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# Kohn–Vogelius formulation and topological sensitivity analysis based method for solving geometric inverse problems

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**Abstract.** In this paper, we propose an alternative approach combining the advantages of the Kohn–Vogelius formulation and the topological sensitivity analysis method for solving geometric inverse problems. The Kohn–Vogelius formulation can rephrase the geometric inverse problem into a shape optimization one minimizing an energy-like function. The sensitivity analysis gives the leading term of the energy-like function variation with respect to the presence of a small geometry perturbation inside the computational domain. The obtained theoretical results lead to build a fast and accurate numerical reconstruction algorithm. The efficiency and accuracy of the proposed algorithm are illustrated by some numerical results.

Keywords: Geometric inverse problem; Kohn–Vogelius formulation; Sensitivity analysis; Energy-like function; Calculus of variations

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## 1. Introduction

In this paper we consider a geometric inverse problem related to the anisotropic Laplace equation. Let $\mathcal{D} \subset \mathbb{R}^2$ denote a bounded domain with smooth boundary $\Sigma = \partial\mathcal{D}$. Inside

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the domain $\mathcal{D}$ we assume the existence of a sub-domain $\mathcal{A}^\star \subset \mathcal{D}$ with boundary $\partial\mathcal{A}^\star$. The geometric inverse problem that we consider can be formulated as follows.

– Given two boundary conditions on the accessible part $\Sigma_a$ of the boundary $\Sigma$: an imposed flux $\mathcal{F} \in H^{-1/2}(\Sigma_a)$ and a measured datum $\varphi_m \in H^{1/2}(\Sigma_a)$.

– Find the unknown boundary $\partial\mathcal{A}^\star$ such that the solution $\phi$ of the anisotropic Laplace equation satisfies the following overdetermined value problem

$$\begin{cases} -\operatorname{div}(\mu(x)\nabla\phi) = \mathcal{Q} & \text{in } \mathcal{D}\setminus\overline{\mathcal{A}^\star}, \\ \mu(x)\nabla\phi.\mathbf{n} = \mathcal{F} & \text{on } \Sigma_a, \\ \phi = \varphi_m & \text{on } \Sigma_a, \\ \phi = 0 & \text{on } \Sigma_i, \\ \phi = \sigma & \text{on } \partial\mathcal{A}^\star, \end{cases} \tag{1}$$

where $\Sigma_a$ and $\Sigma_i$ are two parts of the boundary $\Sigma$ (accessible and inaccessible parts) such that $\Sigma = \Sigma_a \cup \Sigma_i$ and $\Sigma_a \cap \Sigma_i = \emptyset$. The parameter $\mu$ is a smooth scalar positive function ($\mu$ and $\nabla\mu$ belong to $L^\infty(\mathcal{D})$) describing the physical properties of the medium $\mathcal{D}$ and $\mathcal{Q} \in L^2(\mathcal{D})$ is a given source term. In this formulation the domain $\mathcal{D}\setminus\overline{\mathcal{A}^\star}$ is unknown since the free boundary $\partial\mathcal{A}^\star$ is unknown. This problem is ill-posed in the sense of Hadamard [14].

A standard way of solving this geometric inverse problem is to transform it into a shape optimization one. Typically it leads to an optimization problem of the form

$$\min_{\mathcal{A}\subset\mathcal{D}} \mathcal{E}(\mathcal{D}\setminus\overline{\mathcal{A}}, \phi_{\mathcal{A}}),$$
$$\text{subject to} \quad E(\phi_{\mathcal{A}}) = 0,$$

where $\mathcal{A} \longmapsto \mathcal{E}(\mathcal{D}\setminus\overline{\mathcal{A}})$ is a shape function depending on the domain $\mathcal{A}$ via the solution $\phi_{\mathcal{A}}$ to a given partial differential equation $E(\phi) = 0$.

In [15], the shape optimization problem is formulated as follows

$$\min_{\mathcal{A}\subset\mathcal{D}} \frac{1}{2}\int_{\Sigma_a} |\mu(x)\nabla\phi.\mathbf{n} - \mathcal{F}|^2 d\Sigma,$$

where $\phi$ is the solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\mu(x)\nabla\phi) = \mathcal{Q} & \text{in } \mathcal{D}\setminus\overline{\mathcal{A}}, \\ \phi = \varphi_m & \text{on } \Sigma_a, \\ \phi = 0 & \text{on } \Sigma_i, \\ \phi = \sigma & \text{on } \partial\mathcal{A}. \end{cases}$$

In [8,18], the shape optimization problem consists in minimizing the following boundary quadratic function,

$$\min_{\mathcal{A}\subset\mathcal{D}} \frac{1}{2}\int_{\Sigma_a} |\phi - \varphi_m|^2 d\Sigma,$$

where $\phi$ is the solution to the Neumann problem

$$\begin{cases} -\operatorname{div}(\mu(x)\nabla\phi) = \mathcal{Q} & \text{in } \mathcal{D}\setminus\overline{\mathcal{A}}, \\ \mu(x)\nabla\phi.\mathbf{n} = \mathcal{F} & \text{on } \Sigma_a, \\ \phi = 0 & \text{on } \Sigma_i, \\ \phi = \sigma & \text{on } \partial\mathcal{A}. \end{cases}$$

The majority of the developed methods for solving this optimization type problems are based on the shape differentiation techniques. It is proved in [3,6] that this kind of geometric inverse problems are severely ill-posed (i.e. unstable), for both Dirichlet and Neumann conditions on the boundary $\partial\mathcal{A}$. Thus they have to use some regularization methods to solve them numerically.

We propose here an alternative approach combining the advantages of the Kohn–Vogelius formulation [5] and the topological sensitivity analysis method [1,2,4,7,9–13,16,17,19].

The Kohn–Vogelius formulation is a self regularization technique and rephrase the geometrical inverse problem into a shape optimization one. It leads to define for any given domain $\mathcal{A} \subset \mathcal{D}$ two forward problems. The first one is associated to the Neumann datum $\mathcal{F}$, which will be named as the "Neumann problem":

$$(\mathcal{P}_n) \begin{cases} \text{Find } \phi_n \in H^1(\mathcal{D} \setminus \overline{\mathcal{A}}) \text{ solving} \\ -\operatorname{div}(\mu(x)\nabla\phi_n) = \mathcal{Q} \quad \text{in } \mathcal{D} \setminus \overline{\mathcal{A}} \\ \mu(x)\nabla\phi_n.\mathbf{n} = \mathcal{F} \quad \text{on } \Sigma_a \\ \phi_n = 0 \quad \text{on } \Sigma_i, \\ \phi_n = \sigma \quad \text{on } \partial\mathcal{A}. \end{cases} \tag{2}$$

The second one is associated to the Dirichlet (measured) datum $\varphi_m$

$$(\mathcal{P}_d) \begin{cases} \text{Find } \phi_d \in H^1(\mathcal{D} \setminus \overline{\mathcal{A}}) \text{ solving} \\ -\operatorname{div}(\mu(x)\nabla\phi_d) = \mathcal{Q} \quad \text{in } \mathcal{D} \setminus \overline{\mathcal{A}} \\ \phi_d = \varphi_m \quad \text{on } \Sigma_a \\ \phi_d = 0 \quad \text{on } \Sigma_i, \\ \phi_d = \sigma \quad \text{on } \partial\mathcal{A}. \end{cases} \tag{3}$$

One can remark that if $\partial\mathcal{A}$ coincides with the actual boundary $\partial\mathcal{A}^\star$ then the misfit between the solutions vanishes, $\phi_d = \phi_n$. According to this observation, we propose an identification process based on the minimization of the following energy type functional

$$\mathcal{E}(\mathcal{D} \setminus \overline{\mathcal{A}}) = \int_{\mathcal{D}\setminus\overline{\mathcal{A}}} \mu(x)|\nabla\phi_d - \nabla\phi_n|^2 dx.$$

The optimization problem consists in determining the optimal location and shape of the domain $\mathcal{A}$ solution to the following minimizing problem

$$(\mathcal{O}) \quad \begin{cases} \text{Find the domain } \mathcal{A}^* \subset \mathcal{D} \text{ such that} \\ \mathcal{E}(\mathcal{D} \setminus \overline{\mathcal{A}^*}) \le \mathcal{E}(\mathcal{D} \setminus \overline{\mathcal{A}}), \ \forall \mathcal{A} \subset \mathcal{D}. \end{cases}$$

To solve the minimization problem $(\mathcal{O})$ we shall use the topological sensitivity analysis method. It consists in studying the variation of the function $\mathcal{E}$ with respect to a small geometry perturbation of the domain $\mathcal{D}$. More precisely, let $\chi_{\xi,\rho}$ be a small hole created inside the background domain $\mathcal{D}$. We assume that the perturbation $\chi_{\xi,\rho}$ is centered at an arbitrary point $\xi \in \mathcal{D}$ and has the form $\chi_{\xi,\rho} = \xi + \rho\chi$, where $\rho > 0$ and $\chi \subset \mathbb{R}^2$ is a given, regular and bounded domain containing the origin (its boundary $\partial\chi$ is of class $\mathcal{C}^1$).

The topological sensitivity analysis method consists in measuring the sensitivity of the cost function $\mathcal{E}$ with respect to the presence of a geometry perturbation. It leads to an asymptotic expansion on the form:

$$\mathcal{E}(\mathcal{D} \setminus \overline{\chi_{\xi,\rho}}) = \mathcal{E}(\mathcal{D}) + f(\rho)\mathcal{S}(\xi) + o(f(\rho)), \quad \forall\xi \in \mathcal{D},$$

where $\rho \mapsto f(\rho)$ is a scalar positive function going to zero with $\rho$ and describes the behavior of the variation $\mathcal{E}(\mathcal{D} \setminus \overline{\chi_{\xi,\rho}}) - \mathcal{E}(\mathcal{D})$ with respect to $\rho$. The function $\xi \longmapsto \mathcal{S}(\xi)$ measures the sensitivity of $\mathcal{E}$ with respect to a geometry perturbation around the point $\xi$. The function $\mathcal{S}$ is called the topological sensitivity or the topological gradient.

In order to minimize the shape function $\mathcal{E}$, the best location to insert a small geometry perturbation in $\mathcal{D}$ is where the topological gradient $\mathcal{S}$ is most negative. In fact if $\mathcal{S}(\xi) < 0$, we have $\mathcal{E}(\mathcal{D}\backslash\overline{\chi_{\xi,\rho}}) < \mathcal{E}(\mathcal{D})$ for small $\rho$. Particularly, the solution of the problem $\min_{\chi_{\xi,\rho}\subset\mathcal{D}}\mathcal{E}(\mathcal{D}\backslash\overline{\chi_{\xi,\rho}})$ is given by $\chi^{\star}_{\xi,\rho} = \xi^{\star} + \rho\chi$, such that $\mathcal{S}(\xi^{\star}) < 0$ and $\mathcal{S}(\xi^{\star}) \leq \mathcal{S}(\xi)$, $\forall\xi \in \mathcal{D}$.

Based on this observation, we propose a simple and fast reconstruction algorithm for solving the optimization problem $(\mathcal{O})$. The function $\mathcal{S}$ is used as a descent direction in the domain optimization process. The proposed process consists in building a sequence of geometries $(\mathcal{A}_i)_{i\geq 0}$ with $\mathcal{A}_0 = \emptyset$. At the $i$th iteration, the domain $\mathcal{A}_{i+1}$ is constructed by creating a new geometry perturbation $\chi^i$ in the domain $\mathcal{D}_i = \mathcal{D} \setminus \overline{\mathcal{A}_i}$, i.e. $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \chi^i$. The location and shape of the domain $\chi^i$ is obtained using a topological sensitivity analysis for the energy function $\mathcal{E}$ with respect to the insertion of a small geometry perturbation in the domain $\mathcal{D}_i$.

The main contribution of this paper concerns the theoretical and numerical aspects. In the theoretical part, we derive a sensitivity analysis for the considered energy like function $\mathcal{E}$ with respect to the presence of a small geometry perturbation inside the domain $\mathcal{D}$. The established results are based on a rigorous and simplified mathematical analysis valid for a large class of shape functions and an arbitrary shaped geometry perturbation. In the numerical part, we propose a fast and accurate geometric reconstruction algorithm for solving the shape optimization problem $(\mathcal{O})$. The efficiency and accuracy of the proposed algorithm are illustrated by some numerical results. Particularly, we will show that the regular and simple shaped domain (disc or ellipse) can be reconstructed using only one iteration.

The paper is organized as follows. In the next section, we present the perturbed Neumann and Dirichlet problems and we introduce the energy like function associated to a small geometry perturbation. In Section 3, we discuss the influence of the geometry perturbation on the perturbed solutions. Section 4 is devoted to a simplified formulation of the shape function variation with respect to the creation of the hole $\chi_{\xi,\rho}$ in $\mathcal{D}$. The topological sensitivity analysis for the function $\mathcal{E}$ is derived in Section 5. The proposed numerical algorithm is described in Section 6.

## 2. The perturbed problems

In this section, we present the Neumann and Dirichlet problems in the perturbed domain. In the presence of a small geometry perturbation $\chi_{\xi,\rho}$ inside the domain $\mathcal{D}$, the Neumann problem consists in finding $\phi_n^{\rho} \in H^1(\mathcal{D} \setminus \overline{\chi_{\xi,\rho}})$ solution to

$$(\mathcal{P}_n^{\rho})\begin{cases} -\operatorname{div}\,(\mu(x)\nabla\phi_n^{\rho}) = \mathcal{Q} & \text{in} \quad \mathcal{D}_{\xi,\rho} \\ \mu(x)\nabla\phi_n^{\rho}\cdot\mathbf{n} = \mathcal{F} & \text{on} \quad \Sigma_a \\ \phi_n^{\rho} = 0 & \text{on} \quad \Sigma_i \\ \phi_n^{\rho} = \sigma & \text{on} \quad \partial\chi_{\xi,\rho}, \end{cases} \tag{4}$$

with $\mathcal{D}_{\xi,\rho}$ is the perturbed domain defined as $\mathcal{D}_{\xi,\rho} = \mathcal{D} \setminus \overline{\chi_{\xi,\rho}}$.

In the absence of any perturbation (i.e. $\rho = 0$), the Neumann problem reads: find $\phi_n^0 \in H^1(\mathcal{D})$ solution to

$$(\mathcal{P}_n^{0})\begin{cases} -\operatorname{div}\,(\mu(x)\nabla\phi_n^{0}) = \mathcal{Q} & \text{in} \quad \mathcal{D} \\ \mu(x)\nabla\phi_n^{0}\cdot\mathbf{n} = \mathcal{F} & \text{on} \quad \Sigma_a \\ \phi_n^{0} = 0 & \text{on} \quad \Sigma_i. \end{cases} \tag{5}$$

Similarly, the perturbed Dirichlet problem consists in finding $\phi_d^\rho \in H^1(\mathcal{D} \setminus \overline{\chi_{\xi,\rho}})$ solution to

$$(\mathcal{P}_d^\rho)\begin{cases} -\operatorname{div}(\mu(x)\nabla\phi_d^\rho) = \mathcal{Q} & \text{in } \mathcal{D}_{\xi,\rho} \\ \phi_d^\rho = \varphi_m & \text{on } \Sigma_a \\ \phi_d^\rho = 0 & \text{on } \Sigma_i \\ \phi_d^\rho = \sigma & \text{on } \partial\chi_{\xi,\rho}. \end{cases} \tag{6}$$

The Dirichlet problem in the non perturbed domain reads: find $\phi_d^0 \in H^1(\mathcal{D})$ solution to

$$(\mathcal{P}_d^0)\begin{cases} -\operatorname{div}(\mu(x)\nabla\phi_d^0) = \mathcal{Q} & \text{in } \mathcal{D} \\ \phi_d^0 = \varphi_m & \text{on } \Sigma_a \\ \phi_d^0 = 0 & \text{on } \Sigma_i. \end{cases} \tag{7}$$

We are now ready to introduce the considered energy-like function $\mathcal{E}$. For each created geometry perturbation $\chi_{\xi,\rho}$ inside the domain $\mathcal{D}$, the function $\mathcal{E}$ measures the difference between the Neumann and Dirichlet perturbed solutions. It is defined as

$$\mathcal{E}(\mathcal{D} \setminus \overline{\chi_{\xi,\rho}}) = \int_{\mathcal{D}_{\xi,\rho}} \mu(x)|\nabla\phi_d^\rho(x) - \nabla\phi_n^\rho(x)|^2 dx, \ \forall \chi_{\xi,\rho} \subset \mathcal{D}.$$

In the non perturbed domain (i.e. when $\rho = 0$), the function $\mathcal{E}$ has the expression

$$\mathcal{E}(\mathcal{D}) = \int_{\mathcal{D}} \mu(x)|\nabla\phi_d^0(x) - \nabla\phi_n^0(x)|^2 dx.$$

Then, the variation of the function $\mathcal{E}$ with respect to the presence of a small geometry perturbation is given by

$$\begin{aligned} &\mathcal{E}(\mathcal{D} \setminus \overline{\chi_{\xi,\rho}}) - \mathcal{E}(\mathcal{D}) \\ &= \int_{\mathcal{D}_{\xi,\rho}} \mu(x)|\nabla\phi_d^\rho(x) - \nabla\phi_n^\rho(x)|^2 dx - \int_{\mathcal{D}} \mu(x)|\nabla\phi_d^0(x) - \nabla\phi_n^0(x)|^2 dx. \end{aligned}$$

To drive the expected asymptotic expansion for the function $\mathcal{E}$ and calculate the topological sensitivity function $\mathcal{S}$, we will start our analysis by studying the influence of the geometry perturbation on the Neumann and Dirichlet problems solutions. We will derive in the next section two estimates describing the asymptotic behavior of the variations $\phi_n^\rho - \phi_n^0$ and $\phi_d^\rho - \phi_d^0$ with respect to $\rho$.

## 3. Estimate of the perturbed solutions

In this section, we derive two estimates describing the perturbation caused by the creation of a small hole $\chi_{\xi,\rho}$ inside the domain $\mathcal{D}$.

### *3.1. Estimate of the Neumann perturbed solution*

From $(\mathcal{P}_n^\rho)$ and $(\mathcal{P}_n^0)$, one can check that the variation of the Neumann perturbed solution $v_n^\rho = \phi_n^\rho - \phi_n^0$ satisfies the system

$$\begin{cases} -\operatorname{div}(\mu(x)\nabla v_n^\rho) = 0 & \text{in } \mathcal{D} \setminus \overline{\chi}_{\xi,\rho} \\ \mu(x)\nabla v_n^\rho \cdot \mathbf{n} = 0 & \text{on } \Sigma_a \\ v_n^\rho = 0 & \text{on } \Sigma_i \\ v_n^\rho = \sigma - \phi_n^0 & \text{on } \partial\chi_{\xi,\rho}. \end{cases} \tag{8}$$

Let $\Phi_n^\xi$ be a scalar function defined by

$$\Phi_n^\xi(y) = [\sigma(\xi) - \phi_n^0(\xi)] \log(|y|), \ \forall y \in \mathbb{R}^2.$$

Then, the Neumann perturbed solution $\phi_n^\rho$ satisfies the following estimate.

**Proposition 1.** *There exists a positive constant $c > 0$, independent of $\rho$, such that*

$$\left\| \phi_n^\rho - \phi_n^0 - \frac{1}{\log(\rho)} \ \Phi_n^\xi(x-\xi) \right\|_{1,\mathcal{D}_{\xi,\rho}} \le \frac{c}{\sqrt{-\log(\rho)}}. \tag{9}$$

**Proof.** Since $\xi \notin \mathcal{D}_{\xi,\rho}$, we have $\Delta \Phi_n^\xi = 0$ in $\mathcal{D}_{\xi,\rho}$. Then, the total variation $r_{\xi,\rho}^n(x) = \phi_n^\rho(x) - \phi_n^0(x) - \frac{1}{\log(\rho)} \Phi_n^\xi(x-\xi)$ solves the following boundary value problem in $\mathcal{D} \setminus \overline{\chi}_{\xi,\rho}$

$$\begin{cases} -\operatorname{div}(\mu(x)\nabla r_{\xi,\rho}^n) = \dfrac{1}{\log(\rho)} \nabla\mu(x)\cdot\nabla\Phi_n^\xi(x-\xi) & \text{in } \mathcal{D}\setminus\overline{\chi}_{\xi,\rho} \\ \mu(x)\nabla r_{\xi,\rho}^n\cdot\mathbf{n} = -\dfrac{1}{\log(\rho)}\mu(x)\nabla\Phi_n^\xi(x-\xi)\cdot\mathbf{n} & \text{on } \Sigma_a \\ r_{\xi,\rho}^n = -\dfrac{1}{\log(\rho)}\Phi_n^\xi(x-\xi) & \text{on } \Sigma_i \\ r_{\xi,\rho}^n = [\sigma-\phi_n^0] - [\sigma(\xi)-\phi_n^0(\xi)] - \dfrac{1}{\log(\rho)}\Phi_n^\xi((x-\xi)/\rho) & \text{on } \partial\chi_{\xi,\rho}. \end{cases} \tag{10}$$

The condition imposed on the boundary $\partial\chi_{\xi,\rho}$ is obtained using the log-function property $\log(|x-\xi|) = \log(|x-\xi|/\rho) + \log(\rho)$, $\forall x \neq \xi$.

To derive the desired inequality (9), we will estimate the right hand side and the boundary data in (10) separately. To this end, we introduce the following preliminaries results:

– From the fact that $\chi$ is an open domain containing the origin, there exists $r > 0$ such that:

$\overline{B(0,r)} \subset \chi$.

– The domain $\mathcal{D}$ is bounded in such a way that there exists $R > 0$ such that:

$\overline{\mathcal{D}} \subset B(\xi, R)$, $\forall \xi \in \mathcal{D}$.

– From the fact that $C(0, r\rho, R) = \left\{y \in \mathbb{R}^2;\ r\rho < |y| < R\right\} \subset \mathbb{R}^2 \setminus \{0\}$, it follows that the function $\psi : y \mapsto \log(|y|)$ is smooth in $C(0, r\rho, R)$ and admits the estimate

$$\|\nabla\psi\|_{0,C(0,r\rho,R)} \le c\sqrt{-\log(\rho)},$$

where $c$ is a positive constant, independent of $\rho$.

• *Estimate of the right hand side*: Using the smoothness of the functions $\mu$, $\sigma$ and $\phi_n^0$ near $\xi$ and the fact that $\mathcal{D} \setminus \overline{\chi}_{\xi,\rho} - \xi \subset C(0, r\rho, R)$, we deduce

$$\left\|\nabla\mu(x)\nabla\Phi_n^\xi(x-\xi)\right\|_{0,\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}} \le \|\nabla\mu\|_{L^\infty(\mathcal{D})} \left\|\nabla\Phi_n^\xi\right\|_{0,C(0,r\rho,R)} \le c\sqrt{-\log(\rho)}.$$

Then, it follows

$$\left\| \frac{1}{\log(\rho)} \nabla\mu(x)\nabla\Phi_n^\xi(x-\xi) \right\|_{0,\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}} = O\left(\frac{c}{\sqrt{-\log(\rho)}}\right). \tag{11}$$

• *Estimate of the imposed boundary data on $\Sigma$*: Let $\widetilde{R} > 0$ such that $\overline{\chi_{\xi,\rho}} \subset B(\xi, \widetilde{R})$ and $\overline{B(\xi,\widetilde{R})} \subset \mathcal{D}$. By trace theorem we have

$$\left\|\mu(x)\nabla\Phi_n^\xi\cdot\mathbf{n}\right\|_{-1/2,\Sigma_a} + \left\|\Phi_n^\xi\right\|_{1/2,\Sigma_i} \le c\left\|\Phi_n^\xi\right\|_{1,\mathcal{D}_{\widetilde{R}}},$$

where $\mathcal{D}_{\widetilde{R}} = \mathcal{D} \setminus \overline{B(\xi, \widetilde{R})}$. It is easy to remark that the function $x \mapsto \Phi_n^{\xi}(x - \xi)$ is smooth (of class $\mathcal{C}^1$) in $\mathcal{D}_{\widetilde{R}}$ and the norm $\left\| \Phi_n^{\xi} \right\|_{1,\mathcal{D}_{\widetilde{R}}}$ is uniformly bounded. Then, it follows

$$\left\| \mu(x)\nabla r_{\xi,\rho}^n \cdot \mathbf{n} \right\|_{-1/2,\Sigma_a} + \left\| r_{\xi,\rho}^n \right\|_{1/2,\Sigma_i} = O(\frac{-1}{\log(\rho)}). \tag{12}$$

• *Estimate of the imposed boundary data on $\partial\chi_{\xi,\rho}$*: We have

$$\begin{aligned}\left\| r_{\xi,\rho}^n \right\|_{1/2,\partial\chi_{\xi,\rho}} \leq &\left\| [\sigma - \phi_n^0] - [\sigma - \phi_n^0](\xi) \right\|_{1/2,\partial\chi_{\xi,\rho}} \\ &+ \frac{-1}{\log(\rho)} \left\| \Phi_n^{\xi}((x - \xi)/\rho) \right\|_{1/2,\partial\chi_{\xi,\rho}}.\end{aligned}$$

Using the trace theorem and the fact that $x \mapsto (\sigma - \phi_n^0)(x)$ is smooth in $\chi_{\xi,\rho}$, one can deduce

$$\left\| [\sigma - \phi_n^0] - [\sigma - \phi_n^0](\xi) \right\|_{1/2,\partial\chi_{\xi,\rho}} \leq \left\| [\sigma - \phi_n^0] - [\sigma - \phi_n^0](\xi) \right\|_{1,\chi_{\xi,\rho}} = O(\rho).$$

Let $\chi_r = \chi \setminus \overline{B(0, r)}$, we have $\xi + \rho\chi_r \subset \chi_{\xi,\rho}$. By the change of variable $x = \xi + \rho y$ and the trace theorem, one can derive

$$\left\| \Phi_n^{\xi}((x - \xi)/\rho) \right\|_{1/2,\partial\chi_{\xi,\rho}} \leq \left\| \Phi_n^{\xi} \right\|_{1,\chi_r}.$$

From the fact that $y \mapsto \log(|y|)$ is smooth in $\chi_r$, it follows that the quantity $\left\| \Phi_n^{\xi} \right\|_{1,\chi_r}$ is uniformly bounded and we have

$$\frac{-1}{\log(\rho)} \left\| \Phi_n^{\xi}((x - \xi)/\rho) \right\|_{1/2,\partial\chi_{\xi,\rho}} = O(\frac{-1}{\log(\rho)}).$$

Consequently,

$$\left\| r_{\xi,\rho}^n \right\|_{1/2,\partial\chi_{\xi,\rho}} = O(\frac{-1}{\log(\rho)}). \tag{13}$$

Finally, combining the estimates (11)–(13), one can deduce that there exists $c > 0$ such that

$$\left\| \phi_n^{\rho} - \phi_n^0 - \frac{1}{\log(\rho)} \Phi_n^{\xi} \right\|_{1,\mathcal{D}_{\xi,\rho}} = \left\| r_{\xi,\rho}^n \right\|_{1,\mathcal{D}_{\xi,\rho}} \leq \frac{c}{\sqrt{-\log(\rho)}}.$$

### 3.2. *Estimate of the Dirichlet perturbed solution*

From $(\mathcal{P}_d^{\rho})$ and $(\mathcal{P}_d^0)$, one can check that the variation of the Dirichlet perturbed solution $v_d^{\rho} = \phi_d^{\rho} - \phi_d^0$ solves the following system

$$\begin{cases} -\operatorname{div}(\mu(x)\nabla v_d^{\rho}) = 0 & \text{in } \mathcal{D} \setminus \overline{\chi}_{\xi,\rho} \\ v_d^{\rho} = 0 & \text{on } \Sigma_a \\ v_d^{\rho} = 0 & \text{on } \Sigma_i \\ v_d^{\rho} = \sigma - \phi_d^0 & \text{on } \partial\chi_{\xi,\rho}. \end{cases} \tag{14}$$

Similar to the Neumann case, we introduce a scalar function $\Phi_d^{\xi}$ associated to the Dirichlet problem, defined by

$$\Phi_d^{\xi}(y) = [\sigma(\xi) - \phi_d^0(\xi)] \log(|y|), \ \forall y \in \mathbb{R}^2.$$

We have the following estimate.

**Proposition 2.** *There exists a positive constant $c > 0$, independent of $\rho$, such that*

$$\left\| \phi_d^\rho - \phi_d^0 - \frac{1}{\log(\rho)} \Phi_d^\xi \right\|_{1,\mathcal{D}_{\xi,\rho}} \leq \frac{c}{\sqrt{-\log(\rho)}}. \tag{15}$$

**Proof.** The proof of this estimate can be established using an adaptation of the proof developed in Section 3.2.

**Remark 1.** If the physical property $\mu$ is constant in the domain $\mathcal{D}$ (i.e. $\mathcal{D}$ is a homogeneous material), the source term in (10) vanishes and the asymptotic behavior (with respect to $\rho$) of the perturbed solution will be $O(\frac{-1}{\log(\rho)})$ instead of $O(\frac{1}{\sqrt{-\log(\rho)}})$.

**Corollary 1.** *If the domain $\mathcal{D}$ is occupied by a homogeneous material, then the Neumann and Dirichlet perturbed solutions satisfy the estimates*

$$\left\| \phi_n^\rho - \phi_n^0 - \frac{1}{\log(\rho)} \; \Phi_n^\xi(x-\xi) \right\|_{1,\mathcal{D}_{\xi,\rho}} = O(\frac{-1}{\log(\rho)}),$$

$$\left\| \phi_d^\rho - \phi_d^0 - \frac{1}{\log(\rho)} \; \Phi_d^\xi(x-\xi) \right\|_{1,\mathcal{D}_{\xi,\rho}} = O(\frac{-1}{\log(\rho)}).$$

## 4. Variation of the energy-like function

We present in this section a simplified expression of the variation $\mathcal{E}(\mathcal{D} \setminus \overline{\chi}_{\xi,\rho}) - \mathcal{E}(\mathcal{D})$. The obtained formulation plays important role in the topological sensitivity analysis of $\mathcal{E}$.

The variation of the energy-like function $\mathcal{E}$ reads

$$\begin{aligned}
\mathcal{E}(\mathcal{D} \setminus \overline{\chi}_{\xi,\rho}) - \mathcal{E}(\mathcal{D}) &= \int_{\mathcal{D}_{\xi,\rho}} \mu(x)|\nabla\phi_d^\rho - \nabla\phi_n^\rho|^2 dx - \int_{\mathcal{D}} \mu(x)|\nabla\phi_d^0 - \nabla\phi_n^0|^2 dx \\
&= \int_{\mathcal{D}_{\xi,\rho}} \mu(x)|\nabla\phi_n^\rho|^2 dx - \int_{\mathcal{D}} \mu(x)|\nabla\phi_n^0|^2 dx \\
&\quad + \int_{\mathcal{D}_{\xi,\rho}} \mu(x)|\nabla\phi_d^\rho|^2 dx - \int_{\mathcal{D}} \mu(x)|\nabla\phi_d^0|^2 dx \\
&\quad - 2\left[ \int_{\mathcal{D}_{\xi,\rho}} \mu(x)\nabla\phi_n^\rho \cdot \nabla\phi_d^\rho dx - \int_{\mathcal{D}} \mu(x)\nabla\phi_n^0 \cdot \nabla\phi_d^0 dx \right].
\end{aligned}$$

In order to derive a topological asymptotic expansion for the function $\mathcal{E}$, we start our analysis by the following preliminary result.

**Theorem 1.** *The variation of $\mathcal{E}$ caused by the presence of a small geometry perturbation $\chi_{\xi,\rho}$ inside the background domain $\mathcal{D}$ admits the expression*

$$\begin{aligned}
\mathcal{E}(\mathcal{D} \setminus \overline{\chi}_{\xi,\rho}) - \mathcal{E}(\mathcal{D}) &= \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^\rho - \phi_d^0) \cdot \mathbf{n}\,(\phi_d^\rho - \phi_d^0) ds \\
&\quad - \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0) \cdot \mathbf{n}\,(\phi_n^\rho - \phi_n^0) ds
\end{aligned}$$

$$+\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot \mathbf{n}\,(\phi_d^\rho - \phi_d^0)ds$$
$$+\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot \mathbf{n}\,(\phi_n^\rho - \phi_n^0)ds.$$

**Proof.** The variation $\mathcal{E}(\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}) - \mathcal{E}(\mathcal{D})$ can be decomposed as

$$\mathcal{E}(\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}) - \mathcal{E}(\mathcal{D}) = \Im_n(\rho) + \Im_d(\rho) - 2\Im_m(\rho), \quad \text{with}$$

$\Im_n$ is the Neumann term

$$\Im_n(\rho) = \int_{\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}} \mu(x)|\nabla\phi_n^\rho|^2 dx - \int_{\mathcal{D}} \mu(x)|\nabla\phi_n^0|^2 dx,$$

$\Im_d$ is the Dirichlet term

$$\Im_d(\rho) = \int_{\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}} \mu(x)|\nabla\phi_d^\rho|^2 dx - \int_{\mathcal{D}} \mu(x)|\nabla\phi_d^0|^2 dx,$$

and $\Im_m$ is the mixed term

$$\Im_m(\rho) = \int_{\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}} \mu(x)\nabla\phi_n^\rho\cdot\nabla\phi_d^\rho dx - \int_{\mathcal{D}} \mu(x)\nabla\phi_n^0\cdot\nabla\phi_d^0 dx.$$

Next, we will examine each term separately.
− *The Neumann term:* we have

$$\Im_n(\rho) = \int_{\mathcal{D}_{\xi,\rho}} \mu(x)|\nabla\phi_n^\rho - \nabla\phi_n^0|^2 dx + 2\int_{\mathcal{D}_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\nabla\phi_n^0 dx$$
$$- \int_{\chi_{\xi,\rho}} \mu(x)|\nabla\phi_n^0|^2 dx.$$

From the weak formulation of (8), one can derive

$$\int_{\mathcal{D}_{\xi,\rho}} \mu(x)|\nabla\phi_n^\rho - \nabla\phi_n^0|^2 dx = \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho - \phi_n^0)ds$$

and

$$\int_{\mathcal{D}_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\nabla\phi_n^0 dx$$
$$= \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,\phi_n^0 ds + \int_{\Sigma_i} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,\phi_n^0 ds.$$

The integral on the boundary $\partial\chi_{\xi,\rho}$ can be decomposed as

$$\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,\phi_n^0 ds$$
$$= -\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho - \phi_n^0)ds + \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,\phi_n^\rho ds.$$

Then, it follows

$$\Im_n(\rho) = -\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho - \phi_n^0)ds + 2\int_{\Sigma_i} \mu(x)\nabla(\phi_n^\rho - \phi_n^0)\cdot\mathbf{n}\,\phi_n^0 ds$$

$$+2\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\cdot\mathbf{n}\,\phi_n^\rho ds-\int_{\chi_{\xi,\rho}}\mu(x)|\nabla\phi_n^0|^2dx. \quad (16)$$

− *The Dirichlet term:* the term $\Im_d$ can be decomposed as

$$\Im_d(\rho)=\int_{\mathcal{D}_{\xi,\rho}}\mu(x)|\nabla\phi_d^\rho-\nabla\phi_d^0|^2dx+2\int_{\mathcal{D}_{\xi,\rho}}\mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\nabla\phi_d^0dx$$
$$-\int_{\chi_{\xi,\rho}}\mu(x)|\nabla\phi_d^0|^2dx.$$

Using the weak formulation of (14) and the fact that $\phi_d^\rho-\phi_d^0=0$ on $\partial\mathcal{D}$, we have

$$\int_{\mathcal{D}_{\xi,\rho}}\mu(x)|\nabla\phi_d^\rho-\nabla\phi_d^0|^2dx=\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds$$

and

$$\int_{\mathcal{D}_{\xi,\rho}}\mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\nabla\phi_d^0dx$$
$$=\int_{\mathcal{D}_{\xi,\rho}}\mathcal{Q}\,(\phi_d^\rho-\phi_d^0)dx+\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla\phi_d^0\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds.$$

Then, we derive

$$\Im_d(\rho)=\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds+2\int_{\mathcal{D}_{\xi,\rho}}\mathcal{Q}\,(\phi_d^\rho-\phi_d^0)dx$$
$$+2\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla\phi_d^0\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds-\int_{\chi_{\xi,\rho}}\mu(x)|\nabla\phi_d^0|^2dx. \quad (17)$$

− *The mixed term:* this term can be written as

$$\Im_m(\rho)=\int_{\mathcal{D}_{\xi,\rho}}\mu(x)\nabla\phi_n^\rho\cdot\nabla(\phi_d^\rho-\phi_d^0)dx+\int_{\mathcal{D}_{\xi,\rho}}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\nabla\phi_d^0dx$$
$$-\int_{\chi_{\xi,\rho}}\mu(x)\nabla\phi_n^0\nabla\phi_d^0dx.$$

Applying the Green formula for Eq. (4) and making use of the condition $\phi_d^\rho-\phi_d^0=0$ on $\partial\mathcal{D}$, we obtain

$$\int_{\mathcal{D}_{\xi,\rho}}\mu(x)\nabla\phi_n^\rho\cdot\nabla(\phi_d^\rho-\phi_d^0)dx=\int_{\mathcal{D}_{\xi,\rho}}\mathcal{Q}\,(\phi_d^\rho-\phi_d^0)dx+\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla\phi_n^\rho\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds.$$

From the weak formulation of (8) and the fact that $\phi_d^0-\phi_n^0=0$ on $\Sigma_i$, it follows

$$\int_{\mathcal{D}_{\xi,\rho}}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\nabla\phi_d^0dx$$
$$=\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\cdot\mathbf{n}\,\phi_d^0ds+\int_{\Sigma_i}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\cdot\mathbf{n}\,\phi_n^0ds.$$

Then, we deduce

$$\Im_m(\rho)=\int_{\mathcal{D}_{\xi,\rho}}\mathcal{Q}\,(\phi_d^\rho-\phi_d^0)dx+\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla\phi_n^\rho\cdot\mathbf{n}\,\phi_d^\rho ds+\int_{\Sigma_i}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\cdot\mathbf{n}\,\phi_n^0ds$$
$$-\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla\phi_n^0\cdot\mathbf{n}\,\phi_d^0ds-\int_{\chi_{\xi,\rho}}\mu(x)\nabla\phi_n^0\nabla\phi_d^0dx. \quad (18)$$

– *The total variation:* exploiting the obtained expressions (16)–(18), it follows

$$\begin{aligned}
\mathcal{E}(\mathcal{D}\setminus\overline{\chi}_{\xi,\rho})-\mathcal{E}(\mathcal{D}) &= \Im_n(\rho)+\Im_d(\rho)-2\Im_m(\rho)\\
&=\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds\\
&\quad-\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho-\phi_n^0)ds\\
&\quad+\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^0-\phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho+\phi_d^\rho-2\phi_d^0)ds\\
&\quad-\int_{\chi_{\xi,\rho}}\mu(x)|\nabla\phi_d^0-\nabla\phi_n^0|^2dx.
\end{aligned}$$

Using Green formula and taking into account of the normal orientation

$$\int_{\chi_{\xi,\rho}}\mu(x)|\nabla\phi_d^0-\nabla\phi_n^0|^2dx=-\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^0-\phi_n^0)\cdot\mathbf{n}\,(\phi_d^0-\phi_n^0)ds$$

then, we deduce

$$\begin{aligned}
\mathcal{E}(\mathcal{D}\setminus\overline{\chi}_{\xi,\rho})-\mathcal{E}(\mathcal{D}) &= \int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds\\
&\quad-\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_n^\rho-\phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho-\phi_n^0)ds\\
&\quad+\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^0-\phi_n^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds\\
&\quad+\int_{\partial\chi_{\xi,\rho}}\mu(x)\nabla(\phi_d^0-\phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho-\phi_n^0)ds.
\end{aligned}$$

## 5. Topological sensitivity analysis

We are now ready to derive the sensitivity analysis for the energy-like function $\mathcal{E}$ with respect to the presence of a small geometry perturbation $\chi_{\xi,\rho}$ inside the domain $\mathcal{D}$. As it is described in the following theorem, the sensitivity function $\mathcal{S}$ depends on the physical property of the domain, the boundary condition on $\partial\chi_{\xi,\rho}$ and the solutions of the non-perturbed Dirichlet and Neumann problems.

**Theorem 2.** *The function $\mathcal{E}$ defined by*

$$\mathcal{E}(\mathcal{D}\setminus\overline{\chi}_{\xi,\rho})=\int_{\mathcal{D}\setminus\overline{\chi}_{\xi,\rho}}\mu(x)|\nabla\phi_d^\rho-\nabla\phi_n^\rho|^2dx,$$

*admits the following topological asymptotic expansion*

$$\mathcal{E}(\mathcal{D}\setminus\overline{\chi}_{\xi,\rho})=\mathcal{E}(\mathcal{D})+\frac{-1}{\log(\rho)}\mathcal{S}(\xi)+o\left(\frac{-1}{\log(\rho)}\right),$$

*with $\mathcal{S}$ is the topological gradient given by*

$$\mathcal{S}(x)=2\pi\mu(x)\Big[|\phi_d^0(x)-\sigma(x)|^2-|\phi_n^0(x)-\sigma(x)|^2\Big],\forall x\in\mathcal{D}.$$

**Proof.** From Theorem 1, we have

$$\begin{aligned}\mathcal{E}(\mathcal{D}\setminus\overline{\chi}_{\xi,\rho})-\mathcal{E}(\mathcal{D}) = &\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds\\ &-\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^\rho-\phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho-\phi_n^0)ds \qquad (19)\\ &+\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0-\phi_n^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds\\ &+\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0-\phi_n^0)\cdot\mathbf{n}\,(\phi_n^\rho-\phi_n^0)ds. \qquad (20)\end{aligned}$$

Next we will derive an asymptotic expansion for each term.

– *Asymptotic expansion for the first term in (19):*

Using the fact that $\phi_d^\rho=\sigma$ on $\partial\chi_{\xi,\rho}$, we have

$$\begin{aligned}&\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^\rho-\phi_d^0)\cdot\mathbf{n}\,(\phi_d^\rho-\phi_d^0)ds\\ &=\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^\rho-\phi_d^0-\frac{1}{\log(\rho)}\,\Phi_d^\xi)\cdot\mathbf{n}\,(\sigma-\phi_d^0)ds\\ &\quad+\frac{1}{\log(\rho)}\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla\Phi_d^\xi\cdot\mathbf{n}\,(\sigma-\phi_d^0)ds. \qquad (21)\end{aligned}$$

By trace theorem, it follows

$$\begin{aligned}&\left|\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^\rho-\phi_d^0-\frac{1}{\log(\rho)}\,\Phi_d^\xi)\cdot\mathbf{n}\,(\sigma-\phi_d^0)ds\right|\\ &\le\left\|\mu(x)\nabla(\phi_d^\rho-\phi_d^0-\frac{1}{\log(\rho)}\,\Phi_d^\xi)\cdot\mathbf{n}\right\|_{-1/2,\partial\chi_{\xi,\rho}}\left\|\sigma-\phi_d^0\right\|_{1/2,\partial\chi_{\xi,\rho}}\\ &\le\left\|\phi_d^\rho-\phi_d^0-\frac{1}{\log(\rho)}\,\Phi_d^\xi\right\|_{1,\mathcal{D}_{\xi,\rho}}\left\|\sigma-\phi_d^0\right\|_{1,\chi_{\xi,\rho}}.\end{aligned}$$

Using Proposition 2 and the fact that $\sigma-\phi_d^0$ is uniformly bounded in $\chi_{\xi,\rho}$, one can easily deduce

$$\left|\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^\rho-\phi_d^0-\frac{1}{\log(\rho)}\,\Phi_d^\xi)\cdot\mathbf{n}\,(\sigma-\phi_d^0)ds\right|=o\left(\frac{-1}{\log(\rho)}\right).$$

To examine the last term in (21), we begin by the following decomposition

$$\begin{aligned}\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla\Phi_d^\xi\cdot\mathbf{n}\,(\sigma-\phi_d^0)ds &= \mu(\xi)\int_{\partial\chi_{\xi,\rho}} \nabla\Phi_d^\xi\cdot\mathbf{n}\,ds(\sigma(\xi)-\phi_d^0(\xi))\\ &+\int_{\partial\chi_{\xi,\rho}} \nabla\Phi_d^\xi\cdot\mathbf{n}\left[\mu(x)(\sigma-\phi_d^0)(x)-\mu(\xi)(\sigma-\phi_d^0)(\xi)\right]ds.\end{aligned}$$

By a similar argument as in (13), one can establish

$$\int_{\partial\chi_{\xi,\rho}} \nabla\Phi_d^\xi\cdot\mathbf{n}\left[\mu(x)(\sigma-\phi_d^0)(x)-\mu(\xi)(\sigma-\phi_d^0)(\xi)\right]ds=O(\rho).$$

From the fact that $y \mapsto -\frac{1}{2\pi}\log(|y|)$ is the fundamental solution of the Laplace operator and taking into account of the normal orientation, one can derive

$$\int_{\partial\chi_{\xi,\rho}} \nabla \Phi_d^{\xi} \cdot \mathbf{n}\, ds = -2\pi[\sigma(\xi) - \phi_d^0(\xi)].$$

Consequently, the first term in (19) admits the following expansion

$$\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^{\rho} - \phi_d^0)\cdot\mathbf{n}\,(\phi_d^{\rho} - \phi_d^0)ds = \frac{-2\pi}{\log(\rho)}\mu(\xi)\Big[\sigma(\xi) - \phi_d^0(\xi)\Big]^2 + o\left(\frac{-1}{\log(\rho)}\right).$$

– *Asymptotic expansion for the second term in (19):*

The asymptotic expansion for the second term in (19) can be established with the help of the same technique developed in the previous paragraph. Then, one can easily obtain

$$\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_n^{\rho} - \phi_n^0)\cdot\mathbf{n}\,(\phi_n^{\rho} - \phi_n^0)ds = \frac{-2\pi}{\log(\rho)}\mu(\xi)\Big[\sigma(\xi) - \phi_d^0(\xi)\Big]^2 + o\left(\frac{-1}{\log(\rho)}\right).$$

– *Estimate of the term (20):*

From the fact that $\phi_n^{\rho} = \phi_d^{\rho} = \sigma$ on $\partial\chi_{\xi,\rho}$, the term (20) can be rewritten as

$$\begin{aligned}&\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot\mathbf{n}\,(\phi_d^{\rho} - \phi_d^0)ds + \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot\mathbf{n}\,(\phi_n^{\rho} - \phi_n^0)ds\\ &= \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot\mathbf{n}\,(2\sigma - \phi_d^0 - \phi_n^0)ds.\end{aligned}$$

Recall that $-\operatorname{div}(\mu(x)\nabla(\phi_d^0 - \phi_n^0)) = 0$ in $\chi_{\xi,\rho}$. Then, by Green formula one can derive

$$\begin{aligned}&\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot\mathbf{n}\,(2\sigma - \phi_d^0 - \phi_n^0)ds\\ &= \int_{\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot\nabla(\phi_d^0 + \phi_n^0 - 2\sigma)dx.\end{aligned}$$

Finally, using the smoothness of the functions $\phi_n^0$, $\phi_d^0$ and $\sigma$ near the point $\xi$, one can deduce that the term (20) satisfies the estimate

$$\begin{aligned}&\int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot\mathbf{n}\,(\phi_d^{\rho} - \phi_d^0)ds + \int_{\partial\chi_{\xi,\rho}} \mu(x)\nabla(\phi_d^0 - \phi_n^0)\cdot\mathbf{n}\,(\phi_n^{\rho} - \phi_n^0)ds\\ &= o\left(\frac{-1}{\log(\rho)}\right).\end{aligned}$$

## 6. Numerical experiments

This section is devoted to some numerical investigations. In the first part of this section, we present a numerical validation of the topological asymptotic expansion established in Theorem 2. In the second part, we propose a fast and accurate reconstruction algorithm for solving the geometric inverse problem.

### *6.1. Validation of the asymptotic formula*

It is proved in Theorem 2, that the variation of the function $\mathcal{E}$, with respect to the insertion of a small geometry perturbation $\chi_{\xi,\rho}$ in the domain $\mathcal{D}$, satisfies the estimate

$$\mathcal{E}(\mathcal{D}\setminus\overline{\chi_{\xi,\rho}}) = \mathcal{E}(\mathcal{D}) + \frac{-2\pi}{\log(\rho)}\mathcal{S}(\xi) + o\left(\frac{-1}{\log(\rho)}\right),$$

with

$$\mathcal{S}(x) = \mu(x)\Big[|\phi_d^0(x)-\sigma(x)|^2 - |\phi_n^0(x)-\sigma(x)|^2\Big], \quad \forall x\in\mathcal{D}.$$

In this section, we present a numerical validation of this asymptotic behavior. It consists in studying the variation of the following function

$$\mathcal{V}_\xi(\rho) = \mathcal{E}(\mathcal{D}\setminus\overline{\chi_{\xi,\rho}}) - \mathcal{E}(\mathcal{D}) + \frac{2\pi}{\log(\rho)}\mathcal{S}(\xi)$$

with respect to $\frac{-1}{\log(\rho)}$ for some arbitrary locations of the perturbation $\chi_{\xi,\rho}$ in the domain $\mathcal{D}$.

We denote by $\alpha$ the unknown parameter describing the behavior of the function $\rho\mapsto\mathcal{V}_\xi(\rho)$ with respect to $-\log(\rho)$, i.e

$$|\mathcal{V}_\xi(\rho)| = O\left((-\log(\rho))^\alpha\right).$$

Then, one can observe that $\alpha$ can be characterized as the slope of the line approximating the variation $\rho\mapsto\log(|\mathcal{V}_\xi(\rho)|)$ with respect to $\log(-\log(\rho))$.

The numerical simulations are done using the following data:

– The background domain is defined by the square $\mathcal{D}=[-1,\ 1]\times[-1,\ 1]$.

– The (arbitrary chosen) locations $\xi_i$ of the considered perturbation $\chi_\rho^i = \xi_i + \rho B(0,1)$ are described in the following table

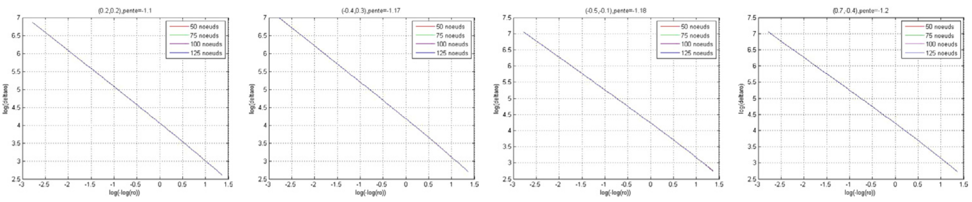
| Perturbation $\chi_\rho^i$ | $\chi_\rho^1$ | $\chi_\rho^2$ | $\chi_\rho^3$ | $\chi_\rho^4$ |
|---|---|---|---|---|
| Location $\xi_i$ | $\xi_1=(0.2, 0.2)$ | $\xi_2=(-0.4, 0.3)$ | $\xi_3=(-0.5,-0.1)$ | $\xi_4=(0.7,-0.4)$ |

Our numerical algorithm is based on the following main steps:

- Step 1:
  – compute the solutions $\phi_d^0$ and $\phi_n^0$ in the domain $\mathcal{D}$.
  – compute $\mathcal{E}(\mathcal{D})$.
- Step 2: for each perturbation $\chi_\rho^i = \xi_i+\rho B(0,1),\ i=1,\ \dots,4$:
  – compute $\mathcal{S}(\xi_i)$,
  – choose $\rho_0^i = \max\{\rho>0,\ \text{such that}\ \xi_i+\rho_0^i B(0,1)\subset\mathcal{D}\}$,
  – compute an approximation of the function $\rho\mapsto\mathcal{E}(\mathcal{D}\setminus\overline{\chi_\rho^i}),\ \rho\in\,]0,\ \rho_0^i]$
- Step 3: deduce a numerical approximation of the function $\rho\mapsto\log(|\mathcal{V}_{\xi^i}(\rho)|),\ \rho\in\,]0,\ \rho_0^i]$.

The obtained results are described in Fig. 1. For each considered perturbation $\chi_\rho^i = \xi_i+\rho B(0,1),\ i=1,\ \dots,4$, we plot the variation of the function $\rho\mapsto\log(|\mathcal{V}_{\xi^i}(\rho)|)$ with respect to $\log(-\log(\rho))$ using different mesh step.

From the plotted curves in Fig. 1, we deduce the slopes $\alpha_i,\ i=1,\ \dots,4$ (see the following table) describing the behavior of the function $\rho\mapsto\mathcal{V}_{\xi^i}(\rho)$ with respect to $-\log(\rho)$.



(a) Perturbation $\chi_\rho^1$. (b) Perturbation $\chi_\rho^2$. (c) Perturbation $\chi_\rho^3$. (d) Perturbation $\chi_\rho^4$.

**Fig. 1.** Variation of $\log(|\mathcal{V}_{\xi^i}(\rho)|)$ with respect to $\log(-\log(\rho))$.

| The perturbation $\chi_\rho^i$: | $\chi_\rho^1$ | $\chi_\rho^2$ | $\chi_\rho^3$ | $\chi_\rho^4$ |
|---|---|---|---|---|
| The slope $\alpha_i, i = 1, \ldots, 4$ | $\alpha_1 = -1.1$ | $\alpha_2 = -1.17$ | $\alpha_3 = -1.18$ | $\alpha_4 = -1.2$ |

The obtained slopes $\alpha_i$ describing the behavior of the function $\rho \mapsto \mathcal{V}_{\xi^i}(\rho)$ with respect to $-\log(\rho)$ for the considered perturbations $\chi_\rho^i$, $i = 1, \ldots, 4$..

For each considered perturbation $\chi_\rho^i$, one can observe here that the obtained slope $\alpha_i$ satisfies the inequality: $\alpha_i < -1$, $i = 1, \ldots, 4$, which confirm the behavior predicted by the theoretical result

$$\mathcal{V}_\xi(\rho) = o\left(\frac{-1}{\log(\rho)}\right).$$

### 6.2. *Reconstruction procedures*

In this section, we aim to build a numerical procedure for detecting an unknown geometry $\mathcal{A}^\star$ inserted in a given domain $\mathcal{D}$ from overdetermined boundary data on $\partial\mathcal{D}$. We start our analysis by considering a simple and smooth shaped geometry. We will show in this particular case that $\mathcal{A}^\star$ can be reconstructed using only one iteration. An iterative reconstruction procedure is proposed in Section 6.2.2 for detecting more complicated geometry. The proposed reconstruction algorithms are based on the topological asymptotic expansion established in Theorem 2.

#### 6.2.1. *One-iteration reconstruction procedure*

In this paragraph, we propose a fast and accurate reconstruction procedure for detecting simple and smooth geometries $\mathcal{A}^\star$. We will show here that circular and ellipse shaped objects can be reconstructed using only one iteration. The main steps of the proposed procedure are described in the following algorithm.

**The one-iteration algorithm:**

- Solve the problems $(\mathcal{P}_n^0)$ and $(\mathcal{P}_d^0)$ in the initial domain $\mathcal{D}$,
- Compute the topological sensitivity function $\mathcal{S}(x)$, $\forall x \in \mathcal{D}$,
- Determine $t^\star \in [0, 1]$ such that $\mathcal{E}(\mathcal{D} \setminus \overline{\mathcal{A}_{t^\star}}) \leq \mathcal{E}(\mathcal{D} \setminus \overline{\mathcal{A}_t})$, $\forall t \in [0, 1]$, where $\mathcal{A}_t = \{x \in \mathcal{D};\ \mathcal{S}(x) \leq t\delta_{min}\}$ with $\delta_{min}$ is the most negative value of the function $\mathcal{S}$ in $\mathcal{D}$.

In order to show the performances of the proposed one-iteration algorithm, we present some numerical illustrations. We will consider three numerical examples. The first one concerns the reconstruction of circular-shaped objects. In the second example, we examine the numerical reconstruction of various ellipses having different locations and sizes. In the third example, we discuss the case of geometry with corners.

**(a) Reconstruction of circular-shaped objects:** the unknown domain $\mathcal{A}^\star$ is described by a disc inserted in the square $\mathcal{D} = [0,\ 1] \times [0,\ 1]$. In this example, we examine the numerical reconstruction of various discs having different radius $r$. The obtained results are illustrated in Fig. 2. For each considered radius, we show:
– the negative zone (red zone) described by the function $x \mapsto (x, \mathcal{S}(x))$, $\forall x \in \mathcal{D}$,
– the iso-values of the sensitivity function $\mathcal{S}$ in the presence of the unknown boundary $\partial\mathcal{A}^\star$,
– a zoom on zone containing the isovalue of $\mathcal{S}$ (color lines) approximating the boundary $\partial\mathcal{A}^\star$ (black line).

One can easily observe in Fig. 2, that the one-iteration algorithm gives quite efficient reconstruction results for different sizes of circular-shaped objects.

**(b) Reconstruction of ellipse-shaped objects:** the unknown domain $\mathcal{A}^\star$ is described by an ellipse inserted in the unit disc $\mathcal{D} = B(0, 1)$. In this example, we examine the numerical reconstruction of various ellipses having different locations and sizes. The obtained results are illustrated in Fig. 3. For each case, we plot the iso-values of the sensitivity function $\mathcal{S}$ (color lines) in the presence of the unknown boundary $\partial\mathcal{A}^\star$ (black line). Here again, as one can see in Fig. 3, the one-iteration algorithm gives quite efficient reconstruction results for different locations and sizes of ellipse-shaped objects.

**(c) Reconstruction of geometry with corners:** We apply now the proposed algorithm to detect more complicated geometry. Our aim is to reconstruct geometry containing straight lines and corners from overdetermined boundary data. More precisely, we want to detect the square $\mathcal{A}^\star$ whose vertices are the points $(0.25,\ 0.5)$, $(0.5,\ 0.25)$, $(0.75,\ 0.5)$, $(0.5,\ 0.75)$.

We see in Fig. 4 that the unknown square $\mathcal{A}^\star$ is located in the zone where the topological sensitivity function $\mathcal{S}$ is the most negative (red zone) but the boundary of $\mathcal{A}^\star$ cannot be well approximated by any iso-value curve. One can remark here, that the one-iteration algorithm detects the zone containing the unknown geometry but the reconstruction result is not good. In order to perform this result and get an efficient reconstruction result for geometries with corners, we suggest an iterative reconstruction procedure in the next section.

### *6.2.2. Iterative reconstruction procedure*

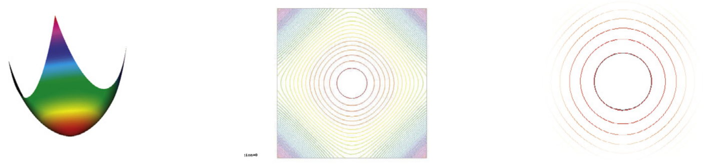
In this section, we propose an iterative process for reconstructing the unknown geometry $\mathcal{A}^\star$. The topological sensitivity $\mathcal{S}$ is used to build a sequence of geometries $(\mathcal{A}_i)_{i\geq 0}$, with $\mathcal{A}_0 = \emptyset$. At the $i$th iteration, the geometry $\mathcal{A}_{i+1}$ is obtained by creating a new hole $\chi^i$ in the domain $\mathcal{D}_i = \mathcal{D} \setminus \overline{\mathcal{A}_i}$, i.e. $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \chi^i$. The location and shape of the created hole $\chi^i$ is defined by the topological sensitivity function $\mathcal{S}$.

At the $i$th iteration, the topological sensitivity $\mathcal{S}^i$ describes the cost function variation with respect to the insertion of a small geometry perturbation in the domain $\mathcal{D}_i$. From Theorem 2, one can deduce

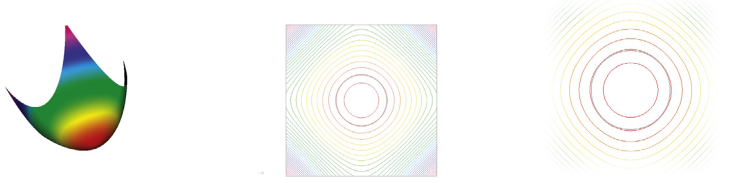
$$\mathcal{E}(\mathcal{D}_i \setminus \overline{\chi_{\xi,\rho}}) - \mathcal{E}(\mathcal{D}_i) = \frac{-1}{\log(\rho)}\mathcal{S}^i(\xi) + o\left(\frac{-1}{\log(\rho)}\right),$$

where the topological gradient $\mathcal{S}^i$ is given by

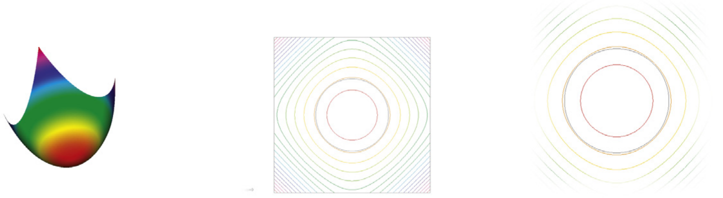
$$\mathcal{S}^i(x) = 2\pi\mu(x)\Big[|\phi_d^i(x) - \sigma(x)|^2 - |\phi_n^i(x) - \sigma(x)|^2\Big], \quad \forall x \in \mathcal{D}_i,$$



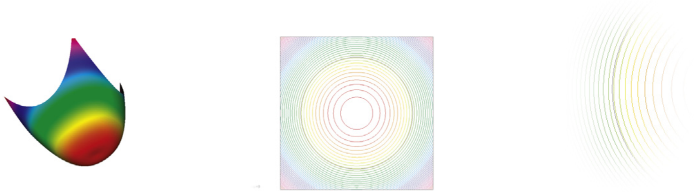
**r=0.1 :** negative zone (red zone) | iso-values of $\mathcal{S}$ in $\mathcal{D}$ | zoom showing the isovalue of $\mathcal{S}$ approximating $\partial\mathcal{A}^\star$ (black line)



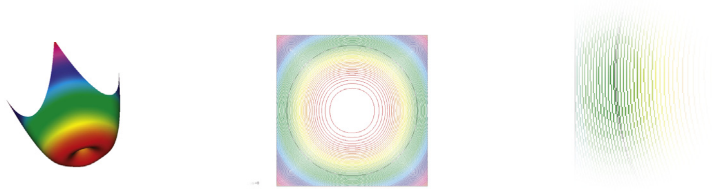
**r=0.15 :** negative zone (red zone) | iso-values of $\mathcal{S}$ in $\mathcal{D}$ | zoom showing the isovalue of $\mathcal{S}$ approximating $\partial\mathcal{A}^\star$ (black line)



**r=0.25 :** negative zone (red zone) | iso-values of $\mathcal{S}$ in $\mathcal{D}$ | zoom showing the isovalue of $\mathcal{S}$ approximating $\partial\mathcal{A}^\star$ (black line)



**r=0.35 :** negative zone (red zone) | iso-values of $\mathcal{S}$ in $\mathcal{D}$ | zoom showing the isovalue of $\mathcal{S}$ approximating $\partial\mathcal{A}^\star$ (black line)



**r=0.45 :** negative zone (red zone) | iso-values of $\mathcal{S}$ in $\mathcal{D}$ | zoom showing the isovalue of $\mathcal{S}$ approximating $\partial\mathcal{A}^\star$ (black line)

**Fig. 2.** Reconstruction of circular-shaped objects $\mathcal{A}^\star$ having different sizes.

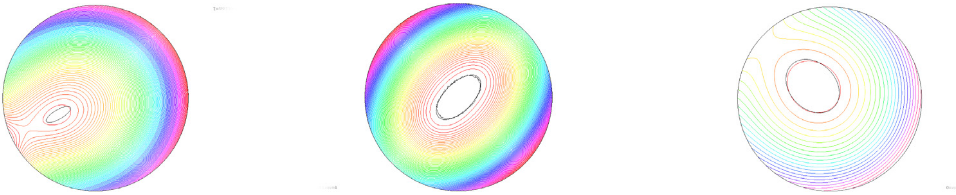


**Fig. 3.** Reconstruction of ellipse-shaped objects having different locations and sizes.

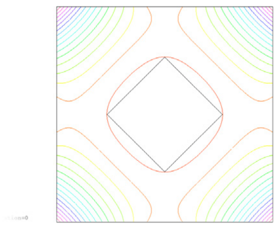


**Fig. 4.** Iso-values of the function $\mathcal{S}$ (colors lines) and the unknown square (black line).

with $\phi_d^i$ and $\phi_n^i$ are the solutions to the following problems

$$(\mathcal{P}_n^i)\begin{cases} -\operatorname{div}(\mu(x)\nabla\phi_n^i) = \mathcal{Q} & \text{in } \mathcal{D}\setminus\overline{\mathcal{A}_i} \\ \mu(x)\nabla\phi_n^i\cdot\mathbf{n} = \mathcal{F} & \text{on } \partial\mathcal{D} \\ \phi_n^i = \sigma & \text{on } \partial\mathcal{A}_i, \end{cases}$$

$$(\mathcal{P}_d^i)\begin{cases} -\operatorname{div}(\mu(x)\nabla\phi_d^i) = \mathcal{Q} & \text{in } \mathcal{D}\setminus\overline{\mathcal{A}_i} \\ \phi_d^i = \varphi_m & \text{on } \partial\mathcal{D} \\ \phi_d^i = \sigma & \text{on } \partial\mathcal{A}_i. \end{cases}$$

In the proposed procedure, the location of the unknown geometry $\chi^i$ is given by the point $\xi_i^\star$ where the topological sensitivity is the most negative i.e. $\mathcal{S}^i(\xi_i^\star) \leq \mathcal{S}^i(x), \forall x \in \mathcal{D}_i$. The shape of $\chi^i$ is defined by a level set curve of the scalar function $\mathcal{S}^i$:

$$\chi^i = \left\{x \in \mathcal{D}_i;\ \mathcal{S}^i(x) \leq c_i \leq 0\right\},$$

where $c_i$ is a constant chosen in such a way the cost function $\mathcal{E}$ decreases as most as possible. Our numerical implementation is based on the following main steps.

**The iterative algorithm:**

- Initialization : choose $\mathcal{D}_0 = \mathcal{D}$, $\mathcal{A}_0 = \emptyset$, and set $i = 0$.
- Repeat until $\mathcal{S}^i \geq 0$ in $\mathcal{D}_i$ :
  - solve the problems $(\mathcal{P}_n^i)$ and $(\mathcal{P}_d^i)$ in $\mathcal{D}_i$,
  - compute the topological sensitivity $\mathcal{S}^i$,
  - determine $\chi^i$ and set $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \chi^i$ and $\mathcal{D}_{i+1} = \mathcal{D}\setminus\overline{\mathcal{A}_{i+1}}$,
  - $i \leftarrow i + 1$.

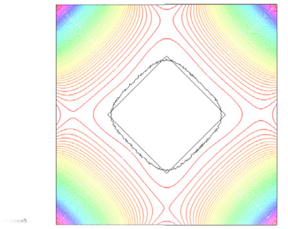
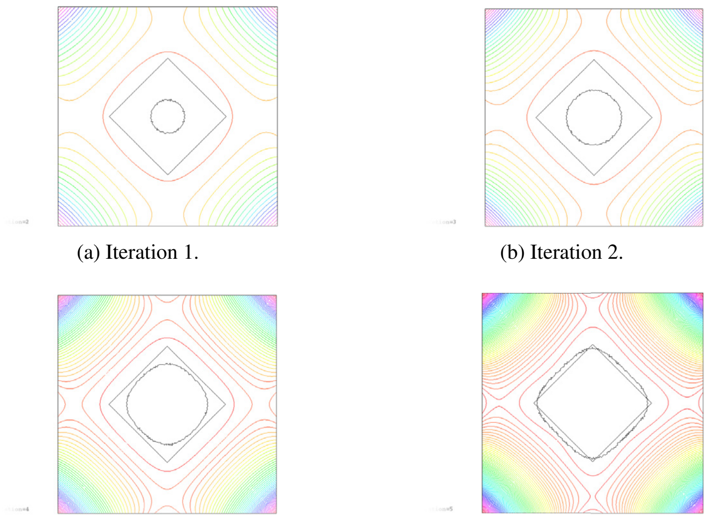


**Fig. 5.** Iterative reconstruction of objects with corners.



(a) Iteration 1.

(b) Iteration 2.

(c) Iteration 3.

(d) Iteration 4.

**Fig. 6.** The geometry $\mathcal{A}_i$ obtained during the optimization process.

In order to test the performances of this iterative process, we apply the proposed algorithm for detecting objects with corners. The unknown domain is described by the square $\mathcal{A}^\star$ whose vertices are the points $(0.25,\ 0.5)$, $(0.5,\ 0.25)$, $(0.75,\ 0.5)$, $(0.5,\ 0.75)$. The obtained result is presented in Fig. 5.

We see in Fig. 5, that the considered square is well approximated by a level-set curve of the topological sensitivity function $\mathcal{S}^4$. One can remark here, that we have obtained a quite reconstruction of an object with corners using only few iterations (only four iterations).

The geometry $\mathcal{A}_i$ obtained during the optimization process is illustrated in Fig. 6.

## 7. Concluding remarks

The presented work is focused on the detection of objects immersed in anisotropic medium from overdetermined boundary data. The proposed approach is based on the Kohn–Vogelius formulation and the topological sensitivity analysis method. The main contributions of this paper concern the theoretical and numerical aspects.

The theoretical part is devoted to a sensitivity analysis for an energy-like function $\mathcal{E}$ with respect to a small geometry perturbation of the background domain. An asymptotic expansion is derived with the help of preliminaries estimates describing the influence of the geometry perturbation on the Neumann and Dirichlet problems solutions.

The numerical part is concerned with some numerical investigations. The unknown geometry is reconstructed using a level-set curve of the topological sensitivity function $\mathcal{S}$.

In this part we have shown that the smooth and simple shaped geometry (like circular or ellipse shaped geometry) can be reconstructed using only one iteration. In the case of more complicated geometry, an iterative reconstruction process is proposed and illustrated by some numerical results.

The considered model can be viewed as a prototype example of geometric inverse problems arising in many industrial problems. The presented approach is general and can be adapted for various Partial Differential Equations.

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