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Infinitely many solutions for systems of *n* fourth order partial differential equations coupled with Navier boundary conditions

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Abstract. In this paper, the existence of infinitely many solutions for a class of systems of n fourth order partial differential equations coupled with Navier boundary conditions is established. The approach is fully based on Ricceri's Variational Principle [B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401–410].

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1. Introduction

In this work, we discuss the existence of infinitely many weak solutions for the nonlinear elliptic system of n fourth order partial differential equations under Navier boundary conditions

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) - \alpha_i\Delta_{p_i}u_i + \beta_i|u_i|^{p_i-2}u_i = \lambda F_{u_i}(x,u_1,\ldots,u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

for $1 \leqslant i \leqslant n$, where $\Delta_{p_i}u_i = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator, $p_i > \max\left\{1, \frac{N}{2}\right\}$ for $1 \leqslant i \leqslant n$, α_i and β_i for $1 \leqslant i \leqslant n$ are positive constants, $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $\lambda > 0$ and $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1,t_2 \cdots ,t_n) \to F(x,t_1, t_2, \cdots ,t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in \Omega$, F_{t_i} is continuous in $\Omega \times \mathbb{R}^n$ for $i = 1,\ldots,n$,

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and $F(x,0,\ldots,0) = 0$ for all $x \in \Omega$. Here, F_{t_i} denotes the partial derivative of F with respect to t_i. The system (1) is called $(p_1,\ldots,p_n)$-biharmonic.

Here and in the sequel, for all $\gamma > 0$ we denote by $K(\gamma)$ the set

$$\left\{(t_1,\ldots,t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \leqslant \gamma \right\}. \tag{2}$$

A special case of our main result is the following theorem.

Theorem 1.1. *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a continuous function for $1 \leqslant i \leqslant n$ such that the differential 1-form $w := \sum_{i=1}^{n} f_i(\xi_1,\ldots,\xi_n)d\xi_i$ is integrable and let F be a primitive of w such that $F(\xi_1,\ldots,\xi_n) \geqslant 0$ in $\mathbb{R}^n$. Fix $p_i > \max\left\{1, \frac{N}{2}\right\}$ for $1 \leqslant i \leqslant n$ and assume that*

$$\liminf_{\xi\to+\infty} \xi^{-p_n} \max_{(t_1,\ldots,t_n)\in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(t_1,\ldots,t_n) = 0$$

and

$$\limsup_{\xi\to+\infty} \xi^{-p_n} F(0,\ldots,0,\xi) = +\infty.$$

Then, the system

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) - \alpha_i\Delta_{p_i}u_i + \beta_i|u_i|^{p_i-2}u_i = f_i(u_1,\ldots,u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases}$$

for $1 \leqslant i \leqslant n$, admits a sequence of pairwise distinct positive weak solutions.

There is an increasing interest in studying fourth-order boundary value problems, because the static form change of a beam can be described by a fourth-order equation, and also a model to study traveling waves in suspension bridges can be described by nonlinear fourth-order equations (for instance, see [22]). More general nonlinear fourth-order elliptic boundary value problems have been studied in recent years. Several results are known concerning the existence of multiple solutions for fourth-order boundary value problems, and we refer the reader to [1,3–5,12,14,18–20,26,25,27] and the references cited therein.

For a discussion about the existence of infinitely many solutions for differential equations, using Ricceri's Variational Principle [31], applying a smooth version of Theorem 2.1 of [7] which is a more precise version of Ricceri's Variational Principle [31] and employing a non-smooth version of Ricceri's Variational Principle [31] due to Marano and Motreanu [28], we refer the reader to the papers [15–17,21,32], [2,4,8–11,14] and [13], respectively. We also refer the reader to the papers [23,24,29,30,34] where the existence of infinitely many solutions for some boundary value problems has been studied by using different approaches (variational methods, perturbation theory, sub-super solutions,...). Here, our motivation comes from the recent papers [4,6].

The outline of the paper is organized as follows: in the forthcoming section, we shall recall our main tool (Theorem 2.1) and some basic notations which we need in the proofs. Whereas, Section 3 is devoted to the existence of infinitely many weak solutions for the system (1). To be precise, our main result (Theorem 3.1), some of its possible consequences, the proofs and some examples to illustrate the results are presented. Finally, in Section 4, as a remarkable consequence of the main result, we shall discuss the existence of infinitely many weak solutions for the nonlinear elliptic system of n doubly eigenvalue fourth order partial differential equations under Navier boundary conditions. Precisely, arguing as in [6], we shall prove that an appropriate oscillating behavior of the nonlinear term, even under small perturbations, ensures again the existence of infinitely many solutions. Two examples of application are pointed out (see Examples 4.1 and 4.2).

2. Preliminaries

Our main tool is the celebrated Ricceri's Variational Principle [31, Theorem 2.5] that we now recall as follows:

Theorem 2.1. *Let X be a reflexive real Banach space, and $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semi-continuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(*a*) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(*b*) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either*

(b_1) I_λ possesses a global minimum, or

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(*c*) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either*

(c_1) there is a global minimum of Φ which is a local minimum of I_λ, or

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.*

Let $X = \prod_{i=1}^{n}(W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega))$, endowed with the norm

$$\|(u_1, \ldots, u_n)\| = \sum_{i=1}^{n} \|u_i\|_{p_i},$$

where

$$\|u_i\|_{p_i} = \left(\int_\Omega |\Delta u_i(x)|^{p_i} dx + \alpha_i \int_\Omega |\nabla u_i(x)|^{p_i} dx + \beta_i \int_\Omega |u_i(x)|^{p_i} dx \right)^{1/p_i},$$

for $1 \leqslant i \leqslant n$.

We say that $u = (u_1, \ldots, u_n)$ is a weak solution to the system (1) if $u = (u_1, \ldots, u_n) \in X$ and

$$\int_\Omega \sum_{i=1}^{n} \Big(|\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta v_i(x) + \alpha_i |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x)$$
$$+ \beta_i |u_i(x)|^{p_i-2} u_i(x) v_i(x) \Big) dx - \lambda \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx = 0$$

for every $(v_1, \ldots, v_n) \in X$.

We also need the following proposition in the proof of Theorem 3.1.

Proposition 2.2. *The operator $T{:}X \to X^*$ defined by*

$$\begin{aligned} T(u_1, \ldots, u_n)(h_1, \ldots, h_n) = & \int_\Omega \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta h_i(x) dx \\ & + \alpha_i \int_\Omega \sum_{i=1}^{n} |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla h_i(x) dx \\ & + \beta_i \int_\Omega \sum_{i=1}^{n} |u_i(x)|^{p_i-2} u_i(x) h_i(x) dx \end{aligned}$$

for every $(u_1, \ldots, u_n), (h_1, \ldots, h_n) \in X$, is strictly monotone.

Proof. Since

$$\begin{aligned} T(u_1,\ldots,u_n)(u_1,\ldots,u_n) &= \sum_{i=1}^{n} \left(\int_\Omega |\Delta u_i(x)|^{p_i} dx + \alpha_i \int_\Omega |\nabla u_i(x)|^{p_i} dx + \beta_i \int_\Omega |u_i(x)|^{p_i} dx \right) \\ &= \sum_{i=1}^{n} \|u_i\|_{p_i}^{p_i}, \end{aligned}$$

T is coercive. Taking into account (2.2) of [33] for $p > 1$ there exists a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geqslant \begin{cases} C_p |x-y|^p & \text{if } p \geqslant 2, \\ C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2, \end{cases}$$

where $\langle .,. \rangle$ denotes the usual inner product in $\mathbb{R}^N$, for every $x,\ y \in \mathbb{R}^N$. Thus, it is easy to see that

$$\begin{aligned}
&\langle T(u_1,\dots,u_n) - T(v_1,\dots,v_n))(u_1 - v_1,\dots,u_n - v_n)\rangle \\
&\geqslant \sum_{i\in I_1} C_{p_i} \int_\Omega (|\Delta u_i(x) - \Delta v_i(x)|^{p_i} dx + \alpha_i |\nabla u_i(x) - \nabla v_i(x)|^{p_i} + \beta_i |u_i(x) - v_i(x)|^{p_i}) dx \\
&\quad + \sum_{i\in I_2} C_{p_i} \int_\Omega \left(\frac{|\Delta u_i(x) - \Delta v_i(x)|^2}{(|\Delta u_i(x)| + |\Delta v_i(x)|)^{2-p_i}} + \alpha_i \frac{|\nabla u_i(x) - \nabla v_i(x)|^2}{(|\nabla u_i(x)| + |\nabla v_i(x)|)^{2-p_i}} \right. \\
&\quad \left. + \beta_i \frac{|u_i(x) - v_i(x)|^2}{(|u_i(x)| + |v_i(x)|)^{2-p_i}} \right) dx > 0
\end{aligned}$$

where $I_1 = \{i \in \{1,\dots,n\}: p_i \geqslant 2\}$ and $I_2 = \{i \in \{1,\dots,n\}: 1 < p_i < 2\}$, for every $(u_1,\dots,u_n),(v_1,\dots,v_n) \in X$, which means that T is strictly monotone. $\square$

Put

$$k = \max \left\{ \sup_{u_i \in W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}}; \quad \text{for} \quad 1 \leqslant i \leqslant n \right\}. \tag{3}$$

For $p_i > \max\left\{1, \frac{N}{2}\right\}$ for $1 \leqslant i \leqslant n$, since the embedding $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ for $1 \leqslant i \leqslant n$ is compact, one has $k < +\infty$.

Fix $x^0 \in \Omega$ and pick $s > 0$ such that

$$S(x^0, s) \subset \Omega$$

where $S(x^0, s)$ denotes the ball with center at x^0 and radius s.

Put

$$\sigma_n = \sigma_n(N, p_n, s) := \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \int_{\frac{s}{2}}^{s} \left| \frac{12(N+1)}{s^3}\xi - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{\xi^3} \right|^{p_n} \xi^{N-1} d\xi,$$

$$\theta_n = \theta_n(N, p_n, x^0, s) := \int_{S(x^0,s)\setminus S\left(x^0,\frac{s}{2}\right)} \left[\sum_{i=1}^{N} \left(\frac{12l\left(x_i - x_i^0\right)}{s^3} - \frac{24\left(x_i - x_i^0\right)}{s^2} + \frac{9}{s}\frac{\left(x_i - x_i^0\right)}{l} \right)^2 \right]^{\frac{p_n}{2}} dx$$

where Γ denotes the Gamma function and $l = \sqrt{\sum_{i=1}^N \left(x_i - x_i^0\right)^2}$,

$$\varrho_n = \varrho_n(N, p_n, s) := \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left(\frac{\left(\frac{s}{2}\right)^N}{N} + \int_{\frac{s}{2}}^{s} \left| \frac{4}{s^3}\xi^3 - \frac{12}{s^2}\xi^2 + \frac{9}{s}\xi - 1 \right|^{p_n} \xi^{N-1} d\xi \right)$$

and

$$L := \frac{1}{k\prod_{i=1}^{n-1} p_i(\sigma_n + \alpha_n\theta_n + \beta_n\varrho_n)}. \tag{4}$$

3. Main results

We formulate our main result as follows.

Theorem 3.1. *Assume that*

(A1) $F(x,0,\dots,0,t_n) \geqslant 0$ *for each* $x \in \Omega \setminus S\left(x^0, \frac{s}{2}\right)$, $t_n \in \mathbb{R}$;
(A2) $\Lambda_0 < L\Lambda_1$ *where*

$$\Lambda_0 : \liminf_{\xi \to +\infty} \xi^{-p_n} \int_\Omega \sup_{(t_1,\dots,t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\dots,t_n)dx,$$

$$\Lambda_1 := \limsup_{\xi \to +\infty} \xi^{-p_n} \int_{S(x^0,\frac{s}{2})} F(x,0,\dots,0,\xi)dx,$$

L is given by (4) and $K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right) = \left\{(t_1,\dots,t_n) \mid \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leqslant \frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right\}$*(see (2)).*

Then, for each

$$\lambda \in \Lambda := \left] \frac{\frac{1}{kL\prod_{i=1}^n p_i}}{\Lambda_1}, \frac{\frac{1}{k\prod_{i=1}^n p_i}}{\Lambda_0} \right[$$

the system (1) has an unbounded sequence of weak solutions in X.

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ for each $u = (u_1,\dots,u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{5}$$

and

$$\Psi(u) = \int_\Omega F(x,u_1(x),\dots,u_n(x))dx. \tag{6}$$

Because $p_i > \max\left\{1, \frac{N}{2}\right\}$ for $1 \leqslant i \leqslant n$, X is compactly embedded in $C^0(\overline{\Omega}) \times \ldots \times C^0(\overline{\Omega})$; it is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1,\dots,u_n) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega \sum_{i=1}^n |\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta v_i(x)dx + \alpha_i \int_\Omega \sum_{i=1}^n |\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x)dx$$

$$+ \beta_i \int_\Omega \sum_{i=1}^n |u_i(x)|^{p_i-2}u_i(x)v_i(x)dx$$

and

$$\Psi'(u)(v) = \int_{\Omega} \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx$$

for every $v = (v_1, \ldots, v_n) \in X$, respectively. Ψ is sequentially weakly upper semicontinuous.

Moreover, Proposition 2.2 establishes Φ' is monotone. So we obtain that Φ is sequentially weakly lower semicontinuous (see [35, Proposition 25.20(d)]). Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $(u_1, \ldots, u_n) \in X$, let $(u_{1m}, \ldots, u_{nm}) \to (u_1, \ldots, u_n)$ weakly in X as $m \to +\infty$; then we have $(u_{1m}, \ldots, u_{nm})$ converges uniformly to $(u_1, \ldots, u_n)$ on $\overline{\Omega}$ as $m \to +\infty$(see [35]). Since $F(x, ., \ldots, .)$ is C^1 in $\mathbb{R}^n$ for every $x \in \overline{\Omega}$, the derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \overline{\Omega}$, so for $1 \leqslant i \leqslant n, F_{u_i}(x, u_{1m}, \ldots, u_{nm}) \to F_{u_i}(x, u_1, \ldots, u_n)$ strongly as $m \to +\infty$; it follows that $\Psi'(u_{1m}, \ldots, u_{nm}) \to \Psi'(u_1, \ldots, u_n)$ strongly as $m \to +\infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [35].

Put $I_\lambda := \Phi - \lambda\Psi$. Clearly, the weak solutions of the system (1) are exactly the solutions of the equation $I'_\lambda(u_1, \ldots, u_n) = 0$. Now, we want to show that

$$\gamma < +\infty.$$

Let $\{\xi_m\}$ be a sequence of positive numbers such that $\xi_m \to +\infty$ as $m \to \infty$ and

$$\lim_{m\to\infty} \xi_m^{-p_n} \int_{\Omega} \sup_{(t_1,\ldots,t_n) \in K\left(\frac{\xi_m^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x, t_1, \ldots, t_n) dx = \Lambda_0.$$

Put $r_m = \frac{\xi_m^{p_n}}{k \prod_{i=1}^{n} p_i}$ for all $m \in \mathbb{N}$. Since, for each $(u_1, \ldots, u_n) \in X$,

$$\sup_{x\in\Omega} |u_i(x)|^{p_i} \leqslant k \|u_i\|_{p_i}^{p_i}$$

for $i = 1, \ldots, n$ (see (3)), we have

$$\sup_{x\in\Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leqslant k \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i}$$

for each $u = (u_1, \ldots, u_n) \in X$, and so

$$\sup_{x\in\Omega} \sum_{i=1}^{n} \frac{|v_i(x)|^{p_i}}{p_i} \leqslant \frac{\xi_m^{p_n}}{\prod_{i=1}^{n} p_i}$$

for all $v = (v_1, \ldots, v_n) \in X$ such that $\sum_{i=1}^{n} \frac{\|v_i\|_{p_i}^{p_i}}{p_i} \leqslant r_m$. Hence, one has

$$\varphi(r_m) = \inf_{u\in\Phi^{-1}(]-\infty,r_m[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r_m])}\Psi(v)\right) - \Psi(u)}{r_m - \Phi(u)} \leqslant \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_m])}\Psi(v)}{r_m}$$
$$\leqslant \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\xi_m^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\ldots,t_n)dx}{\frac{\xi_m^{p_n}}{k\prod_{i=1}^n p_i}}.$$

Therefore,

$$\gamma \leqslant \liminf_{m\to+\infty} \varphi(r_m) \leqslant \left(k\prod_{i=1}^n p_i\right)\Lambda_0 < +\infty. \tag{7}$$

Assumption (A2) together with (7), imply

$$\Lambda \subseteq \left]0, \frac{1}{\gamma}\right[.$$

Fix $\lambda \in \Lambda$. We conclude from (7) that condition (b) of Theorem 2.1 can be applied, and either I_λ has a global minimum or there exists a sequence $\{u_m = (u_{1m},\ldots,u_{nm})\}$ of weak solutions of the system (1) such that $\lim_{m\to\infty}\|(u_{1m},\ldots,u_{nm})\| = +\infty$.

The other step is to show that the functional I_λ has no global minimum. For fixed λ, we claim that the functional $\Phi - \lambda\Psi$ is unbounded from below. Indeed, since

$$\frac{1}{\lambda} < kL\prod_{i=1}^n p_i\Lambda_1,$$

we can choose a sequence $\{d_m\}$ of positive numbers and a positive constant τ such that $d_m \to +\infty$ as $m \to \infty$ and

$$\frac{1}{\lambda} < \tau < kLd_m^{-p_n}\prod_{i=1}^n p_i \int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,d_m)dx, \tag{8}$$

for each $m\in\mathbb{N}$ large enough. Let $\{w_m\}$ be a sequence in X defined by $w_m(x) = (0,\ldots,0,w_{nm}(x))$ such that

$$w_{nm}(x) = \begin{cases} 0 & \text{if } x\in\overline{\Omega}\setminus S(x^0,s),\\ d_m & \text{if } x\in S\left(x^0,\frac{s}{2}\right),\\ d_m\left(\frac{4}{s^3}l^3 - \frac{12}{s^2}l^2 + \frac{9}{s}l - 1\right) & \text{if } x\in S(x^0,s)\setminus S\left(x^0,\frac{s}{2}\right). \end{cases} \tag{9}$$

We have

$$\frac{\partial w_{nm}(x)}{\partial x_i} = \begin{cases} 0 & \text{if } x\in\overline{\Omega}\setminus S(x^0,s)\cup S\left(x^0,\frac{s}{2}\right),\\ d_m\left(\frac{12l(x_i-x_i^0)}{s^3} - \frac{24(x_i-x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i-x_i^0)}{l}\right) & \text{if } x\in S(x^0,s)\setminus S\left(x^0,\frac{s}{2}\right) \end{cases}$$

and

$$\frac{\partial^2 w_{nm}(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x\in\overline{\Omega}\setminus S(x^0,s)\cup S\left(x^0,\frac{s}{2}\right),\\ d_m\left(\frac{12}{s^3}\frac{(x_i-x_i^0)^2+l^2}{l} - \frac{24}{s^2} + \frac{9}{s}\frac{l^2-(x_i-x_i^0)^2}{l^3}\right) & \text{if } x\in S(x^0,s)\setminus S\left(x^0,\frac{s}{2}\right), \end{cases}$$

and so that

$$\sum_{i=1}^{N}\frac{\partial^2 w_{nm}(x)}{\partial x_i^2}=\begin{cases}0 & \text{if } x\in\overline{\Omega}\setminus S(x^0,s)\cup S\left(x^0,\frac{s}{2}\right),\\ d_m\left(\frac{12l(N+1)}{s^3}-\frac{24N}{s^2}+\frac{9}{s}\,\frac{N-1}{l^3}\right) & \text{if } x\in S(x^0,s)\setminus S\left(x^0,\frac{s}{2}\right).\end{cases}$$

For any fixed $m\in\mathbb{N}$, it is easy to see that $w_m=(0,\ldots,0,w_{nm})\in X$ and, in particular, since

$$\int_\Omega|\Delta w_n(x)|^{p_n}dx=d_m^{p_n}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\left|\frac{12(N+1)}{s^3}\xi-\frac{24N}{s^2}+\frac{9(N-1)}{s}\frac{1}{\xi^3}\right|^{p_n}\xi^{N-1}d\xi,$$

$$\int_\Omega|\nabla w_n(x)|^{p_n}dx=\int_{S(x^0,s)\setminus S\left(x^0,\frac{s}{2}\right)}\left[\sum_{i=1}^{N}d_m^2\left(\frac{12l(x_i-x_i^0)}{s^3}-\frac{24(x_i-x_i^0)}{s^2}+\frac{9}{s}\frac{(x_i-x_i^0)}{l}\right)^2\right]^{\frac{p_n}{2}}dx$$

$$=d_m^{p_n}\int_{S(x^0,s)\setminus S(x^0,\frac{s}{2})}\left[\sum_{i=1}^{N}\left(\frac{12l(x_i-x_i^0)}{s^3}-\frac{24(x_i-x_i^0)}{s^2}+\frac{9}{s}\frac{(x_i-x_i^0)}{l}\right)^2\right]^{\frac{p_n}{2}}dx$$

and

$$\int_\Omega|w_n(x)|^{p_n}dx=d_m^{p_n}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{\left(\frac{s}{2}\right)^N}{N}+\int_{\frac{s}{2}}^{s}\left|\frac{4}{s^3}\xi^3-\frac{12}{s^2}\xi^2+\frac{9}{s}\xi-1\right|^{p_n}\xi^{N-1}d\xi\right),$$

we observe that

$$\|w_{nm}\|_{p_n}^{p_n}=(\sigma_n+\alpha_n\theta_n+\beta_n\varrho_n)d_m^{p_n}$$

and so

$$\Phi(w_m)=\frac{(\sigma_n+\alpha_n\theta_n+\beta_n\varrho_n)d_m^{p_n}}{p_n}=\frac{d_m^{p_n}}{kL\prod_{i=1}^{n}p_i}. \tag{10}$$

On the other hand, bearing Assumption (A1) in mind, from (6) one has

$$\Psi(w_m)\geqslant\int_{S(x^0,\frac{s}{2})}F(x,0,\ldots,0,d_m)dx. \tag{11}$$

So, in view of (10), (11) and (8), we obtain

$$I_\lambda(w_m)\leqslant\frac{d_m^{p_n}}{kL\prod_{i=1}^{n}p_i}-\lambda\int_{S(x^0,\frac{s}{2})}F(x,0,\ldots,0,d_m)dx<\frac{d_m^{p_n}}{kL\prod_{i=1}^{n}p_i}(1-\lambda\tau)$$

for every $m\in\mathbb{N}$ large enough. Hence, our claim holds true; it follows that I_λ has no global minimum. Therefore, Theorem 2.1 assures that there is a sequence $\{u_m=(u_{1m},\ldots,u_{nm})\}\subset X$ of critical points of I_λ such that $\lim_{m\to\infty}\|(u_{1m},\ldots,u_{nm})\|=+\infty$, and we have the conclusion. $\square$

We present an example to illustrate Theorem 3.1.

Example 3.1. Let $\Omega\subset\mathbb{R}^2$ be a non-empty open set with a smooth boundary $\partial\Omega$ and consider the increasing sequence of positive real numbers given by

$$a_1=2,\ a_{m+1}=m!(a_m)^{\frac{5}{4}}+2$$

for every $m \geqslant 1$. Define the function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y,t_1,t_2) = \begin{cases} (a_{m+1})^5 e^{x^2+y^2-\frac{1}{1-(t_1-a_{m+1})^2-(t_2-a_{m+1})^2}+1} & \text{if } (x,y,t_1,t_2) \in \Omega \times \bigcup_{m\geqslant 1} S((a_{m+1},a_{m+1}),1), \\ 0 & \text{otherwise,} \end{cases}$$

where $S((a_{m+1},a_{m+1}),1)$ denotes the open unit ball with center at (a_{m+1},a_{m+1}). It is clear that F is a non-negative function such that $F(.,.,t_1,t_2)$ is continuous in Ω for all $(t_1,t_2) \in \mathbb{R}^2$, $F(x,y,.,.)$ is C^1 in $\mathbb{R}^2$ for every $(x,y) \in \Omega$, $F(x,y,0,0) = 0$ for all $(x,y) \in \Omega$ and for every $\varrho > 0$,

$$\sup_{|(t_1,t_2)|\leqslant\varrho} (|F_{t_1}(x,y,t_1,t_2)| + |F_{t_2}(x,y,t_1,t_2)|) \in L^1(\Omega).$$

Now, for every $m \in \mathbb{N}$, one has

$$\max_{(t_1,t_2)\in S((a_{m+1},a_{m+1}),1)} F(x,y,t_1,t_2) = F(x,y,0,a_{m+1}) = (a_{m+1})^5 e^{x^2+y^2}.$$

Since

$$\lim_{m\to+\infty} \frac{F(x,y,0,a_{m+1})}{(a_{m+1})^4} = +\infty,$$

we see that

$$\limsup_{\xi\to+\infty} \xi^{-4} F(x,y,0,\xi)dx = +\infty.$$

Moreover, by choosing $\xi_m = \sqrt[4]{4}(a_{m+1} - 1)$ for every $m \in \mathbb{N}$, one has

$$\sup_{(t_1,t_2)\in K\left(\frac{\xi^4}{16}\right)} F(x,y,t_1,t_2) = (a_m)^5 e^{x^2+y^2}, \quad \forall m \in \mathbb{N}.$$

Then,

$$\lim_{m\to+\infty} \frac{\sup_{(t_1,t_2)\in K\left(\frac{\xi^4}{16}\right)} F(x,y,t_1,t_2)}{4(a_{m+1} - 1)^4} = 0$$

and so

$$\liminf_{\xi\to+\infty} \xi^{-4} \sup_{(t_1,t_2)\in K\left(\frac{\xi^4}{16}\right)} F(x,y,t_1,t_2) = 0.$$

Therefore,

$$0 = \liminf_{\xi\to+\infty} \xi^{-4} \int_\Omega \sup_{(t_1,t_2)\in K\left(\frac{\xi^4}{16}\right)} F(x,y,t_1,t_2)dx < L \limsup_{\xi\to+\infty} \xi^{-4} \int_{S\left(x^0,\frac{\xi}{2}\right)} F(x,y,0,\xi)dx = +\infty.$$

Hence, all the assumptions of Theorem 3.1 are satisfied. So, Theorem 3.1 is applicable to the system

$$\begin{cases} \Delta(|\Delta u|^2\Delta u) = \lambda F_u(x,y,u,v) & \text{in } \Omega, \\ \Delta(|\Delta v|^2\Delta v) = \lambda F_v(x,y,u,v) & \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases}$$

for every $\lambda \in]0, +\infty[$.

Remark 3.1. Arguing as in [4, Remark 3.3] we notice that instead of Assumption (A2) in Theorem 3.1, we are allowed to suppose the following more general condition

(A3) there exist two sequences $\{a_m\}$ and $\{b_m\}$ with $|a_m| < \sqrt[p_n]{L} b_m$ for every $m \in \mathbb{N}$ and $\lim_{m\to+\infty} b_m = +\infty$ such that

$$\lim_{m\to+\infty}\left(b_m^{p_n} - \frac{a_m^{p_n}}{L}\right)^{-1}\int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{b_m^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\ldots,t_n)dx - \int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,a_m)dx$$

$$< L \limsup_{\xi\to+\infty} \xi^{-p_n}\int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,\xi)dx,$$

where L is given by (4) and $K\left(\frac{b_m^{p_n}}{\prod_{i=1}^n p_i}\right) = \left\{(t_1,\ldots,t_n)|\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leqslant \frac{b_m^{p_n}}{\prod_{i=1}^n p_i}\right\}$ (see (2)).

Obviously, from (A3) we obtain (A2), by choosing $a_m = 0$ for all $m \in \mathbb{N}$. Moreover, if we assume (A3) instead of (A2) and set $r_m = \frac{b_m^{p_n}}{k\prod_{i=1}^n p_i}$ for all $m \in \mathbb{N}$, by the same argument as in Theorem 3.1, we obtain

$$\begin{aligned}\varphi(r_m) &= \inf_{u\in\Phi^{-1}(]-\infty,r_m[)} \frac{\sup_{(v\in\Phi^{-1}(]-\infty,r_m])} \Psi(v)) - \Psi(u)}{r_m - \Phi(u)} \\ &\leqslant \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_m])} \Psi(v) - \int_\Omega F(x,w_{1m}(x),\ldots,w_{nm}(x))dx}{r_m - \sum_{i=1}^n \frac{\|w_{im}\|_{p_i}^{p_i}}{p_i}} \\ &\leqslant \left(k\prod_{i=1}^n p_i\right)\frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{b_m^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\ldots,t_n)dx - \int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,a_m)dx}{b_m^{p_n} - \frac{a_m^{p_n}}{L}}\end{aligned}$$

where $w_m(x) = (w_{1m}(x),\ldots w_{nm}(x))$ defined by $w_{1m}(x) = \ldots = w_{(n-1)m}(x) = 0$ for all $x \in \Omega$ and $w_{nm}(x)$ as given in (9) with a_m instead of d_m. We have the same conclusion as in Theorem 3.1 with Λ replaced by

$$\Lambda' :=]\widetilde{\Lambda_1}, \widetilde{\Lambda_2}[$$

where

$$\widetilde{\Lambda_1} := \frac{\frac{1}{kL\prod_{i=1}^n p_i}}{\limsup_{\xi\to+\infty} \xi^{-p_n}\int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,\xi)dx}$$

and

$$\widetilde{\Lambda_2} := \frac{\frac{1}{k\prod_{i=1}^{n} p_i}}{\lim\limits_{m\to+\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{b_m^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x,t_1,\ldots,t_n)dx - \int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,a_m)dx}{b_m^{p_n} - \frac{a_m^{p_n}}{L}}}.$$

It is of interest to list some consequences of Theorem 3.1.

Corollary 3.2. *Assume that assumption (A1) in Theorem* 3.1 *holds. Suppose that*

(B1)
$$\liminf_{\xi\to+\infty} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x,t_1,\ldots,t_n)dx < \frac{1}{k\prod_{i=1}^{n} p_i};$$

(B2)
$$\limsup_{\xi\to+\infty} \xi^{-p_n} \int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,\xi)dx > \frac{1}{kL\prod_{i=1}^{n} p_i},$$

where L is given by (4) *and* $K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right) = \left\{(t_1,\ldots,t_n) | \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \leqslant \frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right\}$ (*see* (2)).

Then, the system

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) - \alpha_i \Delta_{p_i} u_i + \beta_i |u_i|^{p_i-2} u_i = F_{u_i}(x,u_1,\ldots,u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases}$$

for $1 \leqslant i \leqslant n$, *has an unbounded sequence of weak solutions in X.*

Corollary 3.3. *Let* $F:R^n \to R$ *be a* C^1 *function in* R^n *such that* $F(0,\ldots,0) = 0$. *Assume that*

(B3) $F(0,\ldots,0,t_n) \geqslant 0$ *for each* $t_n \in \mathbb{R}$;
(B4)
$$\liminf_{\xi\to+\infty} \xi^{-p_n} \max_{(t_1,\ldots,t_n)\in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(t_1,\ldots,t_n) < \frac{L\left(\frac{s}{2}\right)^N \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}}{m(\Omega)} \limsup_{\xi\to+\infty} \xi^{-p_n} F(0,\ldots,0,\xi),$$

where L is given by (4) *and* $K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right) = \left\{(t_1,\ldots,t_n) | \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \leqslant \frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right\}$ (*see* (2)).

Then, for each

$$\frac{\frac{1}{kL\prod_{i=1}^{n} p_i}}{\left(\frac{s}{2}\right)^N \frac{\pi^{N/2}}{\Gamma\left(1+\frac{N}{2}\right)} \limsup\limits_{\xi\to+\infty} \xi^{-p_n} F(0,\ldots,0,\xi)} < \lambda < \frac{\frac{1}{m(\Omega)k\prod_{i=1}^{n} p_i}}{\liminf\limits_{\xi\to+\infty} \xi^{-p_n} \sup\limits_{(t_1,\ldots,t_n)\in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(t_1,\ldots,t_n)}$$

the system

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) - \alpha_i \Delta_{p_i} u_i + \beta_i |u_i|^{p_i-2} u_i = \lambda F_{u_i}(u_1, \ldots, u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases}$$

for $1 \leqslant i \leqslant n$, has an unbounded sequence of weak solutions in X.

Proof. Set $F(x, t_1, \ldots, t_n) = F(t_1, \ldots, t_n)$ for all $x \in \Omega$ and $t_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n$. Since $m\left(S\left(x^0, \frac{s}{2}\right)\right) = \left(\frac{s}{2}\right)^N \frac{\pi^{N/2}}{\Gamma\left(1+\frac{N}{2}\right)}$, Theorem 3.1 ensures the conclusion. $\square$

Remark 3.2. Theorem 1.1 is an immediate consequence of Corollary 3.3.

By the same way as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.1 instead of (b), the following result holds.

Theorem 3.4. *Assume that Assumption (A1) in Theorem* 3.1 *holds. Suppose that*

(C1)

$$\liminf_{\xi \to 0^+} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x, t_1, \ldots, t_n) dx < L \limsup_{\xi \to 0^+} \xi^{-p_n} \int_{S(x^0, \frac{s}{2})} F(x, 0, \ldots, 0, \xi) dx,$$

where L is given by (4) *and* $K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right) = \left\{(t_1, \ldots, t_n) | \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leqslant \frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right\}$ *(see* (2)*).*

Then, for each

$$\lambda \in \Lambda'' := \left] \frac{\frac{1}{kL\prod_{i=1}^n p_i}}{\limsup_{\xi \to 0^+} \xi^{-p_n} \int_{S(x^0, \frac{s}{2})} F(x, 0, \ldots, 0, \xi) dx}, \frac{\frac{1}{k\prod_{i=1}^n p_i}}{\liminf_{\xi \to 0^+} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x, t_1, \ldots, t_n) dx} \right[,$$

the system (1) *has a sequence of weak solutions, which strongly converges to 0 in X.*

Proof. We take Φ, Ψ and I_λ be as before. In a similar way as in the proof of Theorem 3.1 we verify that $\delta < +\infty$. Let $\{\xi_m\}$ be a sequence of positive numbers such that $\xi_m \to 0^+$ as $m \to +\infty$ and

$$\lim_{m \to \infty} \xi_m^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K\left(\frac{\xi_m^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x, t_1, \ldots, t_n) dx < +\infty.$$

Put $r_m = \frac{\xi_m^{p_n}}{k\prod_{i=1}^n p_i}$ for all $m \in \mathbb{N}$. Fix $\lambda \in \Lambda''$. We claim that the functional I_λ does not have a local minimum at zero. Let $\{d_m\}$ be a sequence of positive numbers and $\tau > 0$ such that $d_m \to 0^+$ as $m \to \infty$ and

$$\frac{1}{\lambda} < \tau < kLd_m^{-p_n} \prod_{i=1}^{n} p_i \int_{S(x^0, \frac{s}{2})} F(x, 0, \ldots, 0, d_m) dx, \tag{12}$$

for each $m \in \mathbb{N}$ large enough. Let $\{w_m\}$ be a sequence in X defined by $w_m(x) = (0, \ldots, 0, w_{nm}(x))$ such that w_{nm} is chosen as in (9). According to (10)–(12), we obtain

$$I_\lambda(w_m) \leqslant \frac{d_m^{p_n}}{kL\prod_{i=1}^{n} p_i} - \lambda \int_{S\left(x^0, \frac{s}{2}\right)} F(x, 0, \ldots, 0, d_m) dx < \frac{d_m^{p_n}}{kL\prod_{i=1}^{n} p_i} (1 - \lambda\tau) < 0,$$

for every $m \in \mathbb{N}$ large enough. Since $I_\lambda(0) = 0$, this confirms our claim. Hence, the part (c) of Theorem 2.1 ensures the existence of a sequence $\{u_m = (u_{1m}, \ldots, u_{nm})\}$ in X of critical points of I_λ such that $\|u_m\| \to 0$ as $m \to \infty$, the proof is complete. □

Remark 3.3. Employing Remark 3.1, we clearly observe that in Theorem 3.4, we are permitted to assume (A3) instead of (C1) by supposing $\lim_{k\to+\infty} b_k = 0$ instead of $\lim_{k\to+\infty} b_k = +\infty$ and replacing $\xi \to +\infty$ with $\xi \to 0^+$, and for every $\lambda \in \Lambda'$, in this case, the system (1) has a sequence of weak solutions, which strongly converges to 0 in X.

Here, we want to point out the following consequence when $n = 1$ and $\alpha_1 = \beta_1 = 0$, which follow from Theorem 3.1.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a L^2-Carathéodory function. Let F be the function defined by $F(x, t) = \int_0^t f(x, s) ds$ for each $(x, t) \in \Omega \times \mathbb{R}$.

Put

$$\sigma = \sigma(N, p, s) := \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \int_{\frac{s}{2}}^{s} \left| \frac{12(N+1)}{s^3} \xi - \frac{24N}{s^2} + \frac{9(N-1)}{s} \frac{1}{\xi^3} \right|^p \xi^{N-1} d\xi. \tag{13}$$

Theorem 3.5. *Let* $p > \max\left\{1, \frac{N}{2}\right\}$. *Suppose that*

(D1) $F(x,t) \geqslant 0$ *for each* $(x, t) \in \left(\Omega \setminus S\left(x^0, \frac{s}{2}\right)\right) \times \mathbb{R}$;

(D2)

$$\liminf_{\xi\to+\infty} \xi^{-p} \int_\Omega \sup_{|t|\leqslant\xi} F(x, t) dx < \frac{1}{k^p \sigma} \limsup_{\xi\to+\infty} \xi^{-p} \int_{S(x^0, \frac{s}{2})} F(x, \xi) dx,$$

where σ *is given by* (13) *and* $k = \sup_{u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{\left(\int_\Omega |\Delta u_i(x)|^p dx\right)^{1/p}}$.

Then, for each

$$\lambda \in \left] \frac{\frac{\sigma}{p}}{\limsup_{\xi\to+\infty} \xi^{-p} \int_{S(x^0, \frac{s}{2})} F(x, \xi) dx}, \frac{\frac{1}{pk^p}}{\liminf_{\xi\to+\infty} \xi^{-p} \int_\Omega \sup_{|t|\leqslant\xi} F(x, t) dx} \right[$$

the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

has an unbounded sequence of weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Remark 3.4. We also observe that in Theorem 3.5, by Theorem 3.4 and replacing $\xi \to +\infty$ with $\xi \to 0^+$, by the same reasoning, we have that for every $\lambda \in \Lambda'''$, in this case, the problem (14) has a sequence of weak solutions, which strongly converges to 0 in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

We illustrate the result as follows:

Example 3.2. Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x,t) = \begin{cases} h(x)t^2(3 - 3\sin(\ln(|t|)) - \cos(\ln(|t|))) & \text{if } (x,t) \in [a,b] \times (\mathbb{R} - \{0\}), \\ 0 & \text{if } (x,t) \in [a,b] \times \{0\} \end{cases}$$

where $h : [a,b] \to \mathbb{R}$ is a positive continuous function.

A direct calculation yields

$$F(x,t) = \begin{cases} h(x)t^3(1 - \sin(\ln(|t|))) & \text{if } (x,t) \in [a,b] \times (\mathbb{R} - \{0\}), \\ 0 & \text{if } (x,t) \in [a,b] \times \{0\}, \end{cases}$$

and so, $\liminf_{\xi \to 0^+} \xi^{-3} \int_a^b \sup_{|t| \leqslant \xi} F(x,t)dx = 0$ and $\limsup_{\xi \to 0^+} \xi^{-3} \int_{a+\alpha}^{b-\beta} F(x,\xi)dx = 2\int_{a+\alpha}^{b-\beta} h(x)dx$ for all positive α and β with $\beta + \alpha < b - a$. Hence, with $p = 3$, taking Remark 3.4 into account, the problem (14), in this case, for every

$$\frac{\sigma(1,3,s)}{6\int_{a+\alpha}^{b-\beta} h(x)dx} < \lambda < +\infty$$

where $0 < s < (b-a)/2$, has a sequence of weak solutions, which strongly converges to 0 in $W^{2,3}([a,b]) \cap W_0^{1,3}([a,b])$.

Finally, in this section, we give the following consequence.

Corollary 3.6. *Let* $p > \max\{1, \frac{N}{2}\}$. *Let* $g_1 : \Omega \to \mathbb{R}$ *be a non-negative continuous function, and set* $G_1(t) = \int_0^t g_1(\xi)d\xi$ *for all* $t \in \mathbb{R}$ *such that*

(E1) $$\liminf_{\xi \to +\infty} \xi^{-p} G_1(\xi) < +\infty;$$

(E2) $$\limsup_{\xi \to +\infty} \xi^{-p} G_1(\xi) = +\infty.$$

Then, for every $\alpha_i \in L^1(\Omega)$ *for* $1 \leqslant i \leqslant n$, *with* $\min_{x\in\Omega}\{\alpha_i(x); 1 \leqslant i \leqslant n\} \geqslant 0$ *and with* $\alpha_1 \neq 0$, *and for every non-negative continuous* $g_i : \mathbb{R} \to \mathbb{R}$ *for* $2 \leqslant i \leqslant n$ *satisfying*

$$\max\left\{\sup_{\xi\in\mathbb{R}} G_i(\xi)dt;\ 2\leqslant i\leqslant n\right\}\leqslant 0$$

and

$$\min\left\{\liminf_{\xi\to+\infty}\frac{G_i(\xi)}{\xi^p};2\leqslant i\leqslant n\right\}>-\infty$$

where $G_i(t)=\int_0^t g_i(\xi)d\xi$ *for all* $t\in\mathbb{R}$ *for* $2\leqslant i\leqslant n$, *for each*

$$\lambda\in\left]0,\ \frac{1}{(pk^p\int_\Omega\alpha_1(x)dx)\ \liminf_{\xi\to+\infty}\ \xi^{-p}G_1(\xi)}\right[$$

where $k=\sup_{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u(x)|}{(\int_\Omega|\Delta u_i(x)|^p dx)^{1/p}}$, *the problem*

$$\begin{cases}\Delta(|\Delta u|^{p-2}\Delta u)=\lambda\sum_{i=1}^{n}\alpha_i(x)g_i(u) & \text{in }\Omega,\\ u=\Delta u=0 & \text{on }\partial\Omega\end{cases}\tag{15}$$

has an unbounded sequence of weak solutions in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.

Proof. Set $f(x,t)=\sum_{i=1}^n\alpha_i(x)g_i(t)$ for all $(x,t)\in\Omega\times\mathbb{R}$. The assumption (E2) in conjunction with the condition

$$\min\left\{\liminf_{\xi\to+\infty}\ \xi^{-p}G_i(\xi);\ 2\leqslant i\leqslant n\right\}>-\infty$$

yields

$$\limsup_{\xi\to+\infty}\ \xi^{-p}\int_{S(x^0,\frac{\xi}{2})}F(x,\xi)dx=\limsup_{\xi\to+\infty}\ \xi^{-p}\sum_{i=1}^{n}\left(G_i(\xi)\int_{S(x^0,\frac{\xi}{2})}\alpha_i(x)dx\right)=+\infty.$$

Moreover, the assumption (E1) together with the condition

$$\max\left\{\sup_{\xi\in\mathbb{R}} G_i(\xi)dt;\ 2\leqslant i\leqslant n\right\}\leqslant 0$$

implies

$$\liminf_{\xi\to+\infty}\xi^{-p}\int_\Omega\sup_{|t|\leqslant\xi}F(x,t)dx\leqslant\left(\int_\Omega\alpha_1(x)dx\right)\liminf_{\xi\to+\infty}\ \xi^{-p}G_1(\xi)<+\infty.$$

Hence, applying Theorem 3.5 we have the result.

Remark 3.5. We point out that by using Corollary 3.6, the problem (15) with $\alpha_1=1$, g_1 be the function h as chosen in [2, Example 1] and $\alpha_2=\ldots=\alpha_n=0$, with $\alpha_1=1$, g_1 be the function f as given in [7, Example 4.1] and $\alpha_2=\ldots=\alpha_n=0$, as well as with $\alpha_1=1$, g_1 be the function f as given in [7, Example 4.2] and $\alpha_2=\ldots=\alpha_n=0$, for every $\lambda\in]0,+\infty[$, has an unbounded sequence of weak solutions in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.

Remark 3.6. In Corollary 3.6, replacing $\xi \to +\infty$ with $\xi \to 0^+$, by the same reasoning, we have that for every $\lambda \in]0, 1/(pk^p \int_\Omega \alpha_1(x)dx) \liminf_{\xi\to 0^+} \xi^{-p} G_1(\xi)[$, the problem (15) has a sequence of weak solutions, which strongly converges to 0 in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Remark 3.7. The statement of the results mainly depends upon the choice of the test function w_m. With our choice of w_m as given in (9), we have the present structure of the results. So, other candidates for the test function w_m can be considered to have other versions of the statement of the results, for instance, see [14].

4. A remarkable consequence

This section is concerned, as a consequence of Theorem 3.1, with the existence of infinitely many solutions for the nonlinear elliptic system of n doubly eigenvalue fourth order partial differential equations under Navier boundary conditions

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) - \alpha_i \Delta_{p_i} u_i + \beta_i |u_i|^{p_i-2} u_i = \lambda H_{u_i}(x, u_1, \dots, u_n) + \mu G_{u_i}(x, u_1, \dots, u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{16}$$

for $1 \leqslant i \leqslant n, p_i > \max\left\{1, \frac{N}{2}\right\}, \alpha_i$ and β_i for $1 \leqslant i \leqslant n$ are positive constants, $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, λ is a positive parameter, μ is a non-negative parameter, $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1,t_2, \cdots ,t_n) \mapsto H(x,t_1,t_2, \cdots ,t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in \Omega$, H_{t_i} is continuous in $\Omega \times \mathbb{R}^n$ for $i = 1, \cdots ,n$, and $H(x,0, \dots ,0) = 0$ for all $x \in \Omega$, $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $G(.,t_1, \dots ,t_n)$ is measurable in Ω for all $(t_1, \dots ,t_n) \in R^n$ and $G(x,., \dots ,.)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$ and satisfies the condition

$$\sup_{|(t_1,\dots,t_n)|\leqslant s} \sum_{i=1}^{n} |G_{t_i}(x, t_1, \dots, t_n)| \leqslant m_s(x), \tag{17}$$

for all $s > 0$ and some $m_s \in L^1$ with $G(.,0, \dots ,0) \in L^1$, and H_{u_i} and G_{u_i} denote the partial derivative of H and G with respect to u_i for $1 \leqslant i \leqslant n$, respectively. Our objective here is to present conditions on H which imply the existence of infinitely many solutions to the system (16).

We state the main result of this section as follows.

Theorem 4.1. *Assume that*

(F1) $H(x,0, \dots ,0,t_n) \geqslant 0$ *for each* $x \in \Omega \setminus S(x^0, \frac{s}{2}), t_n \in \mathbb{R}$;
(F2)

$$\liminf_{\xi\to+\infty} \xi^{-p_n} \int_\Omega \sup_{(t_1,\dots,t_n)\in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} H(x, t_1, \dots, t_n)dx$$
$$< L \limsup_{\xi\to+\infty} \xi^{-p_n} \int_{S(x^0,\frac{s}{2})} H(x, 0, \dots, 0, \xi)dx,$$

where L is given by (4) *and* $K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right) = \left\{(t_1, \ldots, t_n) | \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \leqslant \frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right\}$ (see (2)).

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ *where*

$$\lambda_1 := \frac{\frac{1}{kL\prod_{i=1}^{n} p_i}}{\limsup\limits_{\xi \to +\infty} \xi^{-p_n} \int_{S(x^0, \frac{\xi}{2})} H(x, 0, \ldots, 0, \xi) dx}$$

and

$$\lambda_2 := \frac{\frac{1}{k\prod_{i=1}^{n} p_i}}{\liminf\limits_{\xi \to +\infty} \xi^{-p_n} \int_{\Omega} \sup\limits_{(t_1, \ldots, t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} H(x, t_1, \ldots, t_n) dx},$$

for every nonnegative arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *which is measurable in* Ω *and* C^1 *in* $\mathbb{R}^n$ *satisfying the condition* (17) *such that*

$$G_\infty := \lim_{\xi \to +\infty} \xi^{-p_n} \int_{\Omega} \sup_{(t_1, \ldots, t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} G(x, t_1, \ldots, t_n) dx < +\infty \tag{18}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ *where*

$$\mu_{G,\lambda} := \frac{1}{G_\infty k \prod_{i=1}^{n} p_i} \left(1 - \lambda k \prod_{i=1}^{n} p_i \liminf_{\xi \to +\infty} \xi^{-p_n} \int_{\Omega} \sup_{(t_1, \ldots, t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} H(x, t_1, \ldots, t_n) dx\right),$$

the system (16) *has an unbounded sequence of weak solutions in X.*

Proof. Fix $\bar{\lambda} \in]\lambda_1, \lambda_2[$ and let G be a function satisfying the condition (17). If $\mu = 0$, Theorem 4.1 gives back to Theorem 3.1. We assume $\mu > 0$. Since, $\bar{\lambda} < \lambda_2$, one has

$$\mu_{G,\bar{\lambda}} := \frac{1}{G_\infty k \prod_{i=1}^{n} p_i} \left(1 - \bar{\lambda} k \prod_{i=1}^{n} p_i \liminf_{\xi \to +\infty} \xi^{-p_n} \int_{\Omega} \sup_{(t_1, \ldots, t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^{n} p_i}\right)} H(x, t_1, \ldots, t_n) dx\right) > 0.$$

Fix $\bar{\mu} \in]0, \mu_{G,\bar{\lambda}}[$ and set $v_1 := \lambda_1$ and $v_2 := \lambda_2 / (1 + k\prod_{i=1}^{n} p_i \frac{\bar{\mu}}{\bar{\lambda}} \lambda_2 G_\infty)$. If $G_\infty = 0$, clearly, $v_1 = \lambda_1$, $v_2 = \lambda_2$ and $\bar{\lambda} \in]v_1, v_2[$. If $G_\infty \neq 0$, since $\bar{\mu} < \mu_{G,\bar{\lambda}}$, we obtain

$$\frac{\bar{\lambda}}{\lambda_2} + k \prod_{i=1}^{n} p_i \bar{\mu} G_\infty < 1,$$

and so

$$\frac{\lambda_2}{1 + k\prod_{i=1}^{n} p_i \frac{\bar{\mu}}{\bar{\lambda}} \lambda_2 G_\infty} > \bar{\lambda},$$

namely, $\bar{\lambda} < v_2$. Hence, bearing in mind that $\bar{\lambda} > \lambda_1 = v_1$, one has $\bar{\lambda} \in]v_1, v_2[$.

Now, set

$$F(x, \xi_1, \ldots, \xi_n) = H(x, \xi_1, \ldots, \xi_n) + \frac{\bar{\mu}}{\bar{\lambda}} G(x, \xi_1, \ldots, \xi_n)$$

for all $x \in \Omega$ and $\xi_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n$. Then

$$\begin{aligned} &\xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i})} F(x, t_1, \ldots, t_n) dx \\ &= \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i})} \left[H(x, t_1, \ldots, t_n) + \frac{\bar{\mu}}{\bar{\lambda}} G(x, \xi_1, \ldots, \xi_n) \right] dx \\ &\leqslant \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i})} H(x, t_1, \ldots, t_n) dx + \frac{\bar{\mu}}{\bar{\lambda}} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right)} G(x, t_1, \ldots, t_n) dx. \end{aligned}$$

Taking (18) into account, we get

$$\begin{aligned} &\liminf_{\xi \to +\infty} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i})} F(x, t_1, \ldots, t_n) dx \\ &\quad \leqslant \liminf_{\xi \to +\infty} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i})} H(x, t_1, \ldots, t_n) dx + \frac{\bar{\mu}}{\bar{\lambda}} G_\infty. \end{aligned} \tag{19}$$

Moreover, since G is nonnegative, we obtain

$$\limsup_{\xi \to +\infty} \xi^{-p_n} \int_{S(x^0, \frac{\varsigma}{2})} F(x, 0, \ldots, 0, \xi) dx \geqslant \limsup_{\xi \to +\infty} \xi^{-p_n} \int_{S(x^0, \frac{\varsigma}{2})} H(x, 0, \ldots, 0, \xi) dx. \tag{20}$$

Therefore, (19) together with (20), yields

$$\bar{\lambda} \in]v_1, v_2[\subseteq]\lambda_1, \lambda_2[. \tag{21}$$

So, from (21), Assumption (A2) is verified. Hence, applying Theorem 3.1, we have the conclusion.

Remark 4.1. Under the conditions

$$\liminf_{\xi \to +\infty} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i})} H(x, t_1, \ldots, t_n) dx = 0$$

and

$$\limsup_{\xi \to +\infty} \xi^{-p_n} \int_{S(x^0, \frac{\varsigma}{2})} H(x, 0, \ldots, 0, \xi) dx = +\infty,$$

Theorem 4.1 concludes that for every $\lambda > 0$ and for each $\mu \in [0, 1/kG_\infty \prod_{i=1}^n p_i[$ the system (16) admits infinitely many weak solutions in X. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu > 0$.

Remark 4.2. As we mentioned in the last section, in Theorem 4.1 also, replacing $\xi \to +\infty$ by $\xi \to 0^+$, by the same reasoning, we have that for every $\lambda \in]\lambda_1, \lambda_2[$, in this case, and for every nonnegative arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ which is measurable in Ω and C^1 in $\mathbb{R}^n$ satisfying the condition (17) such that

$$G_0 := \lim_{\xi \to 0^+} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right)} G(x, t_1, \ldots, t_n) dx < +\infty$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ where

$$\mu_{G,\lambda} := \frac{1}{G_0 k \prod_{i=1}^n p_i} \left(1 - \lambda k \prod_{i=1}^n p_i \liminf_{\xi \to 0^+} \xi^{-p_n} \int_\Omega \sup_{(t_1,\ldots,t_n) \in K\left(\frac{\xi^{p_n}}{\prod_{i=1}^n p_i}\right)} H(x, t_1, \ldots, t_n) dx \right),$$

the system (16) has a sequence of weak solutions, which strongly converges to 0 in X.

We present two examples to illustrate Theorem 4.1 as follows.

Example 4.1. Let $\Omega \subset \mathbb{R}^3$ be a non-empty open set with a smooth boundary $\partial\Omega$ and let $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function which is measurable in Ω and C^1 in $\mathbb{R}^3$ satisfying the condition

$$\sup_{|(t_1,t_2,t_3)| \leqslant s} \sum_{i=1}^3 |G_{t_i}(x, y, z, t_1, t_2, t_3)| \leqslant m_s(x, y, z)$$

for all $s > 0$ and some $m_s \in L^1$ with $G(.,.,.,0, \ldots, 0) \in L^1$ such that

$$\lim_{\xi \to +\infty} \xi^{-5} \int_\Omega \sup_{(t_1,t_2,t_2) \in K\left(\frac{\xi^5}{125}\right)} G(x, y, z, t_1, t_2, t_3) dx < +\infty.$$

Consider the increasing sequence of positive real numbers given by

$$a_1 = 2, \ a_{m+1} = m!(a_m)^{\frac{6}{5}} + 2$$

for every $m \geqslant 1$. Define the function $H : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ by

$$H(x, y, z, t_1, t_2, t_3) = \begin{cases} (a_{m+1})^6 e^{x^2+y^2+z^2 - \frac{1}{1 - \sum_{i=1}^3 (t_i - a_{m+1})^2} + 1} & \text{if } \Omega \times \bigcup_{m \geqslant 1} S((a_{m+1}, a_{m+1}, a_{m+1}), 1), \\ 0 & \text{otherwise,} \end{cases}$$

where $S((a_{m+1}, a_{m+1}, a_{m+1}), 1)$ denotes the open unit ball with center at $(a_{m+1}, a_{m+1}, a_{m+1})$. It is clear that F is a non-negative function such that $H(.,.,.,t_1, t_2, t_3)$ is continuous in Ω for all $(t_1, t_2, t_3) \in \mathbb{R}^3$, $H(x, y, z, ., ., .)$ is C^1 in $\mathbb{R}^3$ for every $(x, y, z) \in \Omega$, $H(x, y, z, 0, 0, 0) = 0$ for all $(x, y, z) \in \Omega$ and for every $\varrho > 0$,

$$\sup_{|(t_1,t_2,t_3)|\leqslant\varrho}(|H_{t_1}(x,y,z,t_1,t_2,t_3)|+|H_{t_2}(x,y,z,t_1,t_2,t_3)|+|H_{t_3}(x,y,z,t_1,t_2,t_3)|)\in L^1(\Omega).$$

Now, for every $m\in\mathbb{N}$, the restriction $H(x,y,z,t_1,t_2,t_3)|_{S((a_{m+1},a_{m+1},a_{m+1}),1)}$ attains its maximum in (0,0,a_{m+1}) and one has $F(x,y,z,0,0,a_{m+1})=(a_{m+1})^6e^{x^2+y^2+z^2}$. Since

$$\lim_{m\to+\infty}\frac{H(x,y,z,0,0,a_{m+1})}{(a_{m+1})^5}=+\infty,$$

we see that

$$\limsup_{\xi\to+\infty}\ \xi^{-5}H(x,y,z,0,0,\xi)dx=+\infty.$$

Moreover, by choosing $\xi_m=\sqrt[5]{25}(a_{m+1}-1)$ for every $m\in\mathbb{N}$, one has

$$\sup_{(t_1,t_2,t_3)\in K(\frac{\xi^5}{125})}H(x,y,z,t_1,t_2,t_3)=(a_m)^6e^{x^2+y^2+z^2},\quad\forall m\in\mathbb{N}.$$

Then,

$$\lim_{m\to+\infty}\frac{\sup_{(t_1,t_2,t_3)\in K\left(\frac{\xi^5}{125}\right)}H(x,y,z,t_1,t_2,t_3)}{25(a_{m+1}-1)^5}=0,$$

whereupon

$$\liminf_{\xi\to+\infty}\xi^{-5}\sup_{(t_1,t_2,t_3)\in K\left(\frac{\xi^5}{125}\right)}H(x,,y,z,t_1,t_2,t_3)=0.$$

Therefore,

$$0=\liminf_{\xi\to+\infty}\xi^{-5}\int_\Omega\sup_{(t_1,t_2)\in K\left(\frac{\xi^5}{125}\right)}H(x,y,z,t_1,t_2,t_3)dx$$

$$<L\limsup_{\xi\to+\infty}\ \xi^{-5}\int_{S(x^0,\frac{\xi}{2})}H(x,y,z,0,0,\xi)dx=+\infty.$$

Hence, all the assumptions of Theorem 4.1 are satisfied. So, Theorem 4.1 is applicable to the system

$$\begin{cases}\Delta(|\Delta u|^3\Delta u)-\alpha_1\Delta_5u+\beta_1|u|^3u=\lambda H_u(x,y,z,u,v,w)+\mu G_u(x,y,z,u,v,w) & \text{in }\Omega,\\ \Delta(|\Delta v|^3\Delta v)-\alpha_2\Delta_5v+\beta_2|v|^3v=\lambda H_v(x,y,z,u,v,w)+\mu G_v(x,y,z,u,v,w) & \text{in }\Omega,\\ \Delta(|\Delta w|^3\Delta w)-\alpha_3\Delta_5w+\beta_3|w|^3w=\lambda H_w(x,y,z,u,v,w)+\mu G_w(x,y,z,u,v,w) & \text{in }\Omega,\\ u=\Delta u=v=\Delta v=w=\Delta w=0 & \text{on }\partial\Omega,\end{cases}\tag{22}$$

where α_i,β_i for $i=1,2,3$ are positive constants, for every $(\lambda,\mu)\in]0,+\infty[\times[0,\frac{1}{125kG_\infty}[$.

For instance, choosing

$$G(x,y,z,t_1,t_2,t_3)=\frac{e^{-t_1^+}(t_1^+)^\alpha+e^{-t_2^+}(t_2^+)^\beta+e^{-t_3^+}(t_1^+)^\gamma}{1+\sqrt{x^2+y^2+z^2}},$$

where $t_i^+=\max\{t_i,0\}$ for $i=1,2,3,\alpha,\beta$ and γ are positive real numbers, for all $(x,y,z,t_1,t_2,t_3)\in\Omega\times\mathbb{R}^3$, the system (22), in this case, for every

$(\lambda, \mu) \in]0, +\infty[\times [0, +\infty[$, has an unbounded sequence of weak solutions in $\left(W^{2,5}(\Omega) \cap W_0^{1,5}(\Omega)\right)^3$.

Example 4.2. Let α and β be two positive constants with $\alpha + \beta < 1$, and let f be as given in Example 3.2. By the same argument as in Example 3.2, taking Remark 4.2 into account, we observe that the problem

$$\begin{cases} (|u''|u'')'' = \lambda f(x, u) + \mu g(x, u) & \text{in } (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

where $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is an arbitrary nonnegative L^2-Carathéodory function such that

$$g_0 := \lim_{\xi \to 0^+} \xi^{-3} \int_0^1 \sup_{|t| \leqslant \xi} \int_0^t g(x, \delta) d\delta dx < +\infty$$

for every

$$(\lambda, \mu) \in \left] \frac{\sigma(1, 3, s)}{6 \int_\alpha^{1-\beta} h(x) dx}, +\infty\right[\times \left[0, \frac{8}{g_0}\right[,$$

where $0 < s < \frac{1}{2}$ (we recall that, if $p > 1$, for every $u \in W^{2,p}([0, 1]) \cap W_0^{1,p}([0, 1])$,

$$\max_{x \in [0,1]} |u(x)| \leqslant \frac{1}{2} p^{-\frac{1}{p}} \left(\int_0^1 |u''(x)|^p dx \right)^{\frac{1}{p}}$$

for details see [26, Lemma 2], has a sequence of weak solutions, which strongly converges to 0 in $W^{2,3}([0, 1]) \cap W_0^{1,3}([0, 1])$.

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