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Limit cycles of the sixth-order non-autonomous differential equation

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Abstract We provide sufficient conditions for the existence of periodic solutions of the sixth-order differential equation

$$x^{(6)} + (1+p^2+q^2)\ddddot{x} + (p^2+q^2+p^2q^2)\ddot{x} + p^2q^2x = \varepsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, x^{(5)}),$$

where p and q are rational numbers different from 1, 0, -1 and $p \neq q$, ε is small and F is a nonlinear non-autonomous periodic function. Moreover we provide some applications.

1. Introduction and statement of the main results

One of the main problems in the theory of differential equations is the study of their periodic orbits, their existence, their number and their stability. As usual,

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a limit cycle of a differential equation is a periodic orbit isolated in the set of all periodic orbits of the differential equation.

The objective of this paper is to study the periodic solutions of the sixth-order differential equation

$$
\begin{aligned}
&x^{(6)} + (1+p^2+q^2)\ddddot{x} + (p^2+q^2+p^2q^2)\ddot{x} + p^2q^2x \\
&\quad = \varepsilon F(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},\overset{.....}{x}),
\end{aligned} \tag{1}
$$

where p and q are rational numbers different from -1, 0, 1, and $p \neq q$, ε is a small real parameter, and F is a nonlinear non-autonomous periodic function.

There are many papers studying the periodic orbits of sixth-order differential equations, see for instance in [2–6,7–9,10]. But our main tool for studying the periodic orbits of Eq. (1) is completely different from the tools of the mentioned papers, and consequently the results obtained are distinct and new. We shall use the averaging theory, more precisely Theorem 4. Many of the quoted papers dealing with the periodic orbits of sixth-order differential equations use Schauder's or Leray–Schauder's fixed point theorem, the non-local reduction method or variational methods.

Our main results on the periodic solutions of the sixth-order differential Eq. (1) are the following.

Theorem 1. *Assume that p, q are rational numbers different from 1, 0, −1 and p ≠ q, in differential* Eq. (1). *Let*

$$
\begin{aligned}
\mathcal{F}_1(X_0,Y_0,Z_0,U_0,V_0,W_0) &= -\tfrac{1}{2\pi k}\int_0 \sin t\, F(t,a(t),b(t),c(t),d(t),e(t),f(t))dt,\\
\mathcal{F}_2(X_0,Y_0,Z_0,U_0,V_0,W_0) &= -\tfrac{1}{2\pi k}\int_0^{2\pi k} \cos t\, F(t,a(t),b(t),c(t),d(t),e(t),f(t))dt,\\
\mathcal{F}_3(X_0,Y_0,Z_0,U_0,V_0,W_0) &= \tfrac{1}{2\pi k}\int_0^{2\pi k} \cos\left(\tfrac{m}{n}t\right) F(t,a(t),b(t),c(t),d(t),e(t),f(t))dt,\\
\mathcal{F}_4(X_0,Y_0,Z_0,U_0,V_0,W_0) &= -\tfrac{1}{2\pi k}\int_0^{2\pi k} \sin\left(\tfrac{m}{n}t\right) F(t,a(t),b(t),c(t),d(t),e(t),f(t))dt,\\
\mathcal{F}_5(X_0,Y_0,Z_0,U_0,V_0,W_0) &= \tfrac{1}{2\pi k}\int_0^{2\pi k} \cos\left(\tfrac{r}{s}t\right) F(t,a(t),b(t),c(t),d(t),e(t),f(t))dt,\\
\mathcal{F}_6(X_0,Y_0,Z_0,U_0,V_0,W_0) &= -\tfrac{1}{2\pi k}\int_0^{2\pi k} \sin\left(\tfrac{r}{s}t\right) F(t,a(t),b(t),c(t),d(t),e(t),f(t))dt,
\end{aligned} \tag{2}
$$

be with $p = m/n, q = r/s,$ *where* m, n, s, r *are positive integers* $p \neq q$, $(m,n) = (r,s) = 1$, *let k be the least common multiple of n and s, and*

$$a(t) = \frac{pq(p-q)(p+q)X + q(q-1)(q+1)U + p(p-1)(1+p)W}{pq(-1+p^2)(-1+q^2)(p^2-q^2)},$$
$$b(t) = \frac{(p-q)(p+q)Y + (q-1)(q+1)Z - (p-1)(p+1)V}{(-1+p^2)(-1+q^2)(p^2-q^2)},$$
$$c(t) = \frac{-(p-q)(p+q)X + q(p-1)(p+1)W - p(q-1)(q+1)U}{(-1+p^2)(-1+q^2)(p^2-q^2)},$$
$$d(t) = \frac{-(p-q)(p+q)Y - p^2(q-1)(q+1)Z + q^2(p-1)(p+1)V}{(-1+p^2)(-1+q^2)(p^2-q^2)},$$
$$e(t) = \frac{(p-q)(p+q)X + p^3(q-1)(q+1)U + q^3(p-1)(p+1)W}{(-1+p^2)(-1+q^2)(p^2-q^2)},$$
$$f(t) = \frac{(p-q)(p+q)Y + p^4(q-1)(q+1)Z - q^4(p-1)(p+1)V}{(-1+p^2)(-1+q^2)(p^2-q^2)}, \tag{3}$$

where

$$X(t) = X_0\cos t - Y_0\sin t,\; Y(t) = Y_0\cos t + X_0\sin t,$$
$$Z(t) = Z_0\cos(pt) - U_0\sin(pt),\; U(t) = U_0\cos(pt) + Z_0\sin(pt),$$
$$V(t) = V_0\cos(qt) - W_0\sin(qt),\; W(t) = W_0\cos(qt) + V_0\sin(qt).$$

If the function F is $2\pi k$ – *periodic with respect to the variable t, then for every* $\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*, W_0^*\right)$ *solution of the system*

$$\mathcal{F}_k(X_0, Y_0, Z_0, U_0, V_0, W_0) = 0, \quad k = 1, \ldots, 6, \tag{4}$$

satisfying

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6)}{\partial(X_0, Y_0, Z_0, U_0, V_0, W_0)}\right)|_{(X_0,Y_0,Z_0,U_0,V_0,W_0)=(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*,W_0^*)} \neq 0, \tag{5}$$

the differential Eq. (1) *has a periodic solution* $x(t,\varepsilon)$ *tending to the solution* $x_0\,(t)$ of $x^{(6)} + (1+p^2+q^2)\,\ddddot{x} + (p^2+q^2+p^2q^2)\ddot{x} + p^2q^2x = 0$ *given by*

$$-\frac{1}{pq(-1+p^2)(-1+q^2)(p^2-q^2)}\left(qU_0^*\cos(pt)\right) + qZ_0^*\sin(pt) - q^3U_0^*\cos(pt)$$
$$- q^3Z_0^*\sin(pt) - p^3qX_0^*\cos t + p^3qY_0^*\sin t + pq^3X_0^*\cos t - pq^3Y_0^*\sin t$$
$$- pW_0^*\cos(qt) - pV_0^*\sin(qt) + p^3W_0^*\cos(qt) + p^3V_0^*\sin(qt)), \tag{6}$$

when $\varepsilon \to 0$. *Note that this solution is periodic of period* $2\pi k$.

Theorem 1 is proved in Section 3. Its proof is based on the averaging theory for computing periodic orbits, see Section 2. Two applications of Theorem 1 for studying the periodic solutions of Eq. (1) are given in the following four corollaries. They are proved in Section 4.

The linear differential equation of sixth-order $x^{(6)} + (1+p^2+q^2)\,\ddddot{x} + (p^2+q^2+p^2q^2)\ddot{x} + p^2q^2x = 0$ provides a linear system in $\mathbb{R}^6$ having a six-dimensional centre. Theorem 1 reduces the study of the limit cycles of the differential equation of sixth-order (1) bifurcating from the periodic orbits of that centre to find the nondegenerate zeros of the system of six equations and six unknowns given by (4). The zeros are non-degenerate in the sense that the Jacobian of the system on them must be non-zero. In general, the problem of finding the zeros of six non-linear equations with six unknowns is not easy, but of course it is easier than looking for the periodic orbits directly.

Remark 1.

1. In the case p and q are rational numbers different from $1, 0, -1$ and $p = q$, then we cannot apply Theorem 4 for studying the periodic orbits.
2. In the case $p = q = 1$ (respectively $p = q = 0$), then we cannot applyTheorem 4 for studying the periodic orbits.

Corollary 2. *If* $F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, \overset{.....}{x}) = (x-1)\sin t$, *then the differential* Eq. (1) *with* $p = \frac{1}{2}$, $q = 2$ *has one periodic solution* $x_1(t,\varepsilon)$ *tending to the periodic solution* $x_1(t)$ *given by*

$$x_1(t) = -2\cos(2t),$$

of $x^{(6)} + \frac{21}{4}\ddddot{x} + \frac{21}{4}\ddot{x} + x = 0$ *when* $\varepsilon \to 0$.

Corollary 3. *If* $F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, \overset{.....}{x}) = (x^2-1)\sin t$, *then the differential* Eq. (1) *with* $p = 2$ *and* $q = 3$ *has eighteen periodic solutions* $x_k\,(t,\varepsilon)$ *for* $k = 1, \ldots, 18$ *tending to the periodic solutions*

$$x_1(t) = -\frac{2}{3}\sqrt{3}\cos(2t) + \frac{2}{3}\sqrt{3}\cos(t), x_2(t) = \frac{2}{3}\sqrt{3}\cos(2t) - \frac{2}{3}\sqrt{3}\cos t,$$
$$x_3(t) = \frac{2}{3}\sqrt{3}\cos(2t) + \frac{2}{3}\sqrt{3}\cos(t), x_4(t) = -\frac{2}{3}\sqrt{3}\cos(2t) - \frac{2}{3}\sqrt{3}\cos t,$$
$$x_5(t) = \sqrt{2}\cos t - \frac{1}{3}\sqrt{6}\sin t + \frac{1}{2}\sqrt{2}\cos(3t) + \frac{1}{6}\sqrt{6}\sin(3t),$$
$$x_6(t) = -\sqrt{2}\cos t - \frac{1}{3}\sqrt{6}\sin t - \frac{1}{2}\sqrt{2}\cos(3t) + \frac{1}{6}\sqrt{6}\sin(3t),$$
$$x_7(t) = \sqrt{2}\cos t + \frac{1}{3}\sqrt{6}\sin t + \frac{1}{2}\sqrt{2}\cos(3t) - \frac{1}{6}\sqrt{6}\sin(3t),$$
$$x_8(t) = \sqrt{2}\cos t + \frac{1}{3}\sqrt{6}\sin t - \frac{1}{2}\sqrt{2}\cos(3t) - \frac{1}{6}\sqrt{6}\sin(3t),$$
$$x_9(t) = -\frac{4}{21}\sqrt{42}\sin t - \frac{1}{21}\sqrt{42}\sin(3t),$$
$$x_{10}(t) = \frac{4}{21}\sqrt{42}\sin t + \frac{1}{21}\sqrt{42}\sin(3t),$$

$$x_{11}(t) = -\frac{2}{3}\sqrt{3}\cos(2t) - \frac{2}{21}\sqrt{21}\sin t - \frac{4}{21}\sqrt{21}\sin(3t),$$
$$x_{12}(t) = -\frac{2}{3}\sqrt{3}\cos(2t) + \frac{2}{21}\sqrt{21}\sin t + \frac{4}{21}\sqrt{21}\sin(3t),$$
$$x_{13}(t) = \frac{2}{3}\sqrt{3}\cos(2t) - \frac{2}{21}\sqrt{21}\sin t - \frac{4}{21}\sqrt{21}\sin(3t),$$
$$x_{14}(t) = \frac{2}{3}\sqrt{3}\cos(2t) + \frac{2}{21}\sqrt{21}\sin t + \frac{4}{21}\sqrt{21}\sin(3t),$$
$$x_{15}(t) = -\frac{2}{11}\sqrt{33}\sin(2t) - \frac{2}{33}\sqrt{33}\sin t + \frac{4}{33}\sqrt{33}\sin(3t),$$
$$x_{16}(t) = \frac{2}{11}\sqrt{33}\sin(2t) + \frac{2}{33}\sqrt{33}\sin t - \frac{4}{33}\sqrt{33}\sin(3t),$$
$$x_{17}(t) = \frac{2}{11}\sqrt{33}\sin(2t) - \frac{2}{33}\sqrt{33}\sin t + \frac{4}{33}\sqrt{33}\sin(3t),$$
$$x_{18}(t) = -\frac{2}{11}\sqrt{33}\sin(2t) + \frac{2}{33}\sqrt{33}\sin t - \frac{4}{33}\sqrt{33}\sin(3t),$$

of $x^{(6)} + 14\ddddot{x} + 49\ddot{x} + 36x = 0$ *when* $\varepsilon \to 0$.

2. Basic results on averaging theory

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T-periodic solutions from differential systems of the form

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{7}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{8}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of the system (8) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{9}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (9), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi\,(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

We assume that there exists a k-dimensional submanifold $\mathcal{Z}$ of Ω filled with T-periodic solutions of (8). Then an answer to the problem of bifurcation of

T-periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (7) is given in the following result.

Theorem 4. *Let W be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : \mathrm{Cl}(W) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in \mathrm{Cl}(W)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha)$ of (8) is T-periodic;
(ii) for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (9) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{F} : \mathrm{Cl}(W) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\right). \tag{10}$$

If there exists $a \in W$ with $\mathcal{F}(a) = 0$ and $\det((d\mathcal{F}/d\alpha)(a)) \neq 0$, then there is a T-periodic solution $\varphi(t,\varepsilon)$ of system (7) such that $\varphi(0,\varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 4 goes back to Malkin [8] and Roseau [9], for a shorter proof see [1].

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T-periodic, where $\mathbf{x}(t,\mathbf{z},0)$ denotes the solution of the unperturbed system (8) with $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (7); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 5 (*Perturbations of an isochronous set*). *We assume that there exists an open and bounded set V with $Cl(V) \subset \Omega$ such that for each $\mathbf{z} \in Cl(V)$, the solution $\mathbf{x}(t,\mathbf{z})$ is T-periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t,\mathbf{z}) F_1(t, \mathbf{x}(t,\mathbf{z}))dt. \tag{11}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det((d\mathcal{F}/d\mathbf{z})(a)) \neq 0$, then there exists a T-periodic solution $\varphi(t,\varepsilon)$ of system (7) such that $\varphi(0,\varepsilon) \to a$ as $\varepsilon \to 0$.

For a shorter proof of Theorem 5 see Corollary 1 of [1]. In fact this result goes back to Malkin [8] and Roseau [9].

3. Proof of Theorem 1

If $y = \dot{x},\ z = \ddot{x},\ u = \dddot{x}, v = \ddddot{x}, w = \overset{\cdot\cdot\cdot\cdot\cdot}{x}$, then system (1) can be written as

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= u,\\
\dot{v} &= w,\\
\dot{w} &= -p^2q^2x - (p^2 + q^2 + p^2q^2)z - (1 + p^2 + q^2)w + \varepsilon F(t, x, y, z, u, v, w).
\end{aligned} \tag{12}$$

The unperturbed system has a unique singular point, the origin. The eigenvalues of the linearized system at this singular point are $\pm i$, $\pm pi$ and $\pm qi$. By the linear invertible transformation

$$(X, Y, Z, U, V, W)^T = B(x, y, z, u, v, w)^T,$$

where

$$B = \begin{pmatrix} -p^2q^2 & 0 & -p^2 - q^2 & 0 & -1 & 0\\ 0 & p^2q^2 & 0 & p^2 + q^2 & 0 & 1\\ 0 & q^2 & 0 & 1 + q^2 & 0 & 1\\ pq^2 & 0 & p(1 + q^2) & 0 & p & 0\\ 0 & p^2 & 0 & 1 + p^2 & 0 & 1\\ p^2q & 0 & q(1 + p^2) & 0 & q & 0 \end{pmatrix},$$

we transform the system (12) such that its linear part is real Jordan normal from of the linear part of system (12) with $\varepsilon = 0$, i.e.,

$$\begin{cases} \dot{X} = -Y,\\ \dot{Y} = X - \varepsilon\widetilde{F}(t, a(t), b(t), c(t), d(t), e(t), f(t)),\\ \dot{Z} = -pU + \varepsilon\widetilde{F}(t, a(t), b(t), c(t), d(t), e(t), f(t)),\\ \dot{U} = pZ,\\ \dot{V} = -qW + \varepsilon\widetilde{F}(t, a(t), b(t), c(t), d(t), e(t), f(t)),\\ \dot{W} = qV, \end{cases} \tag{13}$$

where

$$\widetilde{F} = \widetilde{F}(t, a(t), b(t), c(t), d(t), e(t), f(t)) = F(t, x, y, z, u, v, w),$$

with $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$ and $f(t)$ as in (3).

Note that the linear part of the differential system (13) at the origin is in its real Jordan normal from. We shall apply Theorem 5 to the differential system (13). We note that system (13) can be written as system (7) taking

$$x = \begin{pmatrix} X\\ Y\\ Z\\ U\\ V\\ W \end{pmatrix}, \quad F_0(x, t) = \begin{pmatrix} -Y\\ X\\ -\frac{m}{n}U\\ \frac{m}{n}Z\\ -\frac{r}{s}W\\ \frac{r}{s}V \end{pmatrix}, \quad F_1(x, t) = \begin{pmatrix} 0\\ -\widetilde{F}\\ \widetilde{F}\\ 0\\ \widetilde{F}\\ 0 \end{pmatrix}.$$

We shall study the periodic solutions of system (13) in our case, i.e. the periodic solutions of system (13) with $\varepsilon = 0$. These periodic solutions are

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \\ U(t) \\ V(t) \\ W(t) \end{pmatrix} = \begin{pmatrix} X_0 \cos t - Y_0 \sin t \\ Y_0 \cos t + X_0 \sin t \\ Z_0 \cos\left(\frac{m}{n} t\right) - U_0 \sin\left(\frac{m}{n} t\right) \\ U_0 \cos\left(\frac{m}{n} t\right) + Z_0 \sin\left(\frac{m}{n} t\right) \\ V_0 \cos\left(\frac{r}{s} t\right) - W_0 \sin\left(\frac{r}{s} t\right) \\ W_0 \cos\left(\frac{r}{s} t\right) + V_0 \sin\left(\frac{r}{s} t\right) \end{pmatrix}.$$

This set of periodic orbits has dimension six, all having the same period $2\pi k$, where k is defined in the statement of Theorem 1. To look for the periodic solutions of our Eq. (1) we must calculate the zeros $z = (X_0, Y_0, Z_0, U_0, V_0, W_0)$ of the system $\mathcal{F}(z) = 0$, where $\mathcal{F}(z)$ is given by (11). The fundamental matrix $M(t)$ of the differential system (13) with $\varepsilon = 0$, along any periodic solution is

$$M(t) = M_z(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 & 0 & 0 \\ \sin t & \cos t & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{m}{n} t\right) & -\sin\left(\frac{m}{n} t\right) & 0 & 0 \\ 0 & 0 & \sin\left(\frac{m}{n} t\right) & \cos\left(\frac{m}{n} t\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\left(\frac{r}{s} t\right) & -\sin\left(\frac{r}{s} t\right) \\ 0 & 0 & 0 & 0 & \sin\left(\frac{r}{s} t\right) & \cos\left(\frac{r}{s} t\right) \end{pmatrix}.$$

Now computing the function $\mathcal{F}(z)$ given in (11), we got that the system $\mathcal{F}(z) = 0$, can be written as system (4) with the function $\mathcal{F}_k(X_0, Y_0, Z_0, U_0, V_0, W_0)$ given in (2). The zeros $\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*, W_0^*\right)$ of system (4) with respect to the variables X_0, Y_0, Z_0, U_0, V_0, and W_0 provide periodic orbits of system (13) with $\varepsilon \neq 0$ sufficiently small if they are simple, i.e. if (5) holds. Going back through the change of variables, for every simple zero $\left(X_0^* \, Y_0^* \, Z_0^*, \; U_0^*, V_0^*, W_0^*\right)$ of system (4), we obtain a $2\pi k$ periodic solution $x(t)$ of the differential Eq. (1) for $\varepsilon \neq 0$ sufficiently small such that $x(t)$ tends to the periodic solution (6) of $x^{(6)} + (1 + p^2 + q^2)\ddddot{x} + (p^2 + q^2 + p^2q^2)\ddot{x} + p^2q^2x = 0$ when $\varepsilon \to 0$, where k is defined in the statement of Theorem 1. Note that this solution is periodic of period $2\pi k$. This completes the proof of Theorem 1.

4. Proof of Corollaries 2 and 3

Proof of Corollary 2. Consider the function $F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, \overset{\cdot\cdot\cdot\cdot\cdot}{x}) = (x - 1)\sin t$, which corresponds to the case $p = \frac{1}{2}$ and $q = 2$. The functions $\mathcal{F}_i(X_0, Y_0, Z_0, U_0, V_0, W_0)$ for $i = 1, \ldots, 6$ of Theorem 1 are

$$\mathcal{F}_1(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{1}{2} + \frac{1}{90} W_0,$$
$$\mathcal{F}_2(X_0, Y_0, Z_0, U_0, V_0, W_0) = -\frac{1}{90} V_0,$$
$$\mathcal{F}_3(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{8}{45} Z_0,$$
$$\mathcal{F}_4(X_0, Y_0, Z_0, U_0, V_0, W_0) = -\frac{8}{45} U_0,$$
$$\mathcal{F}_5(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{-1}{9} Y_0,$$
$$\mathcal{F}_6(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{1}{9} X_0.$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \mathcal{F}_6 = 0$ has only real solution

$$\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*, W_0^*\right) = (0, 0, 0, 0, 0, -45).$$

Since the Jacobian

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6)}{\partial(X_0, Y_0, Z_0, U_0, V_0, W_0)}\right)\Big|_{(X_0, Y_0, Z_0, U_0, V_0, W_0)=\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*, W_0^*\right)} = \frac{-16}{332150625},$$

by Theorem 1 Eq. (1) has the periodic solution of the statement of the corollary. □

Proof of Corollary 3. Consider the function $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, \dot{\ddddot{x}}, t) = (x^2 - 1)\sin t$, which corresponds to the case $p = 2$ and $q = 3$. The functions $\mathcal{F}_i(X_0, Y_0, Z_0, U_0, V_0, W_0)$ for $i = 1, \ldots, 6$ of Theorem 1 are

$$\begin{aligned}
\mathcal{F}_1(X_0, Y_0, Z_0, U_0, V_0, W_0) = & -\frac{1}{57600} W_0^2 + \frac{1}{2} - \frac{1}{57600} V_0^2 - \frac{1}{1536} Y_0^2 - \frac{1}{3600} U_0^2 \\
& - \frac{1}{4608} X_0^2 - \frac{1}{3600} Z_0^2 + \frac{1}{11520} X_0 W_0 - \frac{1}{11520} Y_0 V_0,
\end{aligned}$$
$$\mathcal{F}_2(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{1}{2304} X_0 Y_0 - \frac{1}{11520} Y_0 W_0 - \frac{1}{11520} X_0 V_0,$$
$$\mathcal{F}_3(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{1}{1440} U_0 Y_0 + \frac{1}{14400} U_0 V_0 - \frac{1}{14400} Z_0 W_0,$$
$$\mathcal{F}_4(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{1}{1440} U_0 W_0 - \frac{1}{1440} Z_0 Y_0 + \frac{1}{14400} Z_0 V_0,$$
$$\mathcal{F}_5(X_0, Y_0, Z_0, U_0, V_0, W_0) = \frac{1}{2304} X_0 Y_0 - \frac{1}{5760} Y_0 W_0 + \frac{1}{3600} U_0 Z_0,$$
$$\mathcal{F}_6(X_0, Y_0, Z_0, U_0, V_0, W_0) = -\frac{1}{4608} Y_0^2 + \frac{1}{7200} U_0^2 - \frac{1}{4608} X_0^2 - \frac{1}{7200} Z_0^2 + \frac{1}{5760} Y_0 V_0.$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \mathcal{F}_6 = 0$ has the eighteen solutions $\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*, W_0^*\right)$ given by

$$\begin{aligned}
&\left(16\sqrt{3}, 0, 0, 20\sqrt{3}, 0, 0\right), \left(-16\sqrt{3}, 0, 0, -20\sqrt{3}, 0, 0\right),\\
&\left(16\sqrt{3}, 0, 0, -20\sqrt{3}, 0, 0\right), \left(-16\sqrt{3}, 0, 0, 20\sqrt{3}, 0, 0\right),\\
&\left(24\sqrt{2}, 8\sqrt{6}, 0, 0, 20\sqrt{6}, 60\sqrt{2}\right), \left(-24\sqrt{2}, 8\sqrt{6}, 0, 0, 20\sqrt{6}, -60\sqrt{2}\right),\\
&\left(24\sqrt{2}, -8\sqrt{6}, 0, 0, -20\sqrt{6}, 60\sqrt{2}\right), \left(-24\sqrt{2}, -8\sqrt{6}, 0, 0, -20\sqrt{6}, -60\sqrt{2}\right),\\
&\left(0, \frac{32}{7}\sqrt{42}, 0, 0, -\frac{40}{7}\sqrt{42}, 0\right), \left(0, -\frac{32}{7}\sqrt{42}, 0, 0, \frac{40}{7}\sqrt{42}, 0\right),\\
&\left(0, \frac{16}{7}\sqrt{21}, 0, 20\sqrt{3}, -\frac{160}{7}\sqrt{21}, 0\right), \left(0, -\frac{16}{7}\sqrt{21}, 0, 20\sqrt{3}, \frac{160}{7}\sqrt{21}, 0\right),\\
&\left(0, \frac{16}{7}\sqrt{21}, 0, -20\sqrt{3}, -\frac{160}{7}\sqrt{21}, 0\right), \left(0, -\frac{16}{7}\sqrt{21}, 0, -20\sqrt{3}, \frac{160}{7}\sqrt{21}, 0\right),\\
&\left(0, \frac{16}{11}\sqrt{33}, \frac{60}{11}\sqrt{33}, 0, \frac{160}{11}\sqrt{33}, 0\right), \left(0, -\frac{16}{11}\sqrt{33}, -\frac{60}{11}\sqrt{33}, 0, -\frac{160}{11}\sqrt{33}, 0\right),\\
&\left(0, \frac{16}{11}\sqrt{33}, -\frac{60}{11}\sqrt{33}, 0, \frac{160}{11}\sqrt{33}, 0\right), \left(0, -\frac{16}{11}\sqrt{33}, \frac{60}{11}\sqrt{33}, 0, -\frac{160}{11}\sqrt{33}, 0\right).
\end{aligned}$$

These 18 solutions of the system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \mathcal{F}_6 = 0$, have been obtained independently using mathematica and maple.

Since the Jacobian (5) for theses eighteen solutions $\left(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*, W_0^*\right)$ is

$$\begin{aligned}
&\frac{-1}{1433272320000}, \frac{-1}{1433272320000}, \frac{-1}{1433272320000}, \frac{-1}{1433272320000}, \frac{-1}{955514880000},\\
&\frac{-1}{955514880000}, \frac{-1}{955514880000}, \frac{-1}{955514880000}, \frac{1}{278691840000}, \frac{1}{278691840000},\\
&\frac{1}{30098718720000}, \frac{1}{30098718720000}, \frac{1}{30098718720000}, \frac{1}{30098718720000},\\
&\frac{1}{19269550080000}, \frac{1}{19269550080000}, \frac{1}{19269550080000}, \frac{1}{19269550080000},
\end{aligned}$$

respectively, we obtain using Theorem 1, the eighteen solutions given in statement of the corollary. □

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