Arab Journal of Mathematical Sciences (2012) **18**, 61–71

King Saud University
Arab Journal of Mathematical Sciences
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ORIGINAL ARTICLE

Introducing an efficient modification of the homotopy perturbation method by using Chebyshev polynomials

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Received 20 March 2011; revised 7 August 2011; accepted 26 September 2011
Available online 13 October 2011

KEYWORDS
Homotopy perturbation method;
Chebyshev polynomials;
Convergence study;
Fourth-order Runge–Kutta method

Abstract In this article an efficient modification of the homotopy perturbation method is presented by using Chebyshev polynomials. Special attention is given to prove the convergence of the method. Some examples are given to verify the convergence hypothesis, and illustrate the efficiency and simplicity of the method. We compared our numerical results against the conventional numerical method, fourth-order Runge–Kutta method (RK4). From the numerical results obtained from these two methods we found that the proposed technique and RK4 are in excellent conformance. From the presented examples, we found that the proposed method can be applied to a wide class of linear and non-linear ODEs.

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Peer review under responsibility of King Saud University.
doi:10.1016/j.ajmsc.2011.09.001

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1. Introduction

The homotopy perturbation method (HPM) was introduced by He [7–9,13–17] in the year 1998. In this method the solution is considered as the summation of an infinite series which converges rapidly to the exact solution. This technique has been employed to solve a large variety of linear and nonlinear differential equations. This scheme is used for solving nonlinear boundary value problems [7]. This method is also adopted for solving the pure strong nonlinear second-order differential equations [8]. Some other applications of this method are as follows: application of He's HPM is described to solve nonlinear integro-differential equations [4], for traveling wave solutions of nonlinear wave equations. Also, this method is used to solve the nonlinear parabolic equation with non local boundary conditions [6]. In general, this method has been successfully applied to solve many types of linear and nonlinear problems in science and engineering by many authors [3,5,11,12,14].

Our main goal in this paper is concerned with the implementation of HPM and its modifications which have efficiently used to solve the ordinary differential equations [1,10]. For this reason, at the beginning of implementation of HPM, Chebyshev orthogonal polynomials are used to expand functions. The obtained results show the advantage from using the proposed modified HPM.

In addition, the proposed modified HPM is numerically performed through Matlab version 7.

2. Solution procedure using the modified HPM

In this section, an efficient modification of HPM is presented by using Chebyshev polynomials.

The well known Chebyshev polynomials [2] are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula:

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \quad n = 1, 2, \ldots$$

The first three Chebyshev polynomials are

$$T_0(x) = 1, \;\; T_1(x) = x, \;\; T_1(x) = 2x^2 - 1.$$

In this paper, we suggest that the non-homogeneous term $f(x)$ can be expressed in Chebyshev series:

$$f(x) \approx \sum_{k=0}^{\infty} c_k T_k(x). \tag{1}$$

Theorem 1 (*Chebyshev truncation theorem*). *The error in approximating f(x) by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$f_m(x) = \sum_{k=0}^{m} c_k T_k(x), \tag{2}$$

then, for all f(x), all m, and all $x \in [-1,1]$, *we have*

$$E_T(m) \equiv |f(x) - f_m(x)| \leqslant \sum_{k=m+1}^{\infty} |c_k|, \tag{3}$$

Proof. The Chebyshev polynomials are bounded by one, that is, $|T_k(x)| \leqslant 1$ for all $x \in [-1,1]$ and for all k. This implies that the k-th term is bounded by $|c_k|$. Subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds gives the theorem.

Now, in order to use these polynomials on the interval $x \in [0,1]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = 2x - 1$. Let the shifted Chebyshev polynomials $T_n(2x-1)$ be denoted by $P_n(x)$. Then $P_n(x)$ can be obtained as follows

$$P_{n+1}(x) = 2(2x-1)P_n(x) - P_{n-1}(x), \quad n = 1,2,\ldots. \tag{4}$$

Now, we use the shifted Chebyshev expansion to expand $f(x)$ in the following form:

$$f(x) \approx f_m(x) = \sum_{k=0}^{m} c_k P_k(x), \tag{5}$$

where the constants coefficients c_k, $k = 0,1,2,\ldots,m$ by using the orthogonal property are defined by:

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(0.5x+0.5)T_0(x)}{\sqrt{1-x^2}}\,\mathrm{d}x, \qquad c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(0.5x+0.5)T_k(x)}{\sqrt{1-x^2}}\,\mathrm{d}x. \tag{6}$$

Now, the proposed modification will implement to solve the following two initial nonlinear ordinary differential equations. □

Model problem 1.

Consider the following nonlinear ordinary differential equation:

$$u'' + xu' + x^2u^3 = f(x) = (2 + 6x^2)e^{x^2} + x^2e^{3x^2}, \quad x \in [0,1], \tag{7}$$

subject to the following initial conditions:

$$u(0) = 1, \qquad u'(0) = 0, \tag{8}$$

with the exact solution $u(x) = e^{x^2}$.

The procedure of the solution follows the following two steps:

Step 1. Expand the function $f(x)$ using Chebyshev polynomials:Using the above consideration, the function $f(x)$ can be approximated by eight terms ($m = 8$) of the expansion (5) as follows:

$$f_C(x) \approx 2.00232 - 0.358488\ x + 18.0328\ x^2 - 86.4534\ x^3 + 416.556\ x^4 - 1042.66\ x^5 + 1502.72\ x^6 - 1134.64x^7 + 366.624x^8.$$

Step 2. Implementation of HPM:The essential idea of HPM is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the equation is in sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the equation goes through a sequence of "deformation", the solution is "close" to that at the previous stage of "deformation". Eventually at $p = 1$, the system takes the original form of equation and the final stage of "deformation" gives the desired solution.Now according to HPM, we can construct the following simple homotopy v:

$$v'' + p[xv' + x^2 v^3 - f(x)] = 0, \tag{9}$$

where $p \in [0, 1]$ is an embedding parameter, we use it to expand the solution as a power series in p in the following form:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \tag{10}$$

setting $p = 1$ results the approximate solution of (9):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{11}$$

For more details on HPM and its convergence, see [3,9].

Substituting from (10) into (9), and equating the terms with the identical powers of p, we can obtain a system of $n + 1$ linear ODEs. Assuming $n = 4$, the system is as follows:

$$\begin{cases} v_0'' = f(x), & v_0(0) = 1, v_0'(0) = 0, \\ v_1'' + xv_0' + x^2 v_0^3 = 0, & v_1(0) = v_1'(0) = 0, \\ v_2'' + xv_1' + x^2\left(3v_1 v_0^2\right) = 0, & v_2(0) = v_2'(0) = 0, \\ v_3'' + xv_2' + x^2\left(3v_2 v_0^2 + 3v_0 v_1^2\right) = 0, & v_3(0) = v_3'(0) = 0, \\ v_4'' + xv_3' + x^2\left(3v_3 v_0^2 + 6v_0 v_1 v_2 + v_1^3\right) = 0, & v_4(0) = v_4'(0) = 0. \end{cases} \tag{12}$$

The solution of the system of equations (12) is in the following form:

$$v_0(x) = \int_0^x \int_0^x f(x)\,\mathrm{d}x\,\mathrm{d}x + v(0) + v'(0)x$$
$$= 1 + 1.00116x^2 - 0.059748x^3 + 1.50273x^4 - 4.32267x^5 + 13.8852x^6 - 24.8252x^7 + 26.8343x^8 - 15.7589x^9 + 4.0736x^{10},$$
$$v_1(x) = -\int_0^x \int_0^x \left[xv_0' + x^2 v_0^3\right] \mathrm{d}x\,\mathrm{d}x$$
$$= -0.250193x^4 + 0.00896221x^5 - 0.30048x^6 + 0.518871x^7 - 1.6219x^8 + 2.59866x^9 - 2.95968x^{10} + 2.209x^{11}.$$

Having v_i, $i = 0, 1, 2, \ldots, 8$ the approximate solution of $u(x)$ is as follows:

$$u_C(x) \cong \sum_{i=0}^{8} v_i$$
$$= 1 + 1.00116x^2 - 0.059748x^3 + 1.25254x^4 - 4.31371x^5 + 13.6181x^6 - 24.3074x^7 + 25.2544x^8 - 13.2109x^9 + 1.27996x^{10}.$$

The absolute error between the exact solution $u(x)$ and the approximate solution $u_C(x)$ using the Chebyshev expansion for $f(x)$ is presented in Fig. 1.

Now, also to perform HPM, we can expand the function $f(x)$ using Taylor series at the point $x = a$:

$$f(x) \cong \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(x - a)^k, \tag{13}$$

for an arbitrary natural number m.

If we expand the function $f(x)$ by the Taylor series (13) about the point $x = 0$ with eight terms, we have:

$$f_T(x) \cong 2 + 9x^2 + 10x^4 + 7.83x^6 + 5.58333x^8 + \mathrm{O}(x^9).$$

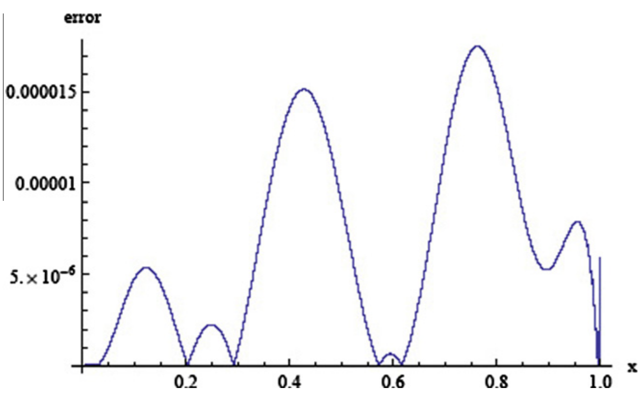

Figure 1 The absolute error $|u(x) - u_C(x)|$.

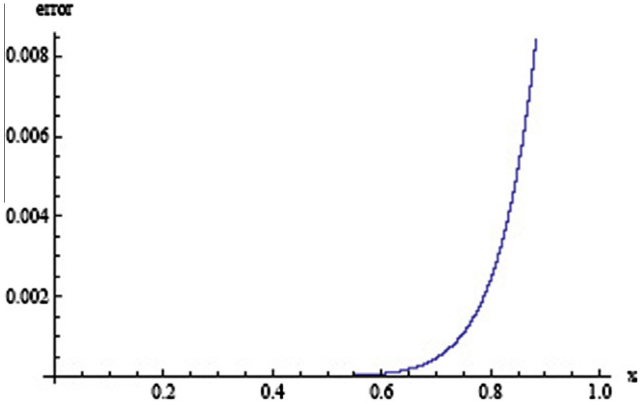

Figure 2 The absolute error $|u(x) - u_T(x)|$.

So, the solution of the system (12) is:

$$\begin{aligned}
v_0(x) &= \int_0^x \int_0^x f(x)\,dx\,dx + v(0) + v'(0)x \\
&= 1 + x^2 + 0.75x^4 + 0.3334x^6 + 0.1399x^8 + 0.0621x^{10}, \\
v_1(x) &= -\int_0^x \int_0^x \left[x v_0' + x^2 v_0^3\right] dx\,dx \\
&= -0.25x^4 - 0.2x^6 - 0.12946x^8 - 0.084661x^{10} + \cdots, \\
v_2(x) &= -\int_0^x \int_0^x \left[x v_1' + x^2\left(3 v_1 v_0^2\right)\right] dx\,dx \\
&= 0.0333333x^6 + 0.348214x^8 + 0.0348413x^{10} + \cdots, \\
v_3(x) &= -\int_0^x \int_0^x \left[x v_2' + x^2\left(3 v_2 v_0^2 + 3 v_0 v_1^2\right)\right] dx\,dx \\
&= -0.00357143x^8 - 0.00420635x^{10} + \cdots
\end{aligned}$$

Having v_i, $i = 0, 1, 2, \ldots, 8$ the approximate solution of $u(x)$ is as follows:

$$\begin{aligned}
u_T(x) &\cong \sum_{i=0}^{8} v_i \\
&= 1 + x^2 + 0.5x^4 + 0.16667x^6 + 0.04167x^8 + 0.008334x^{10} - 0.02619x^{12} \\
&\quad + \cdots
\end{aligned}$$

The absolute error between the exact solution $u(x)$ and the approximate solution $u_T(x)$ using the Taylor expansion for $f(x)$ with $m = 8$ is presented in Fig. 2.

Also, to solve Eq. (7) by the numerical method, fourth-order Runge–Kutta method, we reduce this equation to the following system of ordinary differential equations:

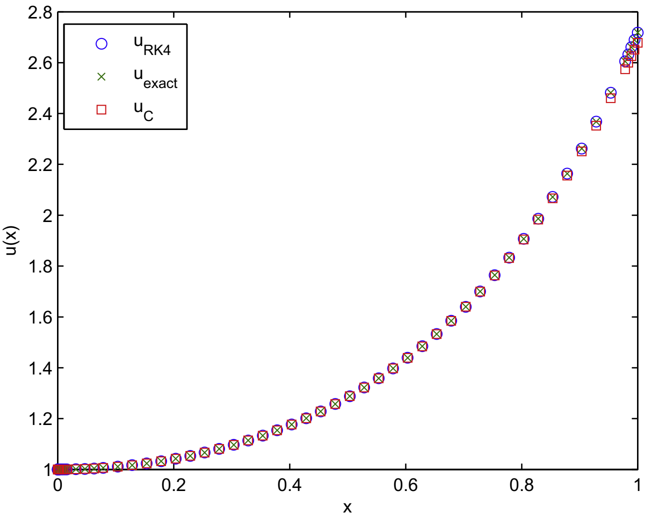

Figure 3 Comparison between the exact solution u_{exact}, u_{RK4} and the solution of our proposed method $u_C(x)$.

$$u'(x) = v(x), \tag{14}$$

$$v'(x) = -xv(x) - x^2u^3(x) + f(x), \tag{15}$$

subject to the following initial conditions:

$$u(0) = 1, \qquad v(0) = 0. \tag{16}$$

Fig. 3 presents a comparison between the exact solution u_x, with fourth-order Runge–Kutta u_{RK4}, and the approximate solution of our proposed method $u_C(x)$. From this figure, we can see that the two methods are in excellent agreement with the exact solution.

Model problem 2.

Consider the following nonlinear ordinary differential equation:

$$u'' + uu' = f(x) = x\sin(2x^2) - 4x^2\sin(x^2) + 2\cos(x^2), \quad x \in [0, 1], \tag{17}$$

subject to the following initial conditions:

$$u(0) = 0, \qquad u'(0) = 0, \tag{18}$$

with the exact solution $u(x) = \sin(x^2)$.

The procedure of the solution follows the following two steps:

Step 1. Expand the function $f(x)$ using Chebyshev polynomials:Using the above consideration, the function $f(x)$ can be approximated by eight terms ($m = 8$) of the expansion (5) as follows:

$$f_C(x) \cong 2 - 0.0003x + 0.008x^2 + 1.892x^3 - 4.308x^4 - 2.3986x^5 + 4.6816x^6 - 6.276x^7 + 3.025x^8.$$

Step 2. Implementation of HPM:According to HPM, we can construct the following simple homotopy v:

$$v'' + p[vv' - f(x)] = 0, \tag{19}$$

where $p \in [0,1]$ is an embedding parameter, we use it to expand the solution in the form (10). Substituting from (10) into (19), and equating the terms with the identical powers of p, we can obtain a system of $n + 1$ linear ordinary differential equations. Assuming $n = 4$ the system is as follows:

$$\begin{cases} v_0'' = f(x), & v_0(0) = v_0'(0) = 0, \\ v_1'' + v_0 v_0' = 0, & v_1(0) = v_1'(0) = 0, \\ v_2'' + (v_1 v_0' + v_0 v_1') = 0, & v_2(0) = v_2'(0) = 0, \\ v_3'' + (v_2 v_0' + v_1 v_1' + v_0 v_2') = 0, & v_3(0) = v_3'(0) = 0, \\ v_4'' + (v_3 v_0' + v_2 v_1' + v_1 v_2' + v_0 v_3') = 0, & v_4(0) = v_4'(0) = 0. \end{cases} \tag{20}$$

The solution of the system of equations (20) is in the following form:

$$v_0(x) = x^2 - 0.00004x^3 + 0.00069x^4 + 0.094575x^5 - 0.14359x^6 - 0.05710x^7 + 0.08359x^8 - 0.08716x^9 + 0.033609x^{10},$$

$$v_1(x) = -0.1x^5 + 6.2996 \times 10^{-6}x^6 - 0.00010x^7 - 0.01182x^8 + 0.01596x^9 + 0.00570x^{10} + \cdots$$

Having v_i, $i = 0, 1, 2, \ldots, 8$ the approximate solution of $u(x)$ is as follows:

$$u_C(x) \cong \sum_{i=0}^{8} v_i = x^2 - 0.000038x^3 + 0.000684x^4 - 0.005425x^5 - 0.143585x^6 - 0.057196x^7 + 0.084268x^8 - 0.071205x^9 + 0.039329x^{10} + \cdots$$

Now, if we expand the function $f(x)$ by the Taylor series (13), we have:

$$f(x) \approx 2 + 2x^3 - 5x^4 - 1.33333x^7 + 0.75x^8 + O(x^9).$$

Table 1 Comparison of the absolute error of $u(x)$ using the Chebyshev expansion and Taylor expansion for $f(x)$ at different values of m ($m = 4$ and 8).

x	$\lvert u(x) - u_{C4}(x)\rvert$	$\lvert u(x) - u_{C8}(x)\rvert$	$\lvert u(x) - u_{T4}(x)\rvert$	$\lvert u(x) - u_{T8}(x)\rvert$
0.0	2.01055e−12	2.13058e−15	1.01055e−09	2.13058e−12
0.2	3.36702e−08	1.83661e−11	1.36702e−06	2.98725e−08
0.4	1.08796e−06	4.90253e−09	5.24571e−04	9.01472e−06
0.8	2.03445e−06	8.73997e−09	0.25348e−04	2.23254e−06
0.6	8.07432e−05	5.94450e−08	7.01854e−03	7.01450e−05
1.0	1.34807e−05	2.87034e−08	2.35475e−03	3.22254e−05

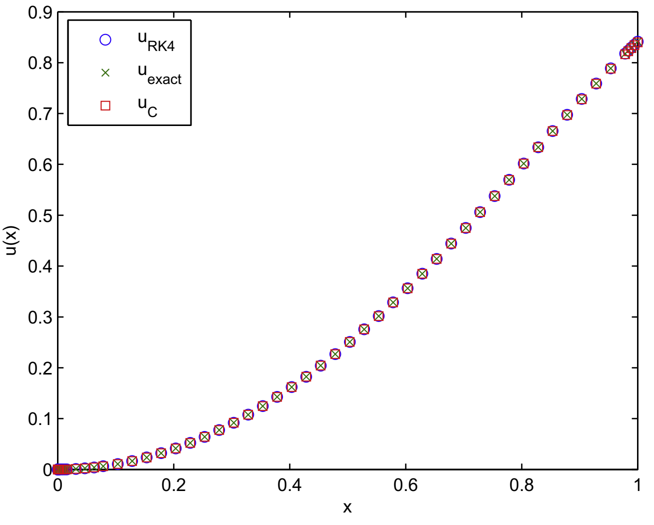

Figure 4 Comparison between the exact solution u_{exact}, u_{RK4} and the solution of our proposed method $u_C(x)$.

So, the solution of the system of equations (20) is:

$$v_0(x) = x^2 + 0.1x^5 - 0.166667x^6 - 0.0185185x^9 + 0.00833333x^{10},$$
$$v_1(x) = -0.1x^5 - 0.0125x^8 + 0.0185185x^9 - 0.000454545x^{11} + 0.0029321x^{12} + \cdots,$$
$$v_2(x) = 0.0125x^8 + 0.00204545x^{11} - 0.0029321x^{12} + 0.000121753x^{14} - 0.000581276x^{15} + \cdots,$$
$$v_3(x) = -0.00159091x^{11} - 0.000324675x^{14} + 0.000457819x^{15} - 0.0000264634x^{17} + \cdots,$$
$$v_4(x) = 0.000202922x^{14} + 0.0000496801x^{17} - 0.0000693145x^{18} + 0.00000511x^{20} + \cdots$$

Having v_i, $i = 0, 1, 2, \ldots, 8$ the approximate solution of $u(x)$ is as follows:

$$u_T(x) \approx \sum_{i=0}^{8} v_i = x^2 - 0.166667x^6 + 0.00833333x^{10} - 0.0017094x^{13} + 0.000106838x^{16} + \cdots$$

A comparison of the absolute error of $u(x)$ using the Chebyshev expansion (columns 2 and 3) and Taylor expansion (columns 4,5) for $f(x)$ with different values of m ($m = 4$ and 8) is presented in Table 1. From this table, it is evident that the overall errors can be made smaller by adding new terms from the series (5) and (13).

Also, to solve Eq. (17) by the numerical method, fourth-order Runge–Kutta method, we reduce this equation to the following system of ODEs:

$$u'(x) = v(x), \tag{21}$$
$$v'(x) = -u(x)v(x) + f(x), \tag{22}$$

subject to the following initial conditions:

$$u(0) = 0, v(0) = 0. \quad (23)$$

Fig. 4, presents a comparison between exact solution u_x, with the numerical method, fourth-order Runge–Kutta and the approximate solution of our proposed method $u_C(x)$. From this figure, we can see that the two methods are in excellent agreement with the exact solution.

3. Conclusion

In this paper an efficient modification of HPM is presented by using Chebyshev polynomials. The convergence analysis of the proposed method is introduced. Also, we presented comparative solutions with proposed method and the numerical method, fourth-order Runge–Kutta method (RK4). We choose the conventional RK4 as our benchmark, as it is widely accepted and used. From the introduced model problems we can conclude that the proposed method can be applied to linear and nonlinear models which represent by ordinary differential equations. The solution obtained using the suggested method is in excellent agreement with the already existing ones and show that this approach can solve the problem effectively. Also, the obtained results demonstrate reliability and efficiency of the proposed method. All numerical results are obtained using Matlab version 7.

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