I - U)^\sharp (I - U) = (I - U)^\sharp (I - U)^\sharp = I,$$

so $U^\sharp U = UU^\sharp = U + U^\sharp$, consequently U is a G-quasi-unitary operator. □

Theorem 4.2. *Let $\mathcal{H}$ be a complex Hilbert space and $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a linear continuous surjective map. Suppose that ϕ preserves G-quasi-unitary operators in both directions. Then there exist $\lambda = \pm 1$ and invertible operator $U \in \mathscr{B}(\mathcal{H})$ satisfying $UU^\sharp = U^\sharp U = \lambda I$ such that*

$$\phi(A) = \lambda UAU^\sharp$$

or

$$\phi(A) = \lambda UA^t U^\sharp$$

for all $A \in \mathscr{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of $\mathcal{H}$.

For the proof of this Theorem, we need the following Lemmas.

Lemma 4.3. *ϕ is injective.*

Proof. If $S \in \mathscr{B}(\mathcal{H})$ is such that $\phi(S) = 0$ then

$$\phi(\lambda S) = 0,$$

for all $\lambda \in \mathbb{C}$. Since 0 is a G-quasi-unitary operator then, λS are G-quasi-unitary operators for all $\lambda \in \mathbb{C}$. We obtain

$$|\lambda|^2 S^\sharp S = |\lambda|^2 S^\sharp S = \bar{\lambda} S^\sharp + \lambda S.$$

Taking successively $\lambda = 1,\ \lambda = 2$ and $\lambda = i$, we get

$$S^\sharp S = 0,$$

$$S^\sharp + S = 0$$

and

$$-iS^\sharp + iS = 0.$$

A simple calculus gives that $S = 0$, hence the proof is complete. □

Lemma 4.4. *ϕ preserves G-projections in both directions.*

Proof. Let p be a G-projection, choose $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = \bar{\lambda} + \lambda$, and $\lambda \neq \bar{\lambda}$. Operators, λp and $\bar{\lambda} p$ are G-quasi-unitary, so are $\lambda\phi(p)$ and $\bar{\lambda}\phi(p)$. It follows that

$$|\lambda|^2 \phi(p)\phi(p)^\sharp = |\lambda|^2 \phi(p)^\sharp \phi(p) = \bar{\lambda}\phi(p)^\sharp + \lambda\phi(p) = \lambda\phi(p)^\sharp + \bar{\lambda}\phi(p).$$

So $\phi(p) = \phi(p)^\sharp$. Taking $\lambda = 2$ further gives $\phi(p)^2 = \phi(p)$ thus $\phi(p)$ is a G-projection. Repeating the same with ϕ^{-1}, completes the proof. □

Lemma 4.5. $\phi(I) = I$

Proof. Let $T \in \mathcal{H}$ such that $\phi(T) = I$ and suppose that $T \neq I$. Then T is a G-projection and $T - I$ is not G-projection. By the hypothesis, since ϕ preserves G-projection operators in both directions, $\phi(T - I) = I - \phi(I)$ is a G-projection but $T - I$ is not. This is a contradiction and hence $\phi(I) = I$. □

We proceed with the proof of Theorem 4.2.

Proof. We will prove that ϕ preserves G-unitary operators in both directions.

If U is G-unitary, by Lemma 4.1, $I - U$ is G-quasi-unitary, since from Lemmas 4.3, 4.4 and 4.5 ϕ is a linear bijective map and unital that preserves G-quasi-unitary operators, this implies that $\phi(I - U) = I - \phi(U)$ is a G-quasi-unitary. It follows by Lemma 4.1 that $\phi(U)$ is G-unitary, consequently, ϕ preserves G-unitary operators in one direction.

Repeating the same with ϕ^{-1}, to get that ϕ preserves G-unitary operators in both directions. Finally, from Theorem 3.4 the desired result follows. □

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