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$\mathcal{I}$-Statistical convergence in 2-normed space

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Abstract. In this paper, we investigate the notion $\mathcal{I}$-statistical convergence and $\mathcal{I}$-statistical Cauchy sequence in 2-normed space, study of their relationship, and make some observations about these classes.

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1. Introduction

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [3] (see also [15]) as follows: let K be a subset of $\mathbb{N}$. Then the asymptotic density of K is denoted by $\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : k \in K\}|$, where the vertical bars denote the cardinality of the enclosed set. A number sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0, \delta(\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}) = 0$. If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to L we write $st\text{-}\lim x_k = L$. Statistical convergence turned out to be one of the most active areas of research in the summability theory after the works of Fridy [4] and Šalát [12].

In [10], Kostyrko et al. introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in [1,2,11,13].

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The concept of 2-normed spaces was initially introduced by Gähler [5] in the 1960s. Since then, this concept has been studied by many authors, see for instance [6–9,14,16].

The notion of ideal statistical convergence and ideal statistical Cauchy sequence has not been studied previously in the setting of 2-normed spaces. Motivated by this fact, in this paper, as a variant of statistical convergence, the notions of ideal statistical convergence and ideal statistical Cauchy sequence are introduced in a 2-normed space and some important results are established.

Throughout, by a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ we shall mean a sequence of real numbers. Throughout the paper, $\mathbb{N}$ will denote the set of all natural numbers.

We now recall some notation and basic definitions used in the paper.

A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if **(i)** $\emptyset \in \mathcal{I}$; **(ii)** $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; **(iii)** $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$. A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter $\mathbb{N}$ if **(i)** $\emptyset \notin F$; **(ii)** $A,B \in F$ imply $A \cap B \in F$; **(iii)** $A \in F$, $A \subset B$ imply $B \in F$.

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 1. [10]Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if, for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geqslant \varepsilon\} \in \mathcal{I}$.

Definition 2. [5]Let X be a real vector space of dimension d, where $2 \leqslant d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies (*i*) $\|x,y\| = 0$ if and only if x and y are linearly dependent; (*ii*) $\|x,y\| = \|y,x\|$; (*iii*) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$, $\alpha \in \mathbb{R}$; (*iv*) $\|x,y + z\| \leqslant \|x,y\| + \|x,z\|$. The pair $(X,\|.,.\|)$ is then called a 2-normed space.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\|:=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Definition 3. [16]Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. A sequence (x_n) of X is said to be $\mathcal{I}$-convergent to x, if for each $\varepsilon > 0$ and nonzero z in X the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x, z\| \geqslant \varepsilon\}$ belongs to $\mathcal{I}$.

If (x_n) is $\mathcal{I}$-convergent to x then we write $\mathcal{I}\text{-}\lim_{n\to\infty}\|x_n - x,z\| = 0$ or $\mathcal{I}\text{-}\lim_{n\to\infty}\|x_n,z\| = \|x,z\|$ for each nonzero z in X. The vector x is $\mathcal{I}$-limit of the sequence (x_n).

Further we will give some examples of ideals and corresponding $\mathcal{I}$-convergences.

(I) Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence coincides with the usual convergence [5].

(II) Put $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$ convergence coincides with the statistical convergence [6].

2. $\mathcal{I}$-Statistical convergence and $\mathcal{I}$-statistical Cauchy sequence on 2-normed spaces

In this section we deal with the ideal statistical convergence and ideal statistical Cauchy sequence on 2-normed spaces and prove some important results. Throughout the paper we assume X to be a 2-normed space having dimension d, where $2 \leqslant d < \infty$.

Following the line of Savaş et al. [13] we now introduce the following definition using ideals.

Definition 4. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$.The sequence (x_k) of X is said to be $\mathcal{I}$-statistically convergent to ξ, if for each $\varepsilon > 0$, $\delta > 0$ and nonzero z in X the set

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - \xi, z\| \geqslant \varepsilon\}| \geqslant \delta\right\} \in \mathcal{I},$$

or equivalently if for each $\varepsilon > 0$

$$\delta_{\mathcal{I}}(A_n(\varepsilon)) = \mathcal{I} - \lim \delta_n(A_n(\varepsilon)) = 0,$$

where $A_n(\varepsilon) = \{k \leqslant n : \|x_k - \xi, z\| \geqslant \varepsilon\}$ and $\delta_n(A_n(\varepsilon)) = \frac{|A_n(\varepsilon)|}{n}$.

If (x_k) is $\mathcal{I}$-convergent to ξ then we write $\mathcal{I}$-*st*-$\lim_{k\to\infty}\|x_k - \xi, z\| = 0$ or $\mathcal{I}$ -*st*-$\lim_{k\to\infty}\|x_k, z\| = \|\xi, z\|$. The number ξ is $\mathcal{I}$-limit of the sequence (x_k).

Remark 1. If $\{x_k\}$ is any sequence in X and ξ is any element of X, then the set

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - \xi, z\| \geqslant \varepsilon\}| \geqslant \delta\right\} = \emptyset,$$

since if $z = \overrightarrow{0}$ (0 vector), $\|x_k - \xi, z\| = 0 \ngeqslant \varepsilon$ so the above set is empty.

Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}$-statistically convergent is the statistical convergence introduced by [3,15].

We next provide a proof of the fact that uniqueness of the limit of $\mathcal{I}$-statistically convergent sequence in 2-normed space $(X, \|\cdot\cdot\|)$ can be verified as follows.

Theorem 1. *Let $\mathcal{I}$ be an admissible ideal and $L, L' \in X$. For each $z \in X$, If $\mathcal{I}$-st-lim$_{k\to\infty}\|x_k, z\| = \|L, z\|$ and $\mathcal{I}$-st-lim$_{k\to\infty}\|x_k, z\| = \|L', z\|$, then $L = L'$.*

Proof:. Assume $L \neq L'$. Then $L - L' \neq \overrightarrow{0}$, so there exists a nonzero $z \in X$, such that $L - L'$ and z are linearly independent (such a z exists since $d \geqslant 2$). Therefore for every $\varepsilon > 0$ and $\delta > 0$

$$\frac{1}{n}|\{k \leqslant n : \|L - L', z\| \geqslant \varepsilon\}| = 2\delta, \quad \text{with } \delta > 0.$$

Now

$$\begin{aligned} 2\delta &= \frac{1}{n}|\{k \leqslant n : \|(L - x_k) + (x_k - L'), z\| \geqslant \varepsilon\}| \\ &\leqslant \frac{1}{n}|\{k \leqslant n : \|x_k - L', z\| \geqslant \varepsilon\}| + \frac{1}{n}|\{k \leqslant n : \|x_k - L, z\| \geqslant \varepsilon\}|. \end{aligned}$$

So

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - L', z\| \geqslant \varepsilon\}| < \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - L, z\| \geqslant \varepsilon\}| \geqslant \delta\right\}.$$

But $\delta_{\mathcal{I}}\left(\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - L', z\| \geqslant \varepsilon\}| < \delta\right\}\right) = 0$ then contradicting the fact that $x_n \to L'(\mathcal{I}\text{-}stat)$. □

Theorem 2. *Let $\mathcal{I}$ be an admissible ideal. For each $z \in X$,*

(i) *If* $\mathcal{I}$-st-$\lim_{k\to\infty}\|x_k, z\| = \|x, z\|, \mathcal{I}$-$st$-$lim_{k\to\infty}\| \quad y_k,z\| = \|y,z\|$ *then* $\mathcal{I}$-st-$lim_{k\to\infty}\|x_k + y_k,z\| = \|x + y,z\|$;
(ii) $\mathcal{I}$-st-$\lim_{k\to\infty}\|ax_k, z\| = \|ax, z\|, a \in \mathbb{R}$;

Proof:. (i) Assumed that $\mathcal{I}$-st-$\lim_{k\to\infty}\| x_k,z\| = \|x,z\|$ and $\mathcal{I}$-st-$\lim_{k\to\infty}\|y_k,z\| = \|y,z\|$ for every nonzero $z \in X$. Then $\delta_{\mathcal{I}}(K_1) = 0$ and $\delta_{\mathcal{I}}(K_2) = 0$, where

$$K_1 = K_1(\varepsilon) := \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - x, z\| \geqslant \varepsilon\}| \geqslant \frac{\delta}{2}\right\}$$

and

$$K_2 = K_2(\varepsilon) := \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|y_k - y, z\| \geqslant \varepsilon\}| \geqslant \frac{\delta}{2}\right\}$$

for each $z \in X$. Let

$$K = K(\varepsilon) := \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|(x_k + y_k) - (x + y), z\| \geqslant \varepsilon\}| \geqslant \delta\right\}.$$

To prove that $\delta_{\mathcal{I}}(K) = 0$, it suffices to show that $K \subset K_1 \cup K_2$. Suppose $k_0 \in K$. Then

$$\frac{1}{n}|\{k \leqslant n : \|x_{k_0} + y_{k_0} - (x + y), z\| \geqslant \varepsilon\}| \geqslant \delta. \tag{2.1}$$

Suppose to the contrary, that $k_0 \notin K_1 \cup K_2$. Then $k_0 \notin K_1$ and $k_0 \notin K_2$. If $k_0 \notin K_1$ and $k_0 \notin K_2$ then $\frac{1}{n}|\{k \leqslant n : \|x_{k_0} - x, z\| \geqslant \varepsilon\}| < \frac{\delta}{2}$ and $\frac{1}{n}|\{k \leqslant n : \|y_{k_0} - y, z\| \geqslant \varepsilon\}| < \frac{\delta}{2}$. Then, we get

$$\begin{aligned}\frac{1}{n}|\{k \leqslant n : \|(x_{k_0} + y_{k_0}) - (x + y), z\| \geqslant \varepsilon\}| &\leqslant \frac{1}{n}|\{k \leqslant n : \|x_{k_0} - x, z\| \geqslant \varepsilon\}| \\ &\quad + \frac{1}{n}|\{k \leqslant n : \|y_{k_0} - y, z\| \geqslant \varepsilon\}| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta,\end{aligned}$$

which contradicts (2.1). Hence $k_0 \in K_1 \cup K_2$, that is, $K \subset K_1 \cup K_2$. This gives that $\mathcal{I}$-*st*-$\lim_{k\to\infty}\|x_k + y_k,z\| = \|x + y,z\|$ for every nonzero $z \in X$, and this completes the proof of *(i)*.

(ii) Let $\mathcal{I}$-*st*-$\lim_{k\to\infty}\|x_k, z\| = \|x, z\|, a \in \mathbb{R}$ and $a \neq 0$. Then

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leqslant n : \|x_k - x, z\| \geqslant \frac{\varepsilon}{|a|}\right\}\right| \geqslant \delta\right\} \in \mathcal{I}. \tag{2.2}$$

Then, since $\|ax_k - ax,z\| = |a|\,\|x_k - x,z\|$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|ax_k - ax, z\| \geqslant \varepsilon\}| \geqslant \delta\right\}$$
$$= \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leqslant n : \|x_k - x, z\| \geqslant \frac{\varepsilon}{|a|}\right\}\right| \geqslant \delta\right\}.$$

Hence, from (2.2) equality we get $\mathcal{I}$-*st*-$\lim_{k\to\infty}\|ax_k,z\| = \|ax,z\|$ for every nonzero $z \in X$. □

Recall that we assume X to have dimension d, where $2 \leqslant d < \infty$, unless otherwise stated. Fix $u = \{u_1, \ldots, u_d\}$ to be a basis for X. Then we have the following:

Lemma 1. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_k) in X is $\mathcal{I}$-statistically convergent to x in X if and only if $\mathcal{I}$-st-$\lim_{k\to\infty}\|x_k - x,u_i\| = 0$ for every $i = 1, \ldots, d$.*

Proof:. It suffices to prove that if $\mathcal{I}$-*st*-$\lim_{k\to\infty}\|x_k - x,u_i\| = 0$ for every $i = 1, \ldots, d$, then we have $\mathcal{I}$-*st*-$\lim_{k\to\infty}\|x_k - x,z\| = 0$ for every nonzero $z \in X$. But this is clear since every $z \in X$ can be written as $z = \alpha_1 u_1 + \ldots + \alpha_d u_d$ for some $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$, and by the triangle inequality we have

$$\|x_k - x, z\| \leqslant |\alpha_1|\|x_k - x, u_1\| + \ldots + |\alpha_d|\|x_k - x, u_d\|$$

for all $k \in \mathbb{N}$. Then

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - x, z\| \geqslant \varepsilon\}| \geqslant \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leqslant n : \|x_k - x, u_1\| \geqslant \frac{\varepsilon}{|\alpha_1|}\right\}\right| \geqslant \delta\right\}$$
$$\cup \cdots \cup \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leqslant n : \|x_k - x, u_d\| \geqslant \frac{\varepsilon}{|\alpha_d|}\right\}\right| \geqslant \delta\right\}.$$

Since the right hand side of the above inclusion belongs to ideal, so does the left hand side. Consequently, we get $\mathcal{I}$-*st*-$\lim_{k\to\infty}\|x_k - x,z\| = 0$ for every nonzero $z \in X$. This proves the result. □

Following Lemma 2, we have :

Lemma 2. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_k) in X is $\mathcal{I}$-statistically convergent to x in X if and only if $\mathcal{I}$-st-$\lim_{k\to\infty}\max\{\|x_k - x,u_i\| = 0, i = 1, \ldots, d\} = 0$.*

This simple fact enlightens us to define a norm on X as follows: With respect to the basis $u = \{u_1, \dots, u_d\}$, we can define a norm on X, which we shall denote it by $\|.\|_\infty$, by

$$\|x\|_\infty := \max\{\|x, u_i\| : i = 1, \dots, d\}.$$

Using the derived norm $\|\cdot\|_\infty$, Lemma 3 now reads:

Lemma 3. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_k) in X is $\mathcal{I}$-statistically convergent to x in X if and only if $\mathcal{I}$-st-$\lim_{k\to\infty}\| x_k - x, u_i\|_\infty = 0$.*

Associated to the derived norm $\|\cdot\|_\infty$, we can define the balls $B_u(x,\varepsilon)$ centered at x having radius ε by

$$B_u(x, \varepsilon) := \{y : \|x - y\|_\infty \leqslant \varepsilon\},$$

where $\|x - y\|_\infty := \max\{\|x - y, u_j\|, j = 1, \dots, d.\}$.

Using these balls, Lemma 4 becomes:

Lemma 4. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_k) in X is $\mathcal{I}$-statistically convergent to x in X if and only if $\delta_{\mathcal{I}}(A_n(\varepsilon)) = 0$, where $A_n(\varepsilon) = \{k \leqslant n : x_k \notin B_u(x,\varepsilon)\}$.*

We introduce the $\mathcal{I}$-statistical Cauchy sequence criterion in 2-normed space $(X,\|\cdot\cdot\|)$.

Definition 5. A sequence $\{x_k\}$in 2-normed space $(X,\|\cdot\cdot\|)$ is said to be $\mathcal{I}$-statistically Cauchy sequence in X if for every $\varepsilon > 0$, $\delta > 0$ and every nonzero $z \in X$ there exists a number N such that

$$\delta_{\mathcal{I}}\left(\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - x_N, z\| \geqslant \varepsilon\}| \geqslant \delta\right\}\right) = 0,$$

i.e., for every nonzero $z \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leqslant n : \|x_k - x_N, z\| \geqslant \varepsilon\}| \geqslant \delta\right\} \in \mathcal{I}.$$

Now we give a similar result as in [16] (see Lemma 2.5).

Theorem 3. *In any 2-normed space $(X,\|\cdot\cdot\|)$, any $\mathcal{I}$-statistically Cauchy sequence is $\mathcal{I}$-statistically convergent if and only if any $\mathcal{I}$-statistically Cauchy sequence with respect to $\|.\|_\infty$ is $\mathcal{I}$-statistically convergent.*

Proof:. Recall that $\mathcal{I}$-statistically convergence in the 2-norm is equivalent to that in the $\|.\|_\infty$ norm. That is,

$$\mathcal{I}\text{-}st\text{-}\lim_{k\to\infty}\|x_k - x, z\| = 0, \qquad \forall z \in X \Longleftrightarrow \mathcal{I}\text{-}st\text{-}\lim_{k\to\infty}\|x_k - x\|_\infty = 0.$$

It is sufficient to show that (x_k) is a $\mathcal{I}$-statistically Cauchy sequence with respect to the 2-norm iff it is $\mathcal{I}$- statistically Cauchy sequence with respect to the norm $\|.\|_\infty$. But it can be done easily very similar to that in ([16], Lemma 2.5) with only minor changes using the ideal. □

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