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Implicit iterative method for approximating a common solution of split equilibrium problem and fixed point problem for a nonexpansive semigroup

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Abstract. In this paper, we introduce and study an implicit iterative method to approximate a common solution of split equilibrium problem and fixed point problem for a nonexpansive semigroup in real Hilbert spaces. Further, we prove that the nets generated by the implicit iterative method converge strongly to the common solution of split equilibrium problem and fixed point problem for a nonexpansive semigroup. This common solution is the unique solution of a variational inequality problem and is the optimality condition for a minimization problem. Furthermore, we justify our main result through a numerical example. The results presented in this paper extend and generalize the corresponding results given by Plubtieng and Punpaeng [S. Plubtieng, R. Punpaeng, Fixed point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces, Math. Comput. Model. 48 (2008) 279–286] and Cianciaruso et al. [F. Cianciaruso, G. Marino, L. Muglia, Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert space, J. Optim. Theory Appl. 146 (2010) 491–509].

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1. Introduction

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively.

In 1994, Blum and Oettli [2] introduced and studied the following equilibrium problem (in short, EP): Find $x \in C$ such that

$$F(x, y) \geqslant 0, \quad \forall y \in C, \tag{1.1}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction.

The EP (1.1) includes variational inequality problems, optimization problems, Nash equilibrium problems, saddle point problems, fixed point problems, and complementary problems as special cases. In other words, EP (1.1) is an unify model for several problems arising in science, engineering, optimization, economics, etc.

In the last two decades, EP (1.1) has been generalized and extensively studied in many directions due to its importance; see, for example [14,16–19] and references therein, for the literature on the existence of solution of the various generalizations of EP (1.1). Some iterative methods have been studied for solving various classes of equilibrium problems, see for example [4,10,13,20–23,30,31] and references therein. Recently, some iterative methods for finding a common solution for system of equilibrium problems have been studied in the same space, see for example [9,28]. In general, the equilibrium problems in systems lie in the different spaces. Therefore, in this paper, we consider the following pair of equilibrium problems in different spaces, which is called *split equilibrium problem* (in short, SEP) due to Moudafi [25]:

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the *split equilibrium problem* (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geqslant 0, \quad \forall x \in C \tag{1.2}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geqslant 0, \quad \forall y \in Q. \tag{1.3}$$

When looked separately, (1.2) is the equilibrium problem (EP) and we denoted its solution set by EP(F_1). The SEP (1.2) and (1.3) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the EP (1.2) in H_1 is the solution of another EP (1.3) in another space H_2, we denote the solution set of EP (1.3) by EP(F_2). The solution set of SEP (1.2) and (1.3) is denoted by $\Omega = \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\}$.

SEP (1.2) and (1.3) generalize a multiple-set split feasibility problem. It also includes as special case, the split variational inequality problem [7] which is the generalization of split zero problems and split feasibility problems, see for detail [3,5–7,25,26].

Example 1.1. Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy, \forall x, y \in \mathbb{R}$. Let $C = [0,2]$ and $Q = (-\infty,0]$; let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be defined by $F_1(x,y) = x^2 - y, \forall x,y \in C$ and $F_2(u,v) = (u + 6)(v - u), \forall\, u,v \in Q$ and let, for each $x \in \mathbb{R}$, we define $A(x) = -3x$. It is easy to observe that $\mathrm{EP}\ (F_1) = [\sqrt{2}, 2], A(2) = -6$ and $\mathrm{EP}\ (F_2) = \{-6\}$. Hence $\Omega := \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\} = \{2\} \neq \emptyset$.

Next, we recall that a mapping $T{:}H_1 \to H_1$ is called a *contraction*, if there is $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leqslant \alpha\|x - y\|, \quad \forall x, y \in H_1.$$

If $\alpha = 1$, T is called *nonexpansive*.

A family $S{:=}\{T(s){:}0 \leqslant s < \infty\}$ of mappings from C into itself is called *nonexpansive semigroup* on C if it satisfies the following conditions:

(*i*) $T(0)x = x$ for all $x \in C$;
(*ii*) $T(s + t) = T(s)T(t)$ for all $s,t \geqslant 0$;
(*iii*) $\|T(s)x - T(s)y\| \leqslant \|x - y\|$ for all $x,y \in C$ and $s \geqslant 0$;
(*iv*) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

The set of all the common fixed points of a family S is denoted by $\mathrm{Fix}(S)$, i.e.,

$$\mathrm{Fix}(S) := \{x \in C : T(s)x = x, 0 \leqslant s < \infty\} = \bigcap_{0 \leqslant s < \infty} \mathrm{Fix}(T(s)),$$

where $\mathrm{Fix}(T(s))$ is the set of fixed points of $T(s)$. It is well known that $\mathrm{Fix}(S)$ is closed and convex.

The *fixed point problem* (in short, FPP) for a nonexpansive semigroup S is:

$$\text{Find } x \in C \text{ such that } x \in \mathrm{Fix}(S). \tag{1.4}$$

In 2006, Marino and Xu [24] considered the following implicit iterative scheme for a nonexpansive mapping T:

$$x_t = t\gamma f(x_t) + (I - tB)Tx_t,$$

where f is a contraction mapping with constant α and $B{:}H_1 \to H_1$ is a strongly positive bounded linear self adjoint operator, i.e., if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Bx, x\rangle \geqslant \bar{\gamma}\|x\|^2, \quad \forall x \in H_1,$$

with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $t \in (0,1)$ and proved that the net (x_t) converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z\rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \mathrm{Fix}(T)} \frac{1}{2}\langle Bx, x\rangle - h(x),$$

where h is the potential function for γf.

In 2008, Plubtieng and Punpaeng [27] introduced and studied the following implicit iterative scheme to prove a strong convergence theorem for FPP (1.4):

$$x_t = tf(x_t) + (1 - t)\frac{1}{s_t}\int_0^{s_t} T(s)x_t ds, \tag{1.5}$$

where (x_t) is a continuous net and (s_t) is a positive real divergent net.

Recently, Cianciaruso et al. [8] introduced and studied the following implicit iterative scheme and obtained strong convergence theorem for EP (1.1) and FPP (1.4)

$$\begin{cases} F(u_t, y) + \frac{1}{r_t}\langle y - u_t, u_t - x_t\rangle, \forall y \in C; \\ x_t = t\gamma f(x_t) + (I - tB)\frac{1}{s_t}\int_0^{s_t} T(s)u_t ds, \end{cases} \tag{1.6}$$

where (s_t) and (r_t) are the continuous nets in (0,1).

Motivated by the work of Plubtieng and Punpaeng [27], Cianciaruso et al. [8], Moudafi [25] and by the ongoing research in this direction, we suggest and analyze an implicit iterative method for approximating a common solution of SEP (1.2) and (1.3) and FPP (1.4) for a nonexpansive semigroup in real Hilbert spaces. Further, we prove that the nets generated by the iterative scheme converge strongly to a common solution of SEP (1.2) and (1.3) and FPP (1.4). Furthermore, we justify our main result through a numerical example. The result presented in this paper generalizes the corresponding results given in [8,27].

2. Preliminaries

We recall some concepts and results which are needed in the sequel.

Definition 2.1. A mapping $U : H_1 \to H_1$ is said to be

(i) *monotone*, if

$$\langle Ux - Uy, x - y\rangle \geqslant 0, \quad \forall x, y \in H_1;$$

(ii) α-*inverse strongly monotone* (or, α-*ism*), if there exists a constant $\alpha > 0$ such that

$$\langle Ux - Uy, x - y\rangle \geqslant \alpha\|Ux - Uy\|^2, \quad \forall x, y \in H_1;$$

(iii) *firmly nonexpansive*, if it is 1-*ism*.

Definition 2.2. A mapping $U : H_1 \to H_1$ is said to be *averaged* if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$U := (1 - \alpha)I + \alpha V,$$

where $\alpha \in (0,1)$ and $V : H_1 \to H_1$ is nonexpansive and I is the identity operator on H_1.

We note that the averaged mappings are nonexpansive. Further, the firmly nonexpansive mappings are averaged.

The following are some key properties of averaged operators, see for instance [1,25].

Proposition 2.1. *Let $U : H_1 \to H_1$ be a nonlinear mapping. Then:*

(i) *If $U = (1 - \alpha)D + \alpha V$, where $D : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0,1)$, then U is averaged;*
(ii) *The composite of finitely many averaged mappings is averaged;*
(iii) *If U is τ-ism, then for $\gamma > 0$, γU is $\frac{\tau}{\gamma}$-ism;*

(iv) *U is averaged if and only if, its complement* $I - U$ *is* τ*-ism for some* $\tau > \frac{1}{2}$.

Definition 2.3. For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H_1 onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \leqslant 0, \quad \forall x \in H_1,\ y \in C. \tag{2.1}$$

It is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x,y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leqslant (1/2)\|(T(x) - x) - (T(y) - y)\|^2$$

and therefore, we get, for all $(x,y) \in H_1 \times \mathrm{Fix}(T)$,

$$\langle x - T(x), y - T(x)\rangle \leqslant (1/2)\|T(x) - x\|^2, \tag{2.2}$$

see, e.g. [11, Theorem 2.3] and [12, Theorem 2.1].

Lemma 2.1 [15]. *Assume that T is nonexpansive self mapping of a closed convex subset C of a Hilbert space* H_1. *If T has a fixed point, then* $I - T$ *is demiclosed, i.e., whenever* $\{x_n\}$ *is a sequence in C converging weakly to some* $x \in C$ *and the sequence* $\{(I - T)x_n\}$ *converges strongly to some y, it follows that* $(I - T)x = y$. *Here I is the identity mapping on* H_1.

Lemma 2.2 [29]. *Let C be a nonempty bounded closed convex subset of a Hilbert space* H_1 *and let* $S := \{T(s) : 0 \leqslant s < \infty\}$ *be a nonexpansive semigroup on C. Then for* $t > 0$ *and for every* $0 \leqslant h < \infty$,

$$\limsup_{t\to\infty} \sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)xds - T(h)\left(\frac{1}{t}\int_0^t T(s)xds\right)\right\| = 0.$$

Lemma 2.3 [24]. *Assume that B is a strong positive bounded linear self adjoint operator on a Hilbert space* H_1 *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leqslant \|B\|^{-1}$. *Then* $\|I - \rho B\| \leqslant 1 - \rho\bar{\gamma}$.

Lemma 2.4 [24]. *Let C be a nonempty closed convex subset of a real Hilbert space* H_1, *let* $f : H_1 \to H_1$ *be an* α*-contraction mapping and let B be a strongly positive bounded linear self adjoint operator with coefficient* $\bar{\gamma}$. *Then for every* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, $(B - \gamma f)$ *is strongly monotone with coefficient* $(\bar{\gamma} - \gamma\alpha)$, *i.e.*,

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y\rangle \geqslant (\bar{\gamma} - \gamma\alpha)\|x - y\|^2.$$

Assumption 2.1 [2]. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $F(x,x) = 0, \quad \forall x \in C$;
(ii) F is monotone, i.e., $F(x,y) + F(y,x) \leqslant 0, \quad \forall x \in C$;
(iii) For each $x,y,z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x,y) \leqslant F(x,y)$;
(iv) For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous.

Lemma 2.5 [10]. *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfying Assumption 2.1. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{F_1} : H_1 \to C$ as follows:*

$$T_r^{F_1}x = \left\{ z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) $T_r^{F_1}(x) \neq \emptyset$ *for each* $x \in H_1$*;*
(ii) $T_r^{F_1}$ *is single-valued;*
(iii) $T_r^{F_1}$ *is firmly nonexpansive, i.e.,*

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \leqslant \langle T_r^{F_1}x - T_r^{F_1}y, x - y\rangle, \quad \forall x, \ y \in H_1;$$

(iv) Fix $(T_r^{F_1}) = \mathrm{EP}(F_1)$;
(v) EP (F_1) *is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.1. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(w) = \left\{ d \in Q : F_2(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geqslant 0, \forall e \in Q \right\}.$$

Then, we easily observe that $T_s^{F_2}(w) \neq \emptyset$ for each $w \in Q$; $T_s^{F_2}$ is single-valued and firmly nonexpansive; EP(F_2,Q) is closed and convex and Fix $\left(T_s^{F_2}\right) = \mathrm{EP}(F_2, Q)$, where EP($F_2$,$Q$) is the solution set of the following equilibrium problem:

Find $y^* \in Q$ such that $F_2(y^*,y) \geqslant 0, \ \forall y \in Q$.

We observe that EP(F_2) $\subseteq$ EP(F_2,Q). Further, it is easy to prove that Ω is closed and convex set.

Lemma 2.6 [8]. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1 hold and let $T_r^{F_1}$ be defined as in Lemma 2.5 for $r > 0$. Let $x,y \in H_1$ and $r_1,r_2 > 0$. Then:*

$$\|T_{r_2}^{F_1}y - T_{r_1}^{F_1}x\| \leqslant \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}^{F_1}y - y\|.$$

Lemma 2.7. *The following inequality holds in a real Hilbert space H_1:*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, \ y \in H_1.$$

Notation 1. We use $\to$ for strong convergence and $\rightharpoonup$ for weak convergence.

3. An implicit iterative method

In this section, we prove a strong convergence theorem based on the proposed implicit iterative method for computing the approximate common solution of SEP (1.2) and (1.3) and FPP (1.4) for a nonexpansive semigroup in real Hilbert spaces.

In the following theorem, we denote the identity operator on H_1 as well as H_2 by the same symbol I.

Assume that $\Omega \neq \emptyset$.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A: H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption* 2.1 *and F_2 is upper semicontinuous in the first argument. Let $S = \{T(s) : 0 \leqslant s < \infty\}$ be a nonexpansive semigroup on C such that* $\mathrm{Fix}(S) \cap \Omega \neq \emptyset$. *Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0,1)$ and B be a strongly positive bounded linear self adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Assume (r_t) and (s_t) are the continuous nets of positive real numbers such that $\liminf_{t \to 0} r_t = r > 0$ and $\lim_{t \to 0} s_t = +\infty$. Let the nets (u_t) and (x_t) be implicitly generated by*

$$u_t = T_{r_t}^{F_1}\left(x_t + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_t\right); \tag{3.1}$$

$$x_t = t\gamma f(x_t) + (I - tB)\frac{1}{s_t}\int_0^{s_t} T(s)u_t ds, \tag{3.2}$$

where $\delta \in (0,1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then (x_t) and (u_t) converge strongly to $z \in$ $\mathrm{Fix}(S) \cap \Omega$, where $z = P_{Fix(S) \cap \Omega}(I - B + \gamma f)z$, which is the unique solution of the variational inequality

$$\langle (\gamma f - B)z, x^* - z \rangle \leqslant 0, \quad \forall x^* \in \mathrm{Fix}(S) \cap \Omega. \tag{3.3}$$

Proof. We first show that (x_t) is well defined. For $t \in (0,1)$ such that $t < \|B\|^{-1}$, define a mapping $S_t : H_1 \to H_1$ by

$$S_t x = t\gamma f(x) + (I - tB)\frac{1}{s_t}\int_0^{s_t} T(s)\left(T_{r_t}^{F_1}\left(x + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax\right)\right)ds, \quad \forall x \in H_1.$$

We claim that S_t is contractive with constant $(1 - t(\bar{\gamma} - \gamma\alpha))$. Indeed, since $T_{r_t}^{F_1}$ and $T_{r_t}^{F_2}$ both are firmly nonexpansive, they are averaged. For $\delta \in \left(0, \frac{1}{L}\right)$, the mapping $\left(I + \delta A^*\left(T_{r_t}^{F_2} - I\right)A\right)$ is averaged, see [25]. It follows from Proposition 2.1 (ii) that the mapping $T_{r_t}^{F_1}\left(I + \delta A^*\left(T_{r_t}^{F_2} - I\right)A\right)$ is averaged and hence nonexpansive. Further, for any $x, y \in H_1$, it follows from Lemma 2.3 that

$$\begin{aligned}\|S_tx-S_ty\| &\leqslant \|t\gamma f(x)+(1-tB)\frac{1}{s_t}\int_0^{s_t}T(s)T_{r_t}^{F_1}(x+\delta A^*\big(T_{r_t}^{F_2}-I\big)Ax)ds-t\gamma f(y)\\
&\quad+(1-tB)\frac{1}{s_t}\int_0^{s_t}T(s)T_{r_t}^{F_1}\big(y+\delta A^*\big(T_{r_t}^{F_2}-I\big)Ay\big)ds\| \leqslant t\gamma\|f(x)-f(y)\|\\
&\quad+(1-t\bar{\gamma})\left\|\frac{1}{s_t}\int_0^{s_t}T(s)\big[T_{r_t}^{F_1}\big(x+\delta A^*\big(T_{r_t}^{F_2}-I\big)Ax\big)\right.\\
&\quad\left.-T_{r_t}^{F_1}\big(y+\delta A^*\big(T_{r_t}^{F_2}-I\big)Ay\big)\big]ds\right\|\\
&\leqslant t\gamma\alpha\|x-y\|\\
&\quad+(1-t\bar{\gamma})\big\|T_{r_t}^{F_1}\big(I+\delta A^*\big(T_{r_t}^{F_2}-I\big)A\big)x\\
&\quad-T_{r_t}^{F_1}\big(I+\delta A^*\big(T_{r_t}^{F_2}-I\big)A\big)y\big\|\\
&\leqslant t\gamma\alpha\|x-y\|+(1-t\bar{\gamma})\|x-y\|\\
&=(1-t(\bar{\gamma}-\gamma\alpha))\|x-y\|.\end{aligned}$$

Since $0<(1-t(\bar{\gamma}-\gamma\alpha))<1$, it follows that S_t is a contraction mapping. Therefore, by Banach contraction principle, S_t has the unique fixed point x_t, i.e., x_t is the unique solution of the fixed point Eq. (3.2).

Next, we show that (x_t) is bounded. Let $p\in \mathrm{Fix}(S)\cap\Omega$, we have $p=T_{r_t}^{F_1}p, Ap=T_{r_t}^{F_2}Ap$ and $p=T(s)p$.

We estimate

$$\begin{aligned}\|u_t-p\|^2 &= \big\|T_{r_t}^{F_1}\big(x_t+\delta A^*\big(T_{r_t}^{F_2}-I\big)Ax_t\big)-p\big\|^2\\
&= \big\|T_{r_t}^{F_1}\big(x_t+\delta A^*\big(T_{r_t}^{F_2}-I\big)Ax_t\big)-T_{r_t}^{F_1}p\big\|^2\\
&\leqslant \|x_t+\delta A^*\big(T_{r_t}^{F_2}-I\big)Ax_t-p\|^2\\
&\leqslant \|x_t-p\|^2+\delta^2\big\|A^*\big(T_{r_t}^{F_2}-I\big)Ax_t\big\|^2+2\delta\langle x_t\\
&\quad -p, A^*\big(T_{r_t}^{F_2}-I\big)Ax_t\rangle. \end{aligned} \tag{3.4}$$

Thus, we have

$$\begin{aligned}\|u_t-p\|^2 &\leqslant \|x_t-p\|^2+\delta^2\langle\big(T_{r_t}^{F_2}-I\big)Ax_t, AA^*\big(T_{r_t}^{F_2}-I\big)Ax_t\rangle+2\delta\langle x_t\\
&\quad -p, A^*\big(T_{r_t}^{F_2}-I\big)Ax_t\rangle.\end{aligned} \tag{3.5}$$

Now, we have

$$\begin{aligned}\delta^2\langle\big(T_{r_t}^{F_2}-I\big)Ax_t, AA^*\big(T_{r_t}^{F_2}-I\big)Ax_t\rangle &\leqslant L\delta^2\langle\big(T_{r_t}^{F_2}-I\big)Ax_t, \big(T_{r_t}^{F_2}-I\big)Ax_t\rangle\\
&= L\delta^2\|\big(T_{r_t}^{F_2}-I\big)Ax_t\|^2.\end{aligned} \tag{3.6}$$

Denoting $\Lambda=2\delta\langle x_t-p, A^*\big(T_{r_t}^{F_2}-I\big)Ax_t\rangle$ and using (2.2), we have

$$\begin{aligned}\Lambda &= 2\delta\langle x_t-p, A^*\big(T_{r_t}^{F_2}-I\big)Ax_t\rangle = 2\delta\langle A(x_t-p), \big(T_{r_t}^{F_2}-I\big)Ax_t\rangle\\
&= 2\delta\langle A(x_t-p)+\big(T_{r_t}^{F_2}-I\big)Ax_t-\big(T_{r_t}^{F_2}-I\big)Ax_t, \big(T_{r_t}^{F_2}-I\big)Ax_t\rangle\\
&= 2\delta\Big\{\langle T_{r_t}^{F_2}Ax_t-Ap, \big(T_{r_t}^{F_2}-I\big)Ax_t\rangle-\|\big(T_{r_t}^{F_2}-I\big)Ax_t\|^2\Big\}\\
&\leqslant 2\delta\Big\{\frac{1}{2}\|\big(T_{r_t}^{F_2}-I\big)Ax_t\|^2-\|\big(T_{r_t}^{F_2}-I\big)Ax_t\|^2\Big\}\leqslant -\delta\|\big(T_{r_t}^{F_2}-I\big)Ax_t\|^2.\end{aligned} \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we obtain

$$\|u_t - p\|^2 \leqslant \|x_t - p\|^2 + \delta(L\delta - 1)\left\|\left(T_{r_t}^{F_2} - I\right)Ax_t\right\|^2. \tag{3.8}$$

Since $\delta \in \left(0, \frac{1}{L}\right)$, we obtain

$$\|u_t - p\|^2 \leqslant \|x_t - p\|^2. \tag{3.9}$$

Now, setting $z_t := \frac{1}{s_t}\int_0^{s_t} T(s)u_t ds$, we obtain

$$\|z_t - p\| = \left\|\frac{1}{s_t}\int_0^{s_t} T(s)u_t ds - p\right\| \leqslant \frac{1}{s_t}\int_0^{s_t} \|T(s)u_t - T(s)p\|ds \leqslant \|u_t - p\| \leqslant \|x_t - p\|. \tag{3.10}$$

Further, we estimate

$$\begin{aligned}\|x_t - p\| &= \left\|t\gamma f(x_t) + (1 - tB)\frac{1}{s_t}\int_0^{s_t} T(s)u_t ds - p\right\| \\ &\leqslant t\|\gamma f(x_t) - Bp\| + (1 - t\overline{\gamma})\frac{1}{s_t}\int_0^{s_t} \|T(s)u_t - T(S)p\|ds \\ &\leqslant t[\gamma\|f(x_t) - f(p)\| + \|\gamma f(p) - Bp\|] + (1 - t\overline{\gamma})\|u_t - p\| \\ &\leqslant t\gamma\alpha\|x_t - p\| + t\|\gamma f(p) - Bp\| + (1 - t\overline{\gamma})\|x_t - p\| \\ &\leqslant [1 - t(\overline{\gamma} - \gamma\alpha)]\|x_t - p\| + t\|\gamma f(p) - Bp\| \\ &\leqslant \frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(p) - Bp\|.\end{aligned} \tag{3.11}$$

Hence, the net (x_t) is bounded and consequently, we deduce that the nets (u_t), (z_t) and $(f(x_t))$ are bounded.

Next, we have

$$\|x_t - z_t\| = \|t(\gamma f(x_t) - Bz_t) + (1 - tB)(z_t - z_t)\| \leqslant t\|\gamma f(x_t) - Bz_t\| \to 0 \text{ as } t \to 0. \tag{3.12}$$

Next, we show that $\|x_t - u_t\| \to 0$ as $t \to 0$. It follows from (3.8) and Lemma 2.7 that

$$\begin{aligned}\|x_t - p\|^2 &\leqslant (1 - t\overline{\gamma})^2\left\|\frac{1}{s_t}\int_0^{s_t} T(s)u_t ds - p\right\|^2 + 2t\langle\gamma f(x_t) - Bp + \gamma f(p) - \gamma f(p), x_t - p\rangle \\ &\leqslant (1 + t^2\overline{\gamma}^2 - 2t\overline{\gamma})\|u_t - p\|^2 + 2t\gamma\alpha\|x_t - p\|^2 + 2t\langle\gamma f(p) - Bp, x_t - p\rangle \\ &\leqslant (1 + t^2\overline{\gamma}^2)\|u_t - p\|^2 + 2t\gamma\alpha\|x_t - p\|^2 + 2t\langle\gamma f(p) - Bp, x_t - p\rangle \\ &\leqslant \|u_t - p\|^2 + 2t\gamma\alpha\|x_t - p\|^2 + t\overline{\gamma}^2\|x_t - p\|^2 + 2t\|\gamma f(p) - Bp\|\|x_t - p\| \\ &\leqslant \|x_t - p\|^2 + \delta(L\delta - 1)\left\|\left(T_{r_t}^{F_2} - I\right)Ax_t\right\|^2 + 2t\gamma\alpha\|x_t - p\|^2 \\ &\quad + t\overline{\gamma}^2\|x_t - p\|^2 + 2t\|\gamma f(p) - Bp\|\|x_t - p\|.\end{aligned} \tag{3.13}$$

Since (x_t) is bounded, we may assume that $\varrho := \sup_{0<t<1}\|x_t - p\|$. Therefore, preceding inequality reduces to

$$\delta(1 - L\delta)\left\|\left(T^{F_2}_{r_t} - I\right)Ax_t\right\|^2 \leqslant t\left[2\gamma\alpha\varrho^2 + \bar{\gamma}^2\varrho^2 + 2\|\gamma f(p) - Bp\|\varrho\right].$$

Further, since $\delta(1 - L\delta) > 0$, the preceding inequality implies that

$$\lim_{t\to 0}\left\|\left(T^{F_2}_{r_t} - I\right)Ax_t\right\| = 0. \tag{3.14}$$

Next, we have

$$\begin{aligned}\|u_t - p\|^2 &= \left\|T^{F_1}_{r_t}\left(x_t + \delta A^*\left(T^{F_2}_{r_t} - I\right)Ax_t\right) - p\right\|^2 \\ &= \left\|T^{F_1}_{r_t}\left(x_t + \delta A^*\left(T^{F_2}_{r_t} - I\right)Ax_t\right) - T^{F_1}_{r_t}p\right\|^2 \\ &\leqslant \langle u_t - p, x_t + \delta A^*\left(T^{F_2}_{r_t} - I\right)Ax_t - p\rangle \\ &= \frac{1}{2}\Big\{\|u_t - p\|^2 + \|x_t + \delta A^*\left(T^{F_2}_{r_t} - I\right)Ax_t - p\|^2 \\ &\quad - \left\|(u_t - p) - \left[x_t + \delta A^*\left(T^{F_2}_{r_t} - I\right)Ax_t - p\right]\right\|^2\Big\} \\ &= \frac{1}{2}\Big\{\|u_t - p\|^2 + \|x_t - p\|^2 - \left\|u_t - x_t - \delta A^*\left(T^{F_2}_{r_t} - I\right)Ax_t\right\|^2\Big\} \\ &= \frac{1}{2}\Big\{\|u_t - p\|^2 + \|x_t - p\|^2 - \Big[\|u_t - x_t\|^2 + \delta^2\|A^*\left(T^{F_2}_{r_t} - I\right)Ax_t\|^2 \\ &\quad - 2\delta\langle u_t - x_t, A^*\left(T^{F_2}_{r_t} - I\right)Ax_t\rangle\Big]\Big\} \\ &\leqslant \frac{1}{2}\Big\{\|u_t - p\|^2 + \|x_t - p\|^2 - \|u_t - x_t\|^2 - \delta^2\|A^*\left(T^{F_2}_{r_t} - I\right)Ax_t\|^2 \\ &\quad + 2\delta\|A(u_t - x_t)\|\|\left(T^{F_2}_{r_t} - I\right)Ax_t\|\Big\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|u_t - p\|^2 &\leqslant \|x_t - p\|^2 - \|u_t - x_t\|^2 - \delta^2\|A^*\left(T^{F_2}_{r_t} - I\right)Ax_t\|^2 + 2\delta\|A(u_t \\ &\quad - x_t)\|\|\left(T^{F_2}_{r_t} - I\right)Ax_t\| \\ &\leqslant \|x_t - p\|^2 - \|u_t - x_t\|^2 + 2\delta\|A(u_t - x_t)\|\left\|\left(T^{F_2}_{r_t} - I\right)Ax_t\right\|.\end{aligned} \tag{3.15}$$

Since (x_t) and (u_t) are bounded and A is a bounded linear operator then the net $(A(u_t - x_t))$ is bounded and hence, we may assume that $l := \sup_{0<t<1}\|A(u_t - x_t)\|$. It follows from (3.13) and (3.15) that

$$\begin{aligned}\|x_t - p\|^2 &\leqslant \|u_t - p\|^2 + 2t\gamma\alpha\|x_t - p\|^2 + t\bar{\gamma}^2\|x_t - p\|^2 + 2t\|\gamma f(p) - Bp\|\|x_t - p\| \\ &\leqslant \|x_t - p\|^2 - \|u_t - x_t\|^2 + 2\delta l\left\|\left(T^{F_2}_{r_t} - I\right)Ax_t\right\| + tJ,\end{aligned}$$

where $J := (2\gamma\alpha + \bar{\gamma}^2)\varrho^2 + 2\|\gamma f(p) - Bp\|\varrho$.

Therefore, from (3.14), we obtain

$$\|x_t - u_t\|^2 \leqslant 2\delta l\left\|\left(T^{F_2}_{r_t} - I\right)Ax_t\right\| + tJ \to 0, \text{ as } t \to 0.$$

Next, we have

$$\begin{aligned}\|T(s)x_t-x_t\| &\leqslant \left\|T(s)x_t-T(s)\frac{1}{s_t}\int_0^{s_t}T(s)u_tds\right\|+\left\|T(s)\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-\frac{1}{s_t}\int_0^{s_t}T(s)u_tds\right\| \\ &\quad+\left\|\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-x_t\right\| \\ &\leqslant \left\|x_t-\frac{1}{s_t}\int_0^{s_t}T(s)u_tds\right\|+\left\|T(s)\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-\frac{1}{s_t}\int_0^{s_t}T(s)u_tds\right\| \\ &\quad+\left\|\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-x_t\right\| \\ &\leqslant 2\left\|x_t-\frac{1}{s_t}\int_0^{s_t}T(s)u_tds\right\|+\left\|T(s)\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-\frac{1}{s_t}\int_0^{s_t}T(s)u_tds\right\|. \end{aligned} \tag{3.16}$$

We know x_t and $f(x_t)$ are bounded. Let $K:=\left\{w\in C:\left\|w-p\right\|\leqslant\frac{1}{\bar{\gamma}-\gamma\alpha}\left\|\gamma f(p)-Bp\right\|\right\}$, then K is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $0\leqslant s<\infty$ and contains (x_t). So without loss of generality, we may assume that $S:=\{T(s):0\leqslant s<\infty\}$ is nonexpansive semigroup on K. By Lemma 2.2, we have

$$\lim_{s_t\to\infty}\left\|T(s)\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-\frac{1}{s_t}\int_0^{s_t}T(s)u_tds\right\|=0. \tag{3.17}$$

Using (3.12), (3.16) and (3.17), we obtain

$$\lim_{t\to 0}\|T(s)x_t-x_t\|=0. \tag{3.18}$$

Let $t,t_0\in(0,\|B\|^{-1})$. Then, we have

$$\begin{aligned}\|x_t-x_{t_0}\| &= \Bigg\|(t-t_0)\gamma f(x_t)+t_0\gamma(f(x_t)-f(x_{t_0}))+(t_0-t)\frac{B}{s_t}\int_0^{s_t}T(s)u_tds \\ &\quad+\left(I-t_0B\right)\left[\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-\frac{1}{s_{t_0}}\int_0^{s_{t_0}}T(s)u_{t_0}ds\right]\Bigg\| \\ &\leqslant |t-t_0|\gamma\|f(x_t)-f(p)+f(p)\|+t_0\gamma\alpha\|x_t-x_{t_0}\| \\ &\quad+|t_0-t|\frac{\|B\|}{s_t}\left\|\int_0^{s_t}T(s)u_tds-p+p\right\|+(I-t_0\bar{\gamma})\Bigg\|\frac{1}{s_t}\int_0^{s_t}T(s)u_tds \\ &\quad-\frac{1}{s_{t_0}}\int_0^{s_{t_0}}T(s)u_{t_0}ds\Bigg\|\leqslant|t-t_0|(\gamma\|f(x_t)-f(p)\|+\gamma\|f(p)\|) \\ &\quad+t_0\gamma\alpha\|x_t-x_{t_0}\|+|t_0-t|\|B\|\left\|\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-p\right\|+|t_0-t|\frac{\|B\|}{s_t}\|p\| \\ &\quad+(I-t_0\bar{\gamma})\left\|\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-\frac{1}{s_{t_0}}\int_0^{s_{t_0}}T(s)u_{t_0}ds\right\| \\ &\leqslant|t-t_0|\left(\gamma\alpha\|x_t-p\|+\gamma\|f(p)\|+\|B\|\left\|\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-p\right\|\right) \\ &\quad+|t_0-t|\frac{\|B\|}{s_t}\|p\|+t_0\gamma\alpha\|x_t-x_{t_0}\| \\ &=+(1-t_0\bar{\gamma})\left\|\frac{1}{s_t}\int_0^{s_t}T(s)u_tds-\frac{1}{s_t}\int_0^{s_t}T(s)u_{t_0}ds\right\| \\ &\quad+(1-t_0\bar{\gamma})\left\|\frac{1}{s_t}\int_0^{s_t}T(s)u_{t_0}ds-\frac{1}{s_{t_0}}\int_0^{s_{t_0}}T(s)u_{t_0}ds\right\|. \end{aligned} \tag{3.19}$$

Since $\left\|\frac{1}{s_t}\int_0^{s_t} T(s)u_t ds - p\right\| \leqslant \|u_t - p\| \leqslant \|x_t - p\| \leqslant \varrho$, and if we denote
$M := (\gamma\alpha + \|B\|)\varrho + \gamma\|f(p)\|,$

we obtain

$$\begin{aligned}\|x_t - x_{t_0}\| &\leqslant |t - t_0|M + |t_0 - t|\frac{\|B\|}{s_t}\|p\| + t_0\gamma\alpha\|x_t - x_{t_0}\| + (1 - t_0\bar{\gamma})\|u_t - u_{t_0}\| \\ &\quad + (1 - t_0\bar{\gamma})\left\|\left(\frac{1}{s_t} - \frac{1}{s_{t_0}}\right)\int_0^{s_t} T(s)u_{t_0}ds - \frac{1}{s_{t_0}}\int_{s_t}^{s_{t_0}} T(s)u_{t_0}ds\right\| \\ &\leqslant |t - t_0|M + |t_0 - t|\frac{\|B\|}{s_t}\|p\| + t_0\gamma\alpha\|x_t - x_{t_0}\| + (1 - t_0\bar{\gamma})\|u_t - u_{t_0}\| \\ &\quad + (1 - t_0\bar{\gamma})\left|\frac{1}{s_t} - \frac{1}{s_{t_0}}\right| s_t(\varrho + \|p\|) + (1 - t_0\bar{\gamma})\left\|\frac{1}{s_{t_0}}\int_{s_t}^{s_{t_0}} T(s)u_{t_0}ds\right\|. \end{aligned} \tag{3.20}$$

Since the mapping $T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)$ is nonexpansive then it follows from $u_t = T_{r_t}^{F_1}(x_t + \delta A^*(T_{r_t}^{F_2} - I)Ax_t)$, $u_{t_0} = T_{r_{t_0}}^{F_1}(x_{t_0} + \delta A^*(T_{r_{t_0}}^{F_2} - I)Ax_{t_0})$ and Lemma 2.6 that

$$\begin{aligned}\|u_t - u_{t_0}\| &\leqslant \left\|T_{r_t}^{F_1}(x_t + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_t) - T_{r_t}^{F_1}\left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right)\right\| \\ &\quad + \|T_{r_t}^{F_1}(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}) - T_{r_{t_0}}^{F_1}\left(x_{t_0} + \delta A^*\left(T_{r_{t_0}}^{F_2} - I\right)Ax_{t_0}\right)\| \\ &\leqslant \|x_t - x_{t_0}\| + \left\|\left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right) - \left(x_{t_0} + \delta A^*\left(T_{r_{t_0}}^{F_2} - I\right)Ax_{t_0}\right)\right\| \\ &\quad + \left|1 - \frac{r_{t_0}}{r_t}\right|\left\|T_{r_t}^{F_1}(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}) - \left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right)\right\| \\ &\leqslant \|x_t - x_{t_0}\| + \delta\|A\|\|T_{r_t}^{F_2}Ax_{t_0} - T_{r_{t_0}}^{F_2}Ax_{t_0}\| + \delta_t \leqslant \|x_t - x_{t_0}\| \\ &\quad + \delta\|A\|\left|1 - \frac{r_{t_0}}{r_t}\right|\|T_{r_t}^{F_2}Ax_{t_0} - Ax_{t_0}\| + \delta_t = \|x_t - x_{t_0}\| + \delta\|A\|\sigma_t + \delta_t,\end{aligned} \tag{3.21}$$

where

$$\sigma_t = \left|1 - \frac{r_{t_0}}{r_t}\right|\|T_{r_t}^{F_2}Ax_{t_0} - Ax_{t_0}\|$$

and

$$\delta_t = \left|1 - \frac{r_{t_0}}{r_t}\right|\left\|T_{r_t}^{F_1}\left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right) - \left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right)\right\|$$

Further, it follows from (3.14) that the net $(T_{r_t}^{F_2}Ax_t - Ax_t)$ is convergent and hence bounded. Therefore, we may assume $M_1 := \sup_{0<t<1}\left\|T_{r_t}^{F_2}Ax_t - Ax_t\right\|$. Further, we can observe that the net $\left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right)$ is also bounded and hence, we may assume that $M_2 := \sup_{0<t<1}\left\|T_{r_t}^{F_1}\left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right) - \left(x_{t_0} + \delta A^*\left(T_{r_t}^{F_2} - I\right)Ax_{t_0}\right)\right\|$.

Moreover, since (r_t) is a continuous net of positive real numbers, we can choose a neighborhood U_{t_0} and a positive number c in such a way that $c < r_t$ for $t \in U_{t_0}$, then (3.21) reduces to

$$\|u_t - u_{t_0}\| \leqslant \|x_t - x_{t_0}\| + \left[\delta\|A\|\frac{M_1}{c} + \frac{M_2}{c}\right]|r_t - r_{t_0}|. \tag{3.22}$$

It follows from (3.20) and (3.22) that

$$\begin{aligned}
\|x_t - x_{t_0}\| &\leqslant |t - t_0|M + |t_0 - t|\frac{\|B\|}{s_t}\|p\| + t_0\gamma\alpha\|x_t - x_{t_0}\| + (1 - t_0\bar{\gamma})\|x_t - x_{t_0}\| \\
&\quad + (1 - t_0\bar{\gamma})\left|\frac{1}{s_t} - \frac{1}{s_{t_0}}\right| s_t(\varrho + \|p\|) + (1 - t_0\bar{\gamma})\left\|\frac{1}{s_{t_0}}\int_{s_t}^{s_{t_0}} T(s)u_{t_0}ds\right\| \\
&\quad + (1 - t_0\bar{\gamma})\left[\delta\|A\|\frac{M_1}{c} + \frac{M_2}{c}\right]|r_t - r_{t_0}| \\
&\leqslant \frac{1}{\bar{\gamma} - \gamma\alpha}\left[|t - t_0|M + |t_0 - t|\frac{\|B\|}{s_t}\|p\| + \left|\frac{1}{s_t} - \frac{1}{s_{t_0}}\right| s_t(\varrho + \|p\|)\right. \\
&\quad \left. + (1 - t_0\bar{\gamma})|s_t - s_{t_0}|(\varrho + \|p\|) + (1 - t_0\bar{\gamma})\left[\delta\|A\|\frac{M_1}{c} + \frac{M_2}{c}\right]|r_t - r_{t_0}|\right].
\end{aligned}$$

The continuity of (r_t) and (s_t) shows that (x_t) is a continuous curve. The continuity of (u_t) is followed by (3.22).

Let $\{t_n\}$ be a sequence in (0,1) such that $t_n \to 0$ as $n \to \infty$. Setting $x_n := x_{t_n}$, $u_n := u_{t_n}, s_n := s_{t_n}$, $r_n := r_{t_n}$. Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $w \in C$. It follows from (3.18) and Lemma 2.1 that $w \in \mathrm{Fix}(S)$. Further, we show that $x_{n_j} \to w$ as $j \to \infty$. Indeed, for each n, we have

$$\begin{aligned}
\|x_n - w\|^2 &= \langle t_n\gamma f(x_n), x_n - w\rangle + \left\langle (1 - t_nB)\frac{1}{s_n}\int_0^{s_n} T(s)u_nds - w, x_n - w\right\rangle \\
&\leqslant t_n\langle \gamma f(x_n) - Bw, x_n - w\rangle + (1 - t_n\bar{\gamma})\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_nds - w\right\|\|x_n - w\| \\
&\leqslant t_n\langle \gamma f(x_n) - Bw, x_n - w\rangle + (1 - t_n\bar{\gamma})\|x_n - w\|^2 \\
&\leqslant t_n\gamma\alpha\|x_n - w\|^2 + t_n\langle \gamma f(w) - Bw, x_n - w\rangle + (1 - t_n\bar{\gamma})\|x_n - w\|^2 \\
&\leqslant [1 - t_n(\bar{\gamma} - \gamma\alpha)]\|x_n - w\|^2 + t_n\langle \gamma f(w) - Bw, x_n - w\rangle \\
&\leqslant \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(w) - Bw, x_n - w\rangle.
\end{aligned}$$

In particular, we have

$$\|x_{n_j} - w\|^2 \leqslant \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(w) - Bw, x_{n_j} - w\rangle. \tag{3.23}$$

Since $x_{n_j} \rightharpoonup w$, it follows from (3.23) that $x_{n_j} \to w$ as $j \to \infty$.

Next, we show that $w \in \mathrm{EP}(F_1)$. Since $u_t = T_{r_t}^{F_1}x_t$ then, we have $u_{n_j} = T_{r_{n_j}}^{F_1}x_{n_j}$ and,

$$F_1(u_{n_j}, y) + \frac{1}{r_{n_j}}\langle y - u_{n_j}, u_{n_j} - x_{n_j}\rangle \geqslant 0, \quad \forall y \in C.$$

It follows from the monotonicity of F_1 that

$$\frac{1}{r_{n_j}}\langle y-u_{n_j},u_{n_j}-x_{n_j}\rangle \geqslant F_1(y,u_{n_j})$$

and hence

$$\left\langle y-u_{n_j},\frac{u_{n_j}-x_{n_j}}{r_{n_j}}\right\rangle \geqslant F_1(y,u_{n_j}).$$

Since $\|u_n - x_n\| \to 0$ and $x_{n_j} \to w$, we get $u_{n_j} \to w$. Further, since $\lim\inf_{t\to 0} r_t = r > 0$, $\frac{u_{n_j}-x_{n_j}}{r_{n_j}} \to 0$. It follows from Assumption 2.1 (iv) that $0 \geqslant F_1(y,w), \forall w \in C$. For τ with $0 < \tau \leqslant 1$ and $y \in C$, let $y_\tau = \tau\, y + (1-\tau)w$. Since $y \in C$, $w \in C$, we get $y_\tau \in C$ and hence $F_1(y_\tau,w) \leqslant 0$. So from Assumption 2.1 (i) and (iv) we have

$$0 = F_1(y_\tau,y_\tau) \leqslant \tau F_1(y_\tau,y) + (1-\tau)F_1(y_\tau,w) \leqslant \tau F_1(y_\tau,y).$$

Therefore $0 \leqslant F_1(y_\tau,y)$. From Assumption 2.1 (iii), we have $0 \leqslant F_1(w,y)$. This implies that $w \in \mathrm{EP}(F_1)$.

Next, we show that $Aw \in \mathrm{EP}(F_2)$. Since $x_{n_j} \to w$ and A is a bounded linear operator, $Ax_{n_j} \to Aw$.

Now, setting $v_{n_j} = Ax_{n_j} - T^{F_2}_{r_{n_j}} Ax_{n_j}$. It follows that from (3.14) that $\lim_{j\to\infty} v_{n_j} = 0$ and $Ax_{n_j} - v_{n_j} = T^{F_2}_{r_{n_j}} Ax_{n_j}$.

Therefore from Lemma 2.5, we have

$$F_2(Ax_{n_j}-v_{n_j},z) + \frac{1}{r_{n_j}}\langle z-(Ax_{n_j}-v_{n_j}),(Ax_{n_j}-v_{n_j})-Ax_{n_j}\rangle \geqslant 0, \quad \forall z \in Q.$$

Since F_2 is upper semicontinuous in the first argument, taking lim sup to above inequality as $j \to \infty$ and using $\lim\inf_{t\to 0} r_t = r > 0$, we obtain

$$F_2(Aw,z) \geqslant 0, \quad \forall z \in Q,$$

which means that $Aw \in \mathrm{EP}(F_2)$ and hence $w \in \Omega$.

Next, we show that $w \in \mathrm{Fix}(S) \cap \Omega$ solves the variational inequality (3.3). Since x_t is the unique solution of fixed point Eq. (3.2), we have

$$(B-\gamma f)x_t = -\frac{1}{t}(I-tB)\left[x_t - \frac{1}{s_t}\int_0^{s_t} T(s)u_t ds\right].$$

Hence, for any $q \in \mathrm{Fix}(S) \cap \Omega$, we obtain

$$\begin{aligned}\langle (B-\gamma f)x_t,x_t-q\rangle &= -\frac{1}{t}\left\langle (I-tB)\left[x_t - \frac{1}{s_t}\int_0^{s_t} T(s)T^{F_1}_{r_t}(I+\delta A^*\left(T^{F_2}_{r_t}-I\right)A)x_t ds\right],x_t-q\right\rangle \\ &= -\frac{1}{t}\left[\frac{1}{s_t}\int_0^{s_t}\langle \left(I-T(s)T^{F_1}_{r_t}\left(I+\delta A^*\left(T^{F_2}_{r_t}-I\right)A\right)x_t\right.\right. \\ &\quad -\left(I-T(s)T^{F_1}_{r_t}\left(I+\delta A^*\left(T^{F_2}_{r_t}-I\right)A\right)q,x_t-q\rangle ds\right] \\ &\quad +\frac{1}{s_t}\left\langle B\int_0^{s_t}[x_t-T(s)u_t]ds,x_t-q\right\rangle. \end{aligned} \tag{3.24}$$

Since the mapping $U := T(s)T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)$ is nonexpansive then $(I - U)$ is monotone and hence

$$\frac{1}{s_t}\int_0^{s_t} \langle (I - T(s)T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)x_t - (I - T(s)T_{r_t}^{F_1}(I + \delta A^*(T_{r_t}^{F_2} - I)A)q, x_t - q\rangle ds \geqslant 0.$$

This together with (3.24), we have

$$\langle (B - \gamma f)x_t, x_t - q\rangle \leqslant \left\langle Bx_t - \frac{B}{s_t}\int_0^{s_t} T(s)u_t ds, x_t - q\right\rangle.$$

From (3.2), we have

$$Bx_t - \frac{B}{s_t}\int_0^{s_t} T(s)u_t ds = tB\left(\gamma f(x_t) - \frac{B}{s_t}\int_0^{s_t} T(s)u_t ds\right).$$

Hence, we have

$$\langle (B - \gamma f)x_t, x_t - q\rangle \leqslant t\left\langle B\left(\gamma f(x_t) - \frac{B}{s_t}\int_0^{s_t} T(s)u_t ds\right), x_t - q\right\rangle.$$

Since the nets (x_t),(z_t),(u_t) and $(f(x_t))$ are bounded, on taking the limit $t := t_{n_j} \to 0$, we obtain

$$\langle (B - \gamma f)w, w - q\rangle = \lim_{j\to\infty}\langle (B - \gamma f)x_{n_j}, x_{n_j} - q\rangle \leqslant 0, \tag{3.25}$$

which implies $w = P_{\mathrm{Fix}(S)\cap\Omega}(I + \gamma f - B)$.

To show that the net x_t converges strongly to w, we assume that there is a sequence $\{s_n\} \subset (0,1)$ such that $x_{s_n} \to q$ when $s_n \to 0$ as $n \to \infty$. Following the same steps of the proof given above, we can prove $q \in \mathrm{Fix}(S) \cap \Omega$. Hence, it follows from (3.25) that

$$\langle (B - \gamma f)q, q - w\rangle \leqslant 0. \tag{3.26}$$

Interchanging the role of w and z, we obtain

$$\langle (B - \gamma f)w, w - q\rangle \leqslant 0. \tag{3.27}$$

Adding (3.26) and (3.27) yields

$$(\bar{\gamma} - \gamma\alpha)\|w - q\|^2 \leqslant \langle w - q, (B - \gamma f)w - (B - \gamma f)q\rangle \leqslant 0.$$

By Lemma 2.4, we have $w = q$ and therefore $x_t \to q$.

Thus, we have shown that each cluster point of (x_t) equals w as $t \to 0$. Therefore $x_t \to w$ and $u_t \to w$ as $t \to 0$, where $w \in \mathrm{Fix}(S) \cap \Omega$ is the unique solution of the variational inequality (3.2). This completes the proof. □

As the consequence of Theorem 3.1, we have the following strong convergence results for computing the approximate common solution of EP (1.1) and FPP (1.4) for a nonexpansive semigroup in real Hilbert space.

Corollary 3.1 [8]. *Let H be a real Hilbert space and $C \subseteq H$ be a nonempty closed convex subset. Let $F : C \times C \to \mathbb{R}$ be a bifunction such that Assumption 2.1 hold. Let $S = \{T(s) : 0 \leqslant s < \infty\}$ be a nonexpansive semigroup on C such that* $\mathrm{Fix}(S) \cap \mathrm{EP}(F) \neq \emptyset$. *Let $f : H \to H$ be a contraction mapping with constant $\alpha \in (0,1)$ and*

B be a strongly positive bounded linear self adjoint operator on H with constant $\bar{\gamma} > 0$, such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Assume (r_t) and (s_t) are the continuous nets of positive real numbers such that $\liminf_{t\to 0} r_t = r > 0$ and $\lim_{t\to 0} s_t = +\infty$. Let the nets (u_t) and (x_t) are generated by the implicit iterative scheme (1.6). Then x_t and u_t converge strongly to $z \in \mathrm{Fix}(S) \cap \mathrm{EP}(F)$, where $z = P_{\mathrm{Fix}\,(S)\cap \mathrm{EP}}(F)(I + \gamma f - B)$, which is the unique solution of the variational inequality

$$\langle (\gamma f - B)z, x^* - z\rangle \leqslant 0, \quad \forall x^* \in \mathrm{Fix}(S) \cap \mathrm{EP}(F).$$

Proof. Taking $H_1 = H_2 = H$, $A = 0$, $F_1 = F$ and $B = I$ in Theorem 3.1 then the conclusion of Corollary 3.1 is obtained. □

Further, we have the following consequence of Theorem 3.1.

Corollary 3.2 [27]. *Let H be a real Hilbert space and $C \subseteq H$ be a nonempty closed convex subset. Let $S = \{T(s): 0 \leqslant s < \infty\}$ be a nonexpansive semigroup on C such that $\mathrm{Fix}(S) \neq \emptyset$. Let $f: H \to H$ be a contraction mapping with constant $\alpha \in (0,1)$. Assume (s_t) be a continuous net of positive real number such that $\lim_{t\to 0} s_t = +\infty$. Let the net (x_t) be generated by implicit scheme (1.5). Then x_t converges strongly to $z \in \mathrm{Fix}(S)$, where $z = P_{\mathrm{Fix}(S)} f(z)$, which is the unique solution of the variational inequality*

$$\langle (I - f)z, x^* - z\rangle \geqslant 0, \quad \forall x^* \in \mathrm{Fix}(S).$$

Proof. Taking $H_1 = H_2 = H$, $u_t = x_t$ and $F_1 = F_2 = 0$ in Theorem 3.1 then the conclusion of Corollary 3.2 is obtained. □

Remark 3.1.

1. The algorithm considered in Theorem 3.1 is different from those considered in [3,7,25,26] in the following sense:
   (i) Implicit iterative algorithm has been considered instead of explicit iterative algorithm
   (ii) In our algorithm net (r_t) has been considered in place of fixed r. Further, the approach presented in this paper is different.

2. The use of implicit iterative method presented in this paper for the split monotone variational inclusions considered in Moudafi [25] and Byrne et al. [3] needs further research effort.

4. Numerical Example

Now, we give a numerical example which justifies Theorem 3.1.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy, \forall x, y \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let $C = [0, +\infty)$ and

$Q = (-\infty,0]$; let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be defined by $F_1(x,y) = (x-2)(y-x), \forall x,y \in C$ and $F_2(u,v) = (u+4)(v-u)$, $\forall u,v \in Q$; let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{8}x$, $A(x) = -2x$, $B(x) = 2x$, and let, for each $x \in C$, $T(x) = x$. Let $\{t_n\}$ be a sequence in (0,1) such that $t_n \to 0$ as $n \to \infty$. Setting $x_n := x_{t_n}, u_n := u_{t_n}$, $z_n := z_{t_n}, r_n := r_{t_n} = 1$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$, $\{u_n\} \subset C$, and $\{z_n\} \subset Q$ generated by the iterative schemes

$$z_n = T_{r_n}^{F_2}(Ax_n); \quad u_n = T_{r_n}^{F_1}\left[x_n + \frac{1}{8}A^*(z_n - Ax_n)\right]; \tag{4.1}$$

$$x_n = \frac{1}{n+2}(2)\left(\frac{1}{8}x_n\right) + \left(I - \frac{1}{n+2}B\right)Tu_n, \tag{4.2}$$

where $t_n = \frac{1}{n+2}$ and $r_n = 1$. Then $\{x_n\}$ converges strongly to $2 \in \mathrm{Fix}(T) \cap \Omega$.

Proof. It is easy to prove that the bifunctions F_1 and F_2 satisfy the Assumption 2.1 and F_2 is upper semicontinuous. A is a bounded linear operator on $\mathbb{R}$ with adjoint operator A^* and $\|A\| = \|A^*\| = 2$. Hence $\delta \in \left(0, \frac{1}{4}\right)$, so we can choose $\delta = \frac{1}{8}$. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and B is a strongly positive bounded linear self adjoint operator with constant $\bar{\gamma} = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $\mathrm{Fix}(T) = (0,\infty)$, $\mathrm{EP}(F_1) = \{2\}$, and $\mathrm{EP}(F_2) = \{-4\}$. Hence $\Omega := \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\} = \{2\}$. Consequently, $\mathrm{Fix}(T) \cap \Omega = \{2\} \neq \emptyset$. After simplification, schemes (4.1) and (4.2) reduce to

$$z_n = -(x_n + 2); \quad u_n = \frac{1}{8}(3x_n + 10); \tag{4.3}$$

$$x_n = \frac{1}{4(n+2)}x_n + \left(1 - \frac{2}{n+2}\right)u_n, \tag{4.4}$$

which reduce to the following scheme:

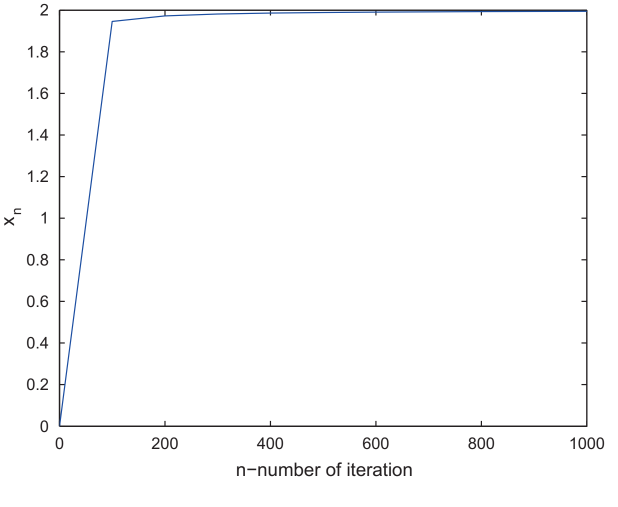

Fig. 1 Convergence of iterative sequence $\{x_n\}$.

$$x_n = \frac{\frac{5}{2}\left[\frac{1}{2} - \frac{1}{n+2}\right]}{\left[\frac{5}{8} + \frac{1}{2(n+2)}\right]}.$$

Following the proof of Theorem 3.1, we obtain that $\{z_n\}$ converges strongly to $-4 \in \mathrm{EP}(F_2)$ and $\{x_n\}$ and $\{u_n\}$ converge strongly to $w = 2 \in \mathrm{Fix}(T) \cap \Omega$ as $n \to \infty$.

Next, using the software Matlab 7.0, we have Fig. 1 which shows that $\{x_n\}$ converges strongly to 2.

The proof is completed. □

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References

[1] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, New York, 2011.
[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.
[3] C. Byrne, Y. Censor, A. Gibali, S. Reich, Weak and strong convergence of algorithms for the split common null point problem, J. Nonlinear Convex Anal. 13 (2012) 759–775.
[4] L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. 214 (2008) 186–201.
[5] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity modulated radiation therapy, Phys. Med. Biol. 51 (2006) 2353–2365.
[6] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in product space, Numer. Algorithms 8 (1994) 221–239.
[7] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, Numer. Algorithms 59 (2012) 301–323.
[8] F. Cianciaruso, G. Marino, L. Muglia, Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert space, J. Optim. Theory Appl. 146 (2010) 491–509.
[9] V. Colao, G.L. Acedo, G. Marino, An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings, Nonlinear Anal. 71 (2009) 2708–2715.
[10] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.
[11] G. Crombez, A hierarchical presentation of operators with fixed points on Hilbert spaces, Numer. Funct. Anal. Optim. 27 (2006) 259–277.
[12] G. Crombez, A geometrical look at iterative methods for operators with fixed points, Numer. Funct. Anal. Optim. 26 (2005) 157–175.
[13] S.D. Flam, A.S. Antipin, Equilibrium programming using proximal-like algorithms, Math. Program. 78 (1997) 29–41.
[14] F. Giannessi, Vector variational inequalities and vector equilibria, Mathematical Theories Nonconvex Optimization and its Applications, vol. 38, Kluwer Academic Publishers, Dordrecht, 2000.
[15] K. Goebel, W.A. Kirk, Topics on Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[16] N. Hadjisavvas, S. Komlósi, S. Schaible, Handbook of Generalized Convexity and Generalized Monotonicity, Springer, Berlin, 2005.
[17] N. Hadjisavvas, S. Schaible, From scaler to vector equilibrium problems in the quasimonotone case, J. Optim. Theory Appl. 96 (1998) 297–309.
[18] K.R. Kazmi, On vector equilibrium problem, Proc. Indian Acad. Sci. Math. Sci. 110 (2000) 213–223.
[19] K.R. Kazmi, A. Raouf, A class of operator equilibrium problems, J. Math. Anal. Appl. 308 (2005) 554–564.
[20] K.R. Kazmi, A. Khaliq, A. Raouf, Iterative approximation of solution of generalized mixed set-valued variational inequality problem, Math. Inequal. Appl. 10 (2007) 677–691.
[21] K.R. Kazmi, F.A. Khan, Existence and iterative approximation of solutions of generalized mixed equilibrium problems, Comput. Math. Appl. 56 (2008) 1314–1321.
[22] K.R. Kazmi, F.A. Khan, Mohammad Shahzad, Two step algorithm for solving regularized generalized mixed variational inequality problem, Bull. Korean Math. Soc. 47 (2010) 675–685.
[23] K.R. Kazmi, S.H. Rizvi, A hybrid extragradient method for approximating the common solutions of a variational inequality, a system of variational inequalities, a mixed equilibrium problem and a fixed point problem, Appl. Math. Comput. 218 (2012) 5439–5452.
[24] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318 (2006) 43–52.
[25] A. Moudafi, Split monotone variational inclusions, J. Optim. Theory Appl. 150 (2011) 275–283.
[26] A. Moudafi, The split common fixed point problem for demicontractive mappings, Inverse Prob. 26 (2010) 055007, 6pp.
[27] S. Plubtieng, R. Punpaeng, Fixed point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces, Math. Comput. Model. 48 (2008) 279–286.
[28] S. Saeidi, Iterative algorithms for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of families and semigroups of nonexpansive mappings, Nonlinear Anal. 70 (2009) 4195–4208.
[29] T. Shimizu, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211 (1997) 71–83.
[30] A. Tada, W. Takahashi, Strong convergence theorem for an equilibrium problem and and a nonexpansive mapping, in: W. Takahashi, T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama, Yokohama, 2006, pp. 609–617.
[31] S. Takahashi, W. Takahashi, Viscosity approximation method for equilibrium problems and fixed point problems in Hilbert space, J. Math. Anal. Appl. 331 (2007) 506–515.