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Hyper-order and fixed points of meromorphic solutions of higher order linear differential equations

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Abstract. The purpose of this paper is to study the growth and fixed points of meromorphic solutions and their derivatives to complex higher order linear differential equations whose coefficients are meromorphic functions. Our results extend the previous results due to Peng and Chen, Xu and Zhang and others.

Keywords: Linear differential equations; Meromorphic solutions; Order of growth; Hyper-order; Fixed points

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1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [11,17]). In addition, we will use notations $\sigma(f)$, $\sigma_2(f)$ to denote respectively the order and the hyper-order of growth of a meromorphic function $f(z)$, $\lambda(f)$, $\overline{\lambda}(f)$, $\overline{\tau}(f)$ to denote respectively the exponents of convergence of the zero-sequence, the sequence of distinct zeros and the sequence of distinct fixed points of $f(z)$. See [2,11,14,17] for notations and definitions.

Consider the second order linear differential equation

$$f'' + A_1(z)\, e^{P(z)} f' + A_0(z)\, e^{Q(z)} f = 0, \tag{1.1}$$

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where $P(z), Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_0(z)\,(\not\equiv 0)$ are entire functions such that $\sigma(A_1) < \deg P(z)$, $\sigma(A_0) < \deg Q(z)$. Gundersen showed in [9, p. 419] that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.1) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.1) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f'' + e^z f' - e^z f = 0$.

In [3], Chen and Shon investigated the case when $\deg P(z) = \deg Q(z)$ and proved the following results.

Theorem A (*[3]*)**.** *Let $A_j(z)\,(\not\equiv 0)\,(j = 0, 1)$ be meromorphic functions with $\sigma(A_j) < 1\,(j = 0, 1)$, a, b be complex numbers such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation*

$$f'' + A_1(z)\,e^{az} f' + A_0(z)\,e^{bz} f = 0 \tag{1.2}$$

has infinite order.

In the same paper, Chen and Shon investigated the fixed points of solutions, their 1st and 2nd derivatives and the differential polynomials and obtained.

Theorem B (*[3]*)**.** *Let $A_j(z)\,(j = 0, 1), a, b, c$ satisfy the additional hypotheses of Theorem A. Let d_0, d_1, d_2 be complex constants that are not all equal to zero. If $f(z) \not\equiv 0$ is any meromorphic solution of Eq. (1.2), then:*

(i) *f, f', f'' all have infinitely many fixed points and satisfy*

$$\overline{\lambda}(f - z) = \overline{\lambda}(f' - z) = \overline{\lambda}(f'' - z) = \infty,$$

(ii) *the differential polynomial*

$$g(z) = d_2 f'' + d_1 f' + d_0 f$$

has infinitely many fixed points and satisfies $\overline{\lambda}(g - z) = \infty$.

In [13], Peng and Chen investigated the order and hyper-order of solutions of some second order linear differential equations and proved the following result.

Theorem C (*[13]*)**.** *Let $A_j(z)\,(\not\equiv 0)\,(j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leqslant |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f\,(\not\equiv 0)$ of the differential equation*

$$f'' + e^{-z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z})\, f = 0$$

has infinite order and $\sigma_2(f) = 1$.

Recently, Xu and Zhang investigated the order, the hyper-order and fixed points of meromorphic solutions of some second order linear differential equations and proved the following results.

Theorem D (*[16]*)**.** *Suppose that $A_j(z)\,(\not\equiv 0)\,(j = 0, 1, 2)$ are meromorphic functions and $\sigma(A_j) < 1$, and a_1, a_2 are two complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose* *that* $|a_1| \leqslant |a_2|$*). Let* a_0 *be a constant satisfying* $a_0 < 0$*. If* $\arg a_1 \neq \pi$ *or* $a_1 < a_0$*, then every meromorphic solution* $f\ (\not\equiv 0)$ *whose poles are of uniformly bounded multiplicities of the equation*

$$f'' + A_0 e^{a_0 z} f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0 \tag{1.3}$$

has infinite order and $\sigma_2(f) = 1$.

Theorem E *([16]). Let* $A_j(z)\, a_j$ *satisfy the additional hypotheses of Theorem D. If* $\varphi\ (\not\equiv 0)$ *is a meromorphic function whose order is less than* 1*, then every meromorphic solution* $f\ (\not\equiv 0)$ *whose poles are of uniformly bounded multiplicities of Eq. (1.3) satisfies*

$$\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty.$$

The main purpose of this paper is to extend and improve the results of Theorems D, E to some higher order linear differential equations. The present article may be understood as an extension and improvement of the recent article of the authors [10]. In fact we will prove the following results.

Theorem 1.1. *Let* $A_j(z)\,(\not\equiv 0)\,(j = 0, 1, 2)$ *and* $B_l(z)\,(l = 2, \ldots, k-1)$ *be meromorphic functions with*

$$\max\{\sigma(A_j)\ \ (j = 0, 1, 2), \sigma(B_l)\ \ (l = 2, \ldots, k-1)\} < 1,$$

a_1*,* a_2 *be complex numbers such that* $a_1 a_2 \neq 0$*,* $a_1 \neq a_2$ *(suppose that* $|a_1| \leqslant |a_2|$*). Let* a_0 *be a constant satisfying* $a_0 < 0$*. If* $\arg a_1 \neq \pi$ *or* $a_1 < a_0$*, then every meromorphic solution* $f\ (\not\equiv 0)$ *whose poles are of uniformly bounded multiplicities of the equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_2 f'' + A_0 e^{a_0 z} f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0 \tag{1.4}$$

satisfies $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$.

Example 1.1. Consider the differential equation

$$f''' + \frac{4}{z} f'' + \left(-\frac{1}{z} - \frac{1}{2} z - 1\right) e^{-z} f' + \left(\left(\frac{1}{z} - \frac{1}{2} z + 2\right) e^{-2z} + e^{-3z}\right) f = 0, \tag{1.5}$$

where $B_2(z) = \frac{4}{z}$, $A_0(z) = -\frac{1}{z} - \frac{1}{2} z - 1$, $a_0 = -1$, $A_1(z) = \frac{1}{z} - \frac{1}{2} z + 2$, $a_1 = -2$, $A_2(z) = 1$, $a_2 = -3$, $a_1 < a_0$. Obviously, the conditions of Theorem 1.1 are satisfied. The meromorphic function $f(z) = \frac{1}{z^2} e^{e^{-z}}$, with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$, is a solution of (1.5).

Motivated by Theorem E, we try to consider the relation between small functions with meromorphic solutions of Eq. (1.4). Indeed, such relationship on higher order differential equations is more difficult than that of second order differential equations. Moreover, the method used in the proof of Theorem E can not deal with the case of higher order linear differential equations.

Theorem 1.2. *Let* $A_j(z)\,(j=0,1,2)\,,B_l(z)\,(l=2,\ldots,k-1),$ $a_0,$ $a_1,$ a_2 *satisfy the additional hypotheses of Theorem 1.1. If* $\varphi\,(\not\equiv 0)$ *is a meromorphic function with order* $\sigma(\varphi)<1$*, then every meromorphic solution* $f\,(\not\equiv 0)$ *whose poles are of uniformly bounded multiplicities of Eq. (1.4) satisfies*

$$\overline{\lambda}(f-\varphi)=\overline{\lambda}(f'-\varphi)=\overline{\lambda}(f''-\varphi)=\infty.$$

Example 1.2. Let $\varphi(z)=-\frac{1}{z^2}$. The function $\varphi(z)$ is a meromorphic function with order $\sigma(\varphi)=0<1$. We have for the solution $f(z)=\frac{1}{z^2}e^{e^{-z}}$ of (1.5)

$$\begin{aligned}
\overline{\lambda}(f-\varphi) &= \overline{\lambda}\left(\frac{1}{z^2}e^{e^{-z}}+\frac{1}{z^2}\right)=\overline{\lambda}\left(e^{e^{-z}}+1\right)=+\infty,\\
\overline{\lambda}(f'-\varphi) &= \overline{\lambda}\left(\left(\frac{1}{z^2}e^{e^{-z}}\right)'+\frac{1}{z^2}\right)=\overline{\lambda}\left(-\frac{2}{z^3}e^{e^{-z}}+\frac{1}{z^2}\left(-e^{-z}\right)e^{e^{-z}}+\frac{1}{z^2}\right)\\
&= \overline{\lambda}\left(\left(-\frac{2}{z}-e^{-z}\right)e^{e^{-z}}+1\right)=+\infty,\\
\overline{\lambda}(f''-\varphi) &= \overline{\lambda}\left(\left(\frac{1}{z^2}e^{e^{-z}}\right)''+\frac{1}{z^2}\right)=\overline{\lambda}\left(\left(\left(-\frac{2}{z^3}-\frac{e^{-z}}{z^2}\right)e^{e^{-z}}\right)'+\frac{1}{z^2}\right)\\
&= \overline{\lambda}\left(\left(\frac{6}{z^4}+\left(\frac{1}{z^2}+\frac{4}{z^3}+\frac{e^{-z}}{z^2}\right)e^{-z}\right)e^{e^{-z}}+\frac{1}{z^2}\right)\\
&= \overline{\lambda}\left(\left(\frac{6}{z^2}+\left(1+\frac{4}{z}+e^{-z}\right)e^{-z}\right)e^{e^{-z}}+1\right)=+\infty.
\end{aligned}$$

By setting $\varphi(z)=z$ in Theorem 1.2, we obtain the following corollary.

Corollary 1.1. *Let* $A_j(z)\,(j=0,1,2),B_l(z)\,(l=2,\ldots,k-1),$ $a_0,$ $a_1,$ a_2 *satisfy the additional hypotheses of Theorem 1.1. If* $f\,(\not\equiv 0)$ *is any meromorphic solution whose poles are of uniformly bounded multiplicities of Eq. (1.4), then* $f,$ f' f'' *all have infinitely many fixed points and satisfy*

$$\overline{\tau}(f)=\overline{\tau}(f')=\overline{\tau}(f'')=\infty.$$

2. Preliminary lemmas

We define the linear measure of a set $E\subset[0,+\infty)$ by $m(E)=\int_0^{+\infty}\chi_E(t)dt$ and the logarithmic measure of a set $F\subset(1,+\infty)$ by $lm(F)=\int_1^{+\infty}\frac{\chi_F(t)}{t}dt$, where χ_H is the characteristic function of a set H.

Lemma 2.1 ([8]). *Let* f *be a transcendental meromorphic function with* $\sigma(f)=\sigma<+\infty$. *Let* $\varepsilon>0$ *be a given constant, and let* $k,$ j *be integers satisfying* $k>j\geqslant 0$. *Then, there* *exists a set* $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ *with linear measure zero, such that, if* $\psi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$*, then there is a constant* $R_0 = R_0(\psi) > 1$*, such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| \geqslant R_0$*, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Lemma 2.2 ([3,12]). *Consider* $g(z) = A(z)e^{az}$*, where* $A(z) \not\equiv 0$ *is a meromorphic function with order* $\sigma(A) = \alpha < 1$*,* a *is a complex constant,* $a = |a| e^{i\varphi}$ $(\varphi \in [0, 2\pi))$*. Set* $E_2 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$*, then* E_2 *is a finite set. Then for any given* $\varepsilon\, (0 < \varepsilon < 1 - \alpha)$ *there is a set* $E_3 \subset [0, 2\pi)$ *that has linear measure zero such that if* $z = re^{i\theta}$*,* $\theta \in [0, 2\pi) \diagdown (E_2 \cup E_3)$*, then we have when* r *is sufficiently large:*

(i) *If* $\cos(\varphi + \theta) > 0$*, then*

$$\exp\{(1-\varepsilon) r\delta(az, \theta)\} \leqslant |g(z)| \leqslant \exp\{(1+\varepsilon) r\delta(az, \theta)\}. \tag{2.2}$$

(ii) *If* $\cos(\varphi + \theta) < 0$*, then*

$$\exp\{(1+\varepsilon) r\delta(az, \theta)\} \leqslant |g(z)| \leqslant \exp\{(1-\varepsilon) r\delta(az, \theta)\}, \tag{2.3}$$

where $\delta(az, \theta) = |a| \cos(\varphi + \theta)$.

Lemma 2.3 ([13]). *Suppose that* $n \geqslant 1$ *is a natural number. Let* $P_j(z) = a_{jn}z^n + \cdots (j = 1, 2)$ *be nonconstant polynomials, where* a_{jq} $(q = 1, \ldots, n)$ *are complex numbers and* $a_{1n}a_{2n} \neq 0$*. Set* $z = re^{i\theta}$*,* $a_{jn} = |a_{jn}| e^{i\theta_j}$*,* $\theta_j \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$*,* $\delta(P_j, \theta) = |a_{jn}| \cos(\theta_j + n\theta)$*. Then there is a set* $E_4 \subset \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right)$ *that has linear measure zero such that if* $\theta_1 \neq \theta_2$*, then there exists a ray* $\arg z = \theta$ *with* $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_4 \cup E_5)$*, satisfying either*

$$\delta(P_1, \theta) > 0, \qquad \delta(P_2, \theta) < 0 \tag{2.4}$$

or

$$\delta(P_1, \theta) < 0, \qquad \delta(P_2, \theta) > 0, \tag{2.5}$$

where $E_5 = \left\{\theta \in \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right) : \delta(P_j, \theta) = 0\right\}$ *is a finite set, which has linear measure zero.*

Remark 2.1 ([13]). We can obtain, in Lemma 2.3, if $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_4 \cup E_5)$ is replaced by $\theta \in \left(\frac{\pi}{2n}, \frac{3\pi}{2n}\right) \setminus (E_4 \cup E_5)$, then it has the same result.

Lemma 2.4 ([3]). *Let* $f(z)$ *be a transcendental meromorphic function of order* $\sigma(f) = \alpha < +\infty$*. Then for any given* $\varepsilon > 0$*, there is a set* $E_6 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ *that has linear measure zero such that if* $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \diagdown E_6$*, then there is a constant* $R_1 = R_1(\theta) > 1$*, such that for all* z *satisfying* $\arg z = \theta$ *and* $|z| \geqslant R_1$*, we have*

$$\exp\left\{-r^{\alpha+\varepsilon}\right\} \leqslant |f(z)| \leqslant \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{2.6}$$

Lemma 2.5 ([6, p. 30]). *Let $n \geqslant 1$, $P_1, P_2, \ldots, P_n$ be nonconstant polynomials with degree d_1, d_2, ..., d_n. Suppose that when $i \neq j$, $\deg(P_i - P_j) = \max\{d_i, d_j\}$. Set $A(z) = \sum_{j=1}^{n} B_j(z) e^{P_j(z)}$, where $B_j(z)\,(\not\equiv 0)$ are meromorphic functions satisfying $\sigma(B_j) < d_j$. Then $\sigma(A) = \max_{1 \leqslant j \leqslant n}\{d_j\}$.*

Using mathematical induction, we can easily prove the following lemma.

Lemma 2.6. *Let $f(z) = g(z)/d(z)$, where $g(z)$ is a transcendental entire function, and let $d(z)$ be the canonical product (or polynomial) formed with the non-zero poles of $f(z)$. Then we have*

$$f^{(n)} = \frac{1}{d}\left[g^{(n)} + D_{n,n-1}g^{(n-1)} + D_{n,n-2}g^{(n-2)} + \cdots + D_{n,1}g' + D_{n,0}g\right] \tag{2.7}$$

and

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + D_{n,n-1}\frac{g^{(n-1)}}{g} + D_{n,n-2}\frac{g^{(n-2)}}{g} + \cdots + D_{n,1}\frac{g'}{g} + D_{n,0}, \tag{2.8}$$

where $D_{n,j}$ are defined as a sum of finite numbers of terms of the type

$$\sum_{(j_1 \cdots j_n)} C_{jj_1 \cdots j_n}\left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n},$$

$C_{jj_1\cdots j_n}$ are constants, and $j + j_1 + 2j_2 + \cdots + nj_n = n$.

Lemma 2.7 ([1]). *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then f satisfies $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$.

The following lemma, due to Gross [7], is important in the factorization and uniqueness theory of meromorphic functions, playing an important role in this paper as well.

Lemma 2.8 ([7,17]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)\,(n \geqslant 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions:*
(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leqslant j < k \leqslant n$;
(iii) *For* $1 \leqslant j \leqslant n$ *and* $1 \leqslant h < k \leqslant n$, $T(r, f_j) = o\left\{T\left(r, e^{g_h(z) - g_k(z)}\right)\right\}$ $(r \to \infty,$ $r \notin E_7)$, *where E_7 is a set of finite linear measure.*
Then $f_j(z) \equiv 0\ (j = 1, \ldots, n)$.

Lemma 2.9 ([15]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)\,(n \geqslant 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions:*
(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv f_{n+1}$;

(ii) *If* $1 \leqslant j \leqslant n+1$ *and* $1 \leqslant k \leqslant n$, *then the order of* f_j *is less than the order of* $e^{g_k(z)}$. *If* $n \geqslant 2, 1 \leqslant j \leqslant n+1$ *and* $1 \leqslant h < k \leqslant n$, *then the order of* f_j *is less than the order of* $e^{g_h - g_k}$.

Then $f_j(z) \equiv 0 \, (j = 1, 2, \ldots, n+1)$.

Lemma 2.10 ([8]). *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_8 \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* i, j $(0 \leqslant i < j \leqslant k)$, *such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_8$, *we have*

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leqslant B \left\{ \frac{T(\alpha r, f)}{r} \left(\log^{\alpha} r\right) \log T(\alpha r, f) \right\}^{j-i}. \tag{2.9}$$

Lemma 2.11 ([9]). *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *and* $\psi : [0, +\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leqslant \psi(r)$ *for all* $r \notin E_9 \cup [0,1]$, *where* $E_9 \subset (1, +\infty)$ *is a set of finite logarithmic measure. Let* $\gamma > 1$ *be a given constant. Then there exists an* $r_1 = r_1(\gamma) > 0$ *such that* $\varphi(r) \leqslant \psi(\gamma r)$ *for all* $r > r_1$.

Lemma 2.12 ([4]). *Let* $A_0, A_1, \ldots, A_{k-1}$ $(k \geqslant 2)$ *be meromorphic functions such that* $\sigma = \max\{\sigma(A_j), \ j = 0, \ldots, k-1\}$. *Then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicity of the differential equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0$$

satisfies $\sigma_2(f) \leqslant \sigma$.

3. Proof of Theorem 1.1

First of all we prove that Eq. (1.4) can't have a meromorphic solution $f \not\equiv 0$ with $\sigma(f) < 1$. Assume a meromorphic solution $f \not\equiv 0$ with $\sigma(f) < 1$. Rewrite (1.4) as

$$A_0 f' e^{a_0 z} + A_1 f e^{a_1 z} + A_2 f e^{a_2 z} = -\left\{ f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_2 f'' \right\}. \tag{3.1}$$

For $a_2 \neq a_0$, by (3.1) and Lemma 2.5, we have

$$\begin{aligned} 1 &= \sigma\left\{A_0 f' e^{a_0 z} + A_1 f e^{a_1 z} + A_2 f e^{a_2 z}\right\} \\ &= \sigma\left[-\left\{f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_2 f''\right\}\right] < 1. \end{aligned}$$

This is a contradiction. For $a_2 = a_0$, by (3.1) and Lemma 2.5, we have

(i) If $A_0 f' + A_2 f \not\equiv 0$, then

$$\begin{aligned} 1 &= \sigma\left\{(A_0 f' + A_2 f) e^{a_0 z} + A_1 f e^{a_1 z}\right\} \\ &= \sigma\left[-\left\{f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_2 f''\right\}\right] < 1. \end{aligned}$$

This is a contradiction.

(ii) If $A_0 f' + A_2 f \equiv 0$, then

$$1 = \sigma\left\{A_1 f e^{a_1 z}\right\} = \sigma\left[-\left\{f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_2 f''\right\}\right] < 1.$$

This is a contradiction. Therefore $\sigma(f) \geqslant 1$.

First step. We prove that $\sigma(f) = +\infty$. Assume that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of Eq. (1.4) with $1 \leqslant \sigma(f) = \sigma < +\infty$. From Eq. (1.4), we know that the poles of $f(z)$ can occur only at the poles of A_j $(j = 0, 1, 2)$ and B_l $(l = 2, \ldots, k-1)$. Note that the multiplicities of poles of f are uniformly bounded, and thus we have [5]

$$\begin{aligned} N(r, f) &\leqslant M_1 \overline{N}(r, f) \leqslant M_1 \left(\sum_{j=0}^{2} \overline{N}(r, A_j) + \sum_{l=2}^{k-1} \overline{N}(r, B_l)\right) \\ &\leqslant M \max\left\{N(r, A_j)\ \ (j = 0, 1, 2), N(r, B_l)\,(l = 2, \ldots, k-1)\right\}, \end{aligned}$$

where M_1 and M are some suitable positive constants. This gives $\lambda\left(\frac{1}{f}\right) \leqslant \alpha = \max\left\{\sigma(A_j)\ \ (j = 0, 1, 2), \sigma(B_l)\,(l = 2, \ldots, k-1)\right\} < 1$. Let $f = g/d$, d be the canonical product formed with the nonzero poles of $f(z)$, with $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) = \beta \leqslant \alpha < 1$, g be an entire function and $1 \leqslant \sigma(g) = \sigma(f) = \sigma < \infty$. Substituting $f = g/d$ into (1.4), by Lemma 2.6 we can get

$$\begin{aligned} &\frac{g^{(k)}}{g} + \left[B_{k-1} + D_{k,k-1}\right]\frac{g^{(k-1)}}{g} + \left[B_{k-2} + B_{k-1} D_{k-1,k-2} + D_{k,k-2}\right]\frac{g^{(k-2)}}{g} \\ &\quad + \cdots + \left[B_2 + D_{k,2} + \sum_{i=3}^{k-1} B_i D_{i,2}\right]\frac{g''}{g} + \left[A_0 e^{a_0 z} + D_{k,1} + \sum_{i=2}^{k-1} B_i D_{i,1}\right]\frac{g'}{g} \\ &\quad + A_0 D_{1,0} e^{a_0 z} + \sum_{i=2}^{k-1} B_i D_{i,0} + D_{k,0} + A_1 e^{a_1 z} + A_2 e^{a_2 z} = 0. \end{aligned} \tag{3.2}$$

By Lemma 2.4, for any given ε $(0 < \varepsilon < 1 - \alpha)$, there is a set $E_6 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ that has linear measure zero such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_6$, then there is a constant $R_1 = R_1(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| \geqslant R_1$, we have

$$|A_0(z)| \leqslant \exp\left\{r^{\alpha+\varepsilon}\right\}, \qquad |B_l(z)| \leqslant \exp\left\{r^{\alpha+\varepsilon}\right\} \quad (l = 2, \ldots, k-1). \tag{3.3}$$

By Lemma 2.1, for any given $\varepsilon\left(0 < \varepsilon < \min\left\{\frac{|a_2| - |a_1|}{|a_2| + |a_1|}, 1 - \alpha\right\}\right)$, there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ of linear measure zero, such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geqslant R_0$, we have

$$\left|\frac{g^{(j)}(z)}{g(z)}\right| \leqslant r^{k(\sigma - 1 + \varepsilon)}, \quad j = 1, \ldots, k, \tag{3.4}$$

$$\left|\frac{d^{(j)}(z)}{d(z)}\right| \leqslant r^{k(\beta - 1 + \varepsilon)}, \quad j = 1, \ldots, k \tag{3.5}$$

and

$$|D_{k,j}| = \left| \sum_{(j_1\cdots j_k)} C_{jj_1\cdots j_k} \left(\frac{d'}{d}\right)^{j_1} \left(\frac{d''}{d}\right)^{j_2} \cdots \left(\frac{d^{(k)}}{d}\right)^{j_k} \right|$$
$$\leqslant \sum_{(j_1\cdots j_k)} |C_{jj_1\cdots j_k}| \left|\frac{d'}{d}\right|^{j_1} \left|\frac{d''}{d}\right|^{j_2} \cdots \left|\frac{d^{(k)}}{d}\right|^{j_k}$$
$$\leqslant \sum_{(j_1\cdots j_k)} |C_{jj_1\cdots j_k}| r^{j_1(\beta-1+\varepsilon)} r^{2j_2(\beta-1+\varepsilon)} \cdots r^{kj_k(\beta-1+\varepsilon)}$$
$$= \sum_{(j_1\cdots j_k)} |C_{jj_1\cdots j_k}| r^{(j_1+2j_2+\cdots+kj_k)(\beta-1+\varepsilon)}. \tag{3.6}$$

By $j_1 + \cdots + kj_k = k - j \leqslant k$ and (3.6), we have

$$|D_{k,j}| \leqslant M r^{k(\beta-1+\varepsilon)}, \tag{3.7}$$

where $M > 0$ is a some constant. Let $z = re^{i\theta}$, $a_1 = |a_1| e^{i\theta_1}$, $a_2 = |a_2| e^{i\theta_2}$, $\theta_1, \theta_2 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Case 1. $\arg a_1 \neq \pi$, which is $\theta_1 \neq \pi$.

(i) Assume that $\theta_1 \neq \theta_2$. By Lemmas 2.2 and 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ (where E_4 and E_5 are defined as in Lemma 2.3, $E_1 \cup E_4 \cup E_5 \cup E_6$ is of linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0, \qquad \delta(a_2 z, \theta) < 0$$

or

$$\delta(a_1 z, \theta) < 0, \qquad \delta(a_2 z, \theta) > 0.$$

When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get

$$|A_1 e^{a_1 z}| \geqslant \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\}, \tag{3.8}$$
$$|A_2 e^{a_2 z}| \leqslant \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} < 1. \tag{3.9}$$

By (3.8) and (3.9) we have

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geqslant |A_1 e^{a_1 z}| - |A_2 e^{a_2 z}| \geqslant \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} - 1$$
$$\geqslant (1 - o(1)) \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\}. \tag{3.10}$$

By (3.2), we get

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \leqslant \left|\frac{g^{(k)}}{g}\right| + |B_{k-1} + D_{k,k-1}| \left|\frac{g^{(k-1)}}{g}\right|$$
$$+ |B_{k-2} + B_{k-1} D_{k-1,k-2} + D_{k,k-2}| \left|\frac{g^{(k-2)}}{g}\right| + \cdots$$

$$+\left|B_2+D_{k,2}+\sum_{i=3}^{k-1}B_iD_{i,2}\right|\left|\frac{g''}{g}\right|+\left[|A_0|\left|e^{a_0z}\right|+\left|D_{k,1}+\sum_{i=2}^{k-1}B_iD_{i,1}\right|\right]\left|\frac{g'}{g}\right|$$

$$+\left|A_0D_{1,0}\right|\left|e^{a_0z}\right|+\sum_{i=2}^{k-1}\left|B_iD_{i,0}\right|+\left|D_{k,0}\right|. \tag{3.11}$$

Since $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, it follows that $\left|e^{a_0z}\right|=e^{a_0r\cos\theta}<1$. Substituting (3.3), (3.4), (3.7) and (3.10) into (3.11), we obtain

$$(1-o(1))\exp\left\{(1-\varepsilon)\,\delta(a_1z,\theta)\,r\right\}\leqslant M_1r^{M_2}\exp\left\{r^{\alpha+\varepsilon}\right\}, \tag{3.12}$$

where $M_1>0$ and $M_2>0$ are some constants. By $\delta(a_1z,\theta)>0$ and $\alpha+\varepsilon<1$ we know that (3.12) is a contradiction. When $\delta(a_1z,\theta)<0$, $\delta(a_2z,\theta)>0$, using a proof similar to the above, we can also get a contradiction.

(ii) Assume that $\theta_1=\theta_2$. By Lemma 2.3, for the above ε, there is a ray $\arg z=\theta$ such that $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$ and $\delta(a_1z,\theta)>0$. Since $|a_1|\leqslant|a_2|$, $a_1\neq a_2$ and $\theta_1=\theta_2$, it follows that $|a_1|<|a_2|$, thus $\delta(a_2z,\theta)>\delta(a_1z,\theta)>0$. For sufficiently large r, we have by Lemma 2.2

$$\left|A_1e^{a_1z}\right|\leqslant\exp\left\{(1+\varepsilon)\,\delta(a_1z,\theta)\,r\right\}, \tag{3.13}$$

$$\left|A_2e^{a_2z}\right|\geqslant\exp\left\{(1-\varepsilon)\,\delta(a_2z,\theta)\,r\right\}. \tag{3.14}$$

By (3.13) and (3.14) we get

$$\begin{aligned}\left|A_1e^{a_1z}+A_2e^{a_2z}\right|&\geqslant\left|A_2e^{a_2z}\right|-\left|A_1e^{a_1z}\right|\\&\geqslant\exp\left\{(1-\varepsilon)\,\delta(a_2z,\theta)\,r\right\}-\exp\left\{(1+\varepsilon)\,\delta(a_1z,\theta)\,r\right\}\\&=\exp\left\{(1+\varepsilon)\,\delta(a_1z,\theta)\,r\right\}\left[\exp\{\eta r\}-1\right],\end{aligned} \tag{3.15}$$

where

$$\eta=(1-\varepsilon)\,\delta(a_2z,\theta)-(1+\varepsilon)\,\delta(a_1z,\theta).$$

Since $0<\varepsilon<\frac{|a_2|-|a_1|}{|a_2|+|a_1|}$, it follows that

$$\begin{aligned}\eta&=(1-\varepsilon)\,|a_2|\cos(\theta_2+\theta)-(1+\varepsilon)\,|a_1|\cos(\theta_1+\theta)\\&=(1-\varepsilon)\,|a_2|\cos(\theta_1+\theta)-(1+\varepsilon)\,|a_1|\cos(\theta_1+\theta)\\&=\left[(1-\varepsilon)\,|a_2|-(1+\varepsilon)\,|a_1|\right]\cos(\theta_1+\theta)\\&=\left[|a_2|-|a_1|-\varepsilon\,(|a_2|+|a_1|)\right]\cos(\theta_1+\theta)>0.\end{aligned}$$

Then, from (3.15), we get

$$\left|A_1e^{a_1z}+A_2e^{a_2z}\right|\geqslant(1-o(1))\exp\left\{\left[(1+\varepsilon)\,\delta(a_1z,\theta)+\eta\right]r\right\}. \tag{3.16}$$

Since $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, it follows that $\left|e^{a_0z}\right|=e^{a_0r\cos\theta}<1$. Substituting (3.3), (3.4), (3.7) and (3.16) into (3.11), we obtain

$$(1-o(1))\exp\left\{\left[(1+\varepsilon)\,\delta(a_1z,\theta)+\eta\right]r\right\}\leqslant M_1r^{M_2}\exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{3.17}$$

By $\delta(a_1z,\theta)>0$, $\eta>0$ and $\alpha+\varepsilon<1$ we know that (3.17) is a contradiction.

Case 2. $a_1 < a_0$, which is $\theta_1 = \pi$.

(i) Assume that $\theta_1 \neq \theta_2$, then $\theta_2 \neq \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ and $\delta(a_2 z, \theta) > 0$. Because $\cos\theta > 0$, we have $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos\theta < 0$. For sufficiently large r, we obtain by Lemma 2.2

$$|A_1 e^{a_1 z}| \leqslant \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} < 1, \tag{3.18}$$

$$|A_2 e^{a_2 z}| \geqslant \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\}. \tag{3.19}$$

By (3.18) and (3.19) we obtain

$$\begin{aligned}|A_1 e^{a_1 z} + A_2 e^{a_2 z}| &\geqslant |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}| \geqslant \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} - 1 \\ &\geqslant (1 - o(1)) \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\}.\end{aligned} \tag{3.20}$$

Since $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows that $|e^{a_0 z}| = e^{a_0 r \cos\theta} < 1$. Substituting (3.3), (3.4), (3.7) and (3.20) into (3.11), we obtain

$$(1 - o(1)) \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} \leqslant M_1 r^{M_2} \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{3.21}$$

By $\delta(a_2 z, \theta) > 0$ and $\alpha + \varepsilon < 1$ we know that (3.21) is a contradiction.

(ii) Assume that $\theta_1 = \theta_2$, then $\theta_1 = \theta_2 = \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, then $\cos\theta < 0$, $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos\theta > 0$, $\delta(a_2 z, \theta) = |a_2| \cos(\theta_2 + \theta) = -|a_2| \cos\theta > 0$. Since $|a_1| \leqslant |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, it follows that $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta)$, for sufficiently large r, we get (3.13), (3.14) and (3.16) hold. Since $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, it follows that $|e^{a_0 z}| = e^{a_0 r \cos\theta} > 1$. Substituting (3.3), (3.4), (3.7) and (3.16) into (3.11), we obtain

$$(1 - o(1)) \exp\{[(1+\varepsilon)\delta(a_1 z, \theta) + \eta] r\} \leqslant M_1 r^{M_2} \exp\left\{r^{\alpha+\varepsilon}\right\} e^{a_0 r \cos\theta}. \tag{3.22}$$

Thus

$$(1 - o(1)) \exp\{\gamma r\} \leqslant M_1 r^{M_2} \exp\left\{r^{\alpha+\varepsilon}\right\}, \tag{3.23}$$

where $\gamma = (1+\varepsilon)\delta(a_1 z, \theta) + \eta - a_0 \cos\theta$. Since $\eta > 0$, $\cos\theta < 0$, $\delta(a_1 z, \theta) = -|a_1| \cos\theta$, $a_1 < a_0$, it follows that

$$\begin{aligned}\gamma &= -(1+\varepsilon)|a_1| \cos\theta - a_0 \cos\theta + \eta = -[(1+\varepsilon)|a_1| + a_0] \cos\theta + \eta \\ &> -[-(1+\varepsilon) a_0 + a_0] \cos\theta + \eta = \varepsilon a_0 \cos\theta + \eta > 0.\end{aligned}$$

By $\alpha + \varepsilon < 1$, we know that (3.23) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = \sigma(g) = +\infty$.

Second step. We prove that $\sigma_2(f) = 1$. By

$$\max\left\{\sigma\left(A_0 e^{a_0 z}\right), \sigma\left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right), \sigma(B_l) \ \ (l = 2, \ldots, k-1)\right\} = 1$$

and Lemma 2.12, we obtain $\sigma_2(f) \leqslant 1$. By Lemma 2.10, we know that there exists a set $E_8 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B [T(2r, f)]^{j+1} \quad (j = 1, \ldots, k). \tag{3.24}$$

By (1.4), we have

$$A_1e^{a_1z}+A_2e^{a_2z}=-\left[\frac{f^{(k)}}{f}+B_{k-1}\frac{f^{(k-1)}}{f}+\cdots+B_2\frac{f''}{f}+A_0e^{a_0z}\frac{f'}{f}\right]. \quad (3.25)$$

Case 1. $\arg a_1\neq\pi$.

(i) $(\theta_1\neq\theta_2)$ In first step, we have proved that there is a ray $\arg z=\theta$ where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$, satisfying

$$\delta(a_1z,\theta)>0,\qquad \delta(a_2z,\theta)<0\quad\text{or}\quad \delta(a_1z,\theta)<0,\qquad \delta(a_2z,\theta)>0.$$

(a) When $\delta(a_1z,\theta)>0$, $\delta(a_2z,\theta)<0$, for sufficiently large r, we get (3.10) holds. Substituting (3.3), (3.10) and (3.24) into (3.25), we obtain for all $z=re^{i\theta}$ satisfying $|z|=r\notin[0,1]\cup E_8$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$

$$(1-o(1))\exp\{(1-\varepsilon)\delta(a_1z,\theta)r\}\leqslant M_0\exp\left\{r^{\alpha+\varepsilon}\right\}[T(2r,f)]^{k+1}, \quad (3.26)$$

where $M_0>0$ is a some constant. Since $\delta(a_1z,\theta)>0$, $\alpha+\varepsilon<1$, then by using Lemma 2.11 and (3.26), we obtain $\sigma_2(f)\geqslant1$, hence $\sigma_2(f)=1$.

(b) When $\delta(a_1z,\theta)<0$, $\delta(a_2z,\theta)>0$, using a proof similar to the above, we can also get $\sigma_2(f)=1$.

(ii) $(\theta_1=\theta_2)$ In first step, we have proved that there is a ray $\arg z=\theta$ where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$, satisfying $\delta(a_2z,\theta)>\delta(a_1z,\theta)>0$ and for sufficiently large r, we get (3.16) holds. Substituting (3.3), (3.16) and (3.24) into (3.25), we obtain for all $z=re^{i\theta}$ satisfying $|z|=r\notin[0,1]\cup E_8$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$

$$(1-o(1))\exp\{[(1+\varepsilon)\delta(a_1z,\theta)+\eta]r\}\leqslant M_1\exp\left\{r^{\alpha+\varepsilon}\right\}[T(2r,f)]^{k+1}, \quad (3.27)$$

where $M_1>0$ is a some constant. Since $\delta(a_1z,\theta)>0$, $\eta>0$, $\alpha+\varepsilon<1$, then by using Lemma 2.11 and (3.27), we obtain $\sigma_2(f)\geqslant1$, hence $\sigma_2(f)=1$.

Case 2. $a_1<a_0$.

(i) $(\theta_1\neq\theta_2)$ In first step, we have proved that there is a ray $\arg z=\theta$ where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$, satisfying $\delta(a_2z,\theta)>0$ and $\delta(a_1z,\theta)<0$ and for sufficiently large r, we get (3.20) holds. Using the same reasoning as in second step (**Case 1** (i)), we can get $\sigma_2(f)=1$.

(ii) $(\theta_1=\theta_2)$ In first step, we have proved that there is a ray $\arg z=\theta$ where $\theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$, satisfying $\delta(a_2z,\theta)>\delta(a_1z,\theta)>0$ and for sufficiently large r, we get (3.16) holds. Substituting (3.3), (3.16) and (3.24) into (3.25), we obtain for all $z=re^{i\theta}$ satisfying $|z|=r\notin[0,1]\cup E_8$, $\theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\setminus(E_1\cup E_4\cup E_5\cup E_6)$

$$\begin{aligned}&(1-o(1))\exp\{[(1+\varepsilon)\delta(a_1z,\theta)+\eta]r\}\\&\quad\leqslant M_2\exp\left\{r^{\alpha+\varepsilon}\right\}e^{a_0r\cos\theta}[T(2r,f)]^{k+1},\end{aligned} \quad (3.28)$$

where $M_2>0$ is a some constant. Thus

$$(1-o(1))\exp\{\gamma r\}\leqslant M_2\exp\left\{r^{\alpha+\varepsilon}\right\}[T(2r,f)]^{k+1}, \quad (3.29)$$

where $\gamma=(1+\varepsilon)\delta(a_1z,\theta)+\eta-a_0\cos\theta$. Since $\gamma>0$, $\alpha+\varepsilon<1$, then by using Lemma 2.11 and (3.29), we obtain $\sigma_2(f)\geqslant1$, hence $\sigma_2(f)=1$. Concluding the above proof, we obtain $\sigma_2(f)=1$. The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

Assume $f\,(\not\equiv 0)$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of Eq. (1.4), then $\sigma(f) = +\infty$ by Theorem 1.1. Set $g_0(z) = f(z) - \varphi(z)$. Then $g_0(z)$ is a meromorphic function and $\sigma(g_0) = \sigma(f) = \infty$. Substituting $f = g_0 + \varphi$ into (1.4), we have

$$\begin{aligned} &g_0^{(k)} + B_{k-1}g_0^{(k-1)} + \cdots + B_2 g_0'' + A_0 e^{a_0 z} g_0' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) g_0 \\ &\quad = -\left[\varphi^{(k)} + B_{k-1}\varphi^{(k-1)} + \cdots + B_2\varphi'' + A_0 e^{a_0 z}\varphi' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right)\varphi\right]. \end{aligned} \tag{4.1}$$

We can rewrite (4.1) in the following form

$$g_0^{(k)} + h_{0,k-1}g_0^{(k-1)} + \cdots + h_{0,2}g_0'' + h_{0,1}g_0' + h_{0,0}g_0 = h_0, \tag{4.2}$$

where

$$h_0 = -\left[\varphi^{(k)} + B_{k-1}\varphi^{(k-1)} + \cdots + B_2\varphi'' + A_0 e^{a_0 z}\varphi' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right)\varphi\right].$$

We prove that $h_0 \not\equiv 0$. In fact, if $h_0 \equiv 0$, then

$$\varphi^{(k)} + B_{k-1}\varphi^{(k-1)} + \cdots + B_2\varphi'' + A_0 e^{a_0 z}\varphi' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right)\varphi = 0.$$

Hence, φ is a solution of Eq. (1.4) with $\sigma(\varphi) = +\infty$ by Theorem 1.1, it is a contradiction. Hence, $h_0 \not\equiv 0$ is proved. By Lemma 2.7 and (4.2) we know that $\overline{\lambda}(g_0) = \overline{\lambda}(f - \varphi) = \sigma(g_0) = \sigma(f) = \infty$.

Now we prove that $\overline{\lambda}(f' - \varphi) = \infty$. Set $g_1(z) = f'(z) - \varphi(z)$. Then $g_1(z)$ is a meromorphic function and $\sigma(g_1) = \sigma(f') = \sigma(f) = \infty$. Set $R(z) = A_1 e^{a_1 z} + A_2 e^{a_2 z}$ and $G(z) = A_0 e^{a_0 z}$. Differentiating both sides of Eq. (1.4), we have

$$\begin{aligned} &f^{(k+1)} + B_{k-1}f^{(k)} + \left(B_{k-1}' + B_{k-2}\right)f^{(k-1)} + \left(B_{k-2}' + B_{k-3}\right)f^{(k-2)} + \cdots \\ &\quad + \left(B_3' + B_2\right)f''' + \left(B_2' + G\right)f'' + \left(G' + R\right)f' + R'f = 0. \end{aligned} \tag{4.3}$$

By (1.4), we obtain

$$f = -\frac{1}{R}\left[f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_2 f'' + Gf'\right]. \tag{4.4}$$

Substituting (4.4) into (4.3), we have

$$\begin{aligned} &f^{(k+1)} + \left(B_{k-1} - \frac{R'}{R}\right)f^{(k)} + \left(B_{k-1}' + B_{k-2} - B_{k-1}\frac{R'}{R}\right)f^{(k-1)} \\ &\quad + \left(B_{k-2}' + B_{k-3} - B_{k-2}\frac{R'}{R}\right)f^{(k-2)} + \cdots + \left(B_3' + B_2 - B_3\frac{R'}{R}\right)f''' \\ &\quad + \left(B_2' + G - B_2\frac{R'}{R}\right)f'' + \left(G' + R - G\frac{R'}{R}\right)f' = 0. \end{aligned} \tag{4.5}$$

We can write (4.5) in the form

$$f^{(k+1)} + h_{1,k-1}f^{(k)} + h_{1,k-2}f^{(k-1)} + \cdots + h_{1,2}f''' + h_{1,1}f'' + h_{1,0}f' = 0, \tag{4.6}$$

where

$$h_{1,0} = G' + R - G\frac{R'}{R},$$
$$h_{1,1} = B_2' + G - B_2\frac{R'}{R},$$
$$h_{1,i} = B_{i+1}' + B_i - B_{i+1}\frac{R'}{R}, \quad (i = 2, \ldots, k-2),$$
$$h_{1,k-1} = B_{k-1} - \frac{R'}{R}.$$

Substituting $f^{(j+1)} = g_1^{(j)} + \varphi^{(j)}$ $(j = 0, \ldots, k)$ into (4.6), we get

$$g_1^{(k)} + h_{1,k-1}g_1^{(k-1)} + h_{1,k-2}g_1^{(k-2)} + \cdots + h_{1,2}g_1'' + h_{1,1}g_1' + h_{1,0}g_1 = h_1, \quad (4.7)$$

where

$$h_1 = -\left[\varphi^{(k)} + h_{1,k-1}\varphi^{(k-1)} + h_{1,k-2}\varphi^{(k-2)} + \cdots + h_{1,2}\varphi'' + h_{1,1}\varphi' + h_{1,0}\varphi\right].$$

We can get

$$h_{1,i}(z) = \frac{N_i(z)}{R(z)}, \quad (i = 0, 1, \ldots, k-1), \quad (4.8)$$

where

$$N_0 = G'R + R^2 - GR', \quad (4.9)$$
$$N_1 = B_2'R + GR - B_2R', \quad (4.10)$$
$$N_i = \left(B_{i+1}' + B_i\right)R - B_{i+1}R', \quad (i = 2, \ldots, k-2), \quad (4.11)$$
$$N_{k-1} = B_{k-1}R - R'. \quad (4.12)$$

Now we prove that $h_1 \not\equiv 0$. In fact, if $h_1 \equiv 0$, then $\frac{h_1}{\varphi} \equiv 0$. Hence, by (4.8) we get

$$\frac{\varphi^{(k)}}{\varphi}R + \frac{\varphi^{(k-1)}}{\varphi}N_{k-1} + \frac{\varphi^{(k-2)}}{\varphi}N_{k-2} + \cdots + \frac{\varphi''}{\varphi}N_2 + \frac{\varphi'}{\varphi}N_1 + N_0 = 0. \quad (4.13)$$

Obviously, $\frac{\varphi^{(j)}}{\varphi}$ $(j = 1, \ldots, k)$ are meromorphic functions with $\sigma\left(\frac{\varphi^{(j)}}{\varphi}\right) < 1$. By (4.9)–(4.12) we can rewrite (4.13) in the form

$$f_1 e^{(a_1+a_0)z} + f_2 e^{(a_2+a_0)z} + f_3 e^{a_1 z} + f_4 e^{a_2 z}$$
$$+ 2A_1A_2 e^{(a_1+a_2)z} + A_1^2 e^{2a_1 z} + A_2^2 e^{2a_2 z} = 0, \quad (4.14)$$

where f_j $(j = 1, 2, 3, 4)$ are meromorphic functions with $\sigma(f_j) < 1$. Set $I = \{a_1 + a_0, a_2 + a_0, a_1, a_2, a_1 + a_2, 2a_1, 2a_2\}$. If $\arg a_1 \neq \pi$ or $a_1 < a_0$, then $a_1 \neq ca_0$ $(0 < c \leqslant 1)$. By the conditions of Theorem 1.1, it is clear that $2a_1 \neq a_1 + a_0$, a_1, $a_1 + a_2$, $2a_2$ and $2a_2 \neq a_2$, $a_1 + a_2$, $2a_1$.

(i) If $2a_1 \neq a_2 + a_0, a_2$, then we write (4.14) in the form

$$A_1^2 e^{2a_1 z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq I \setminus \{2a_1\}$. By Lemmas 2.8 and 2.9, we get $A_1 \equiv 0$, which is a contradiction.
(ii) If $2a_1 = a_2 + a_0$, then $2a_2 \neq \beta$ for all $\beta \in I \setminus \{2a_2\}$, hence we write (4.14) in the form

$$A_2^2 e^{2a_2 z} + \sum_{\beta \in \Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq I \setminus \{2a_2\}$. By Lemmas 2.8 and 2.9, we get $A_2 \equiv 0$, which is a contradiction.
(iii) If $2a_1 = a_2$, then $2a_2 \neq \beta$ for all $\beta \in I \setminus \{2a_2\}$, hence we write (4.14) in the form

$$A_2^2 e^{2a_2 z} + \sum_{\beta \in \Gamma_3} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_3 \subseteq I \setminus \{2a_2\}$. By Lemmas 2.8 and 2.9, we get $A_2 \equiv 0$, which is a contradiction. Hence, $h_1 \not\equiv 0$ is proved. By Lemma 2.7 and (4.7) we know that $\overline{\lambda}(g_1) = \overline{\lambda}(f' - \varphi) = \sigma(g_1) = \sigma(f) = \infty$.

Now we prove that $\overline{\lambda}(f'' - \varphi) = \infty$. Set $g_2(z) = f''(z) - \varphi(z)$. Then $g_2(z)$ is a meromorphic function and $\sigma(g_2) = \sigma(f'') = \sigma(f) = \infty$. Differentiating both sides of Eq. (4.3), we have

$$\begin{aligned} &f^{(k+2)} + B_{k-1} f^{(k+1)} + \left(2B'_{k-1} + B_{k-2}\right) f^{(k)} + \left(B''_{k-1} + 2B'_{k-2} + B_{k-3}\right) f^{(k-1)} \\ &\quad + \left(B''_{k-2} + 2B'_{k-3} + B_{k-4}\right) f^{(k-2)} + \cdots + \left(B''_4 + 2B'_3 + B_2\right) f^{(4)} \\ &\quad + \left(B''_3 + 2B'_2 + G\right) f''' + \left(B''_2 + 2G' + R\right) f'' \\ &\quad + \left(G'' + 2R'\right) f' + R'' f = 0. \end{aligned} \tag{4.15}$$

By (4.4) and (4.15), we have

$$\begin{aligned} &f^{(k+2)} + B_{k-1} f^{(k+1)} + \left(2B'_{k-1} + B_{k-2} - \frac{R''}{R}\right) f^{(k)} \\ &+ \left(B''_{k-1} + 2B'_{k-2} + B_{k-3} - B_{k-1}\frac{R''}{R}\right) f^{(k-1)} \\ &+ \cdots + \left(B''_4 + 2B'_3 + B_2 - B_4\frac{R''}{R}\right) f^{(4)} + \left(B''_3 + 2B'_2 + G - B_3\frac{R''}{R}\right) f''' \\ &\quad + \left(B''_2 + 2G' + R - B_2\frac{R''}{R}\right) f'' + \left(G'' + 2R' - G\frac{R''}{R}\right) f' = 0. \end{aligned} \tag{4.16}$$

Now we prove that $G' + R - G\frac{R'}{R} \not\equiv 0$. Suppose that $G' + R - G\frac{R'}{R} \equiv 0$, then we have

$$f_1 e^{(a_1+a_0)z} + f_2 e^{(a_2+a_0)z} + 2A_1A_2 e^{(a_1+a_2)z} + A_1^2 e^{2a_1 z} + A_2^2 e^{2a_2 z} = 0, \tag{4.17}$$

where $f_j\,(j = 1, 2)$ are meromorphic functions with $\sigma(f_j) < 1$. Set $K = \{a_1 + a_0, a_2 + a_0, a_1 + a_2, 2a_1, 2a_2\}$. If $\arg a_1 \neq \pi$ or $a_1 < a_0$, then $a_1 \neq c a_0\ (0 < c \leqslant 1)$. By the conditions of Theorem 1.1, it is clear that $2a_1 \neq a_1 + a_0$, $a_1 + a_2$, $2a_2$ and $2a_2 \neq a_1 + a_2, 2a_1$.

(i) If $2a_1 \neq a_2 + a_0$, then we write (4.17) in the form

$$A_1^2 e^{2a_1 z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\varGamma_1 \subseteq K \setminus \{2a_1\}$. By Lemmas 2.8 and 2.9, we get $A_1 \equiv 0$, which is a contradiction.
(ii) If $2a_1 = a_2 + a_0$, then $2a_2 \neq a_1 + a_0,\, a_2 + a_0,\, a_1 + a_2,\, 2a_1$. Hence, we write (4.17) in the form

$$A_2^2 e^{2a_2 z} + \sum_{\beta \in \varGamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\varGamma_2 \subseteq K \setminus \{2a_2\}$. By Lemmas 2.8 and 2.9, we get $A_2 \equiv 0$, which is a contradiction. Hence, $G' + R - G\frac{R'}{R} \not\equiv 0$ is proved. Set

$$\psi(z) = G'R + R^2 - GR' \quad \text{and} \quad \phi(z) = G''R + 2R'R - GR''. \tag{4.18}$$

By (4.5) and (4.18), we get

$$\begin{aligned} f' = \frac{-R}{\psi(z)} \Bigg\{ & f^{(k+1)} + \left(B_{k-1} - \frac{R'}{R}\right) f^{(k)} + \left(B'_{k-1} + B_{k-2} - B_{k-1}\frac{R'}{R}\right) f^{(k-1)} \\ & + \left(B'_{k-2} + B_{k-3} - B_{k-2}\frac{R'}{R}\right) f^{(k-2)} + \cdots + \left(B'_3 + B_2 - B_3\frac{R'}{R}\right) f''' \\ & + \left(B'_2 + G - B_2\frac{R'}{R}\right) f'' \Bigg\}. \end{aligned} \tag{4.19}$$

Substituting (4.18) and (4.19) into (4.16), we obtain

$$\begin{aligned} & f^{(k+2)} + \left[B_{k-1} - \frac{\phi}{\psi}\right] f^{(k+1)} + \left[2B'_{k-1} + B_{k-2} - \frac{R''}{R} - \frac{\phi}{\psi}\left(B_{k-1} - \frac{R'}{R}\right)\right] f^{(k)} \\ & + \left[B''_{k-1} + 2B'_{k-2} + B_{k-3} - B_{k-1}\frac{R''}{R} - \frac{\phi}{\psi}\left(B'_{k-1} + B_{k-2} - B_{k-1}\frac{R'}{R}\right)\right] f^{(k-1)} \\ & + \cdots + \left[B''_4 + 2B'_3 + B_2 - B_4\frac{R''}{R} - \frac{\phi}{\psi}\left(B'_4 + B_3 - B_4\frac{R'}{R}\right)\right] f^{(4)} \\ & + \left[B''_3 + 2B'_2 + G - B_3\frac{R''}{R} - \frac{\phi}{\psi}\left(B'_3 + B_2 - B_3\frac{R'}{R}\right)\right] f''' \\ & + \left[B''_2 + 2G' + R - B_2\frac{R''}{R} - \frac{\phi}{\psi}\left(B'_2 + G - B_2\frac{R'}{R}\right)\right] f'' = 0. \end{aligned} \tag{4.20}$$

Set $E_1 = B''_3 + 2B'_2$, $E_i = B''_{i+2} + 2B'_{i+1} + B_i$ $(i = 2, \ldots, k-3)$, $E_{k-2} = 2B'_{k-1} + B_{k-2}$ and $F_i = B'_{i+2} + B_{i+1}$ $(i = 1, \ldots, k-3)$. E_i $(i = 1, \ldots, k-2)$, F_i $(i = 1, \ldots, k-3)$ are meromorphic functions with $\sigma(E_i) < 1$, $\sigma(F_i) < 1$. We can write Eq. (4.20) in the form

$$\begin{aligned} & f^{(k+2)} + h_{2,k-1} f^{(k+1)} + h_{2,k-2} f^{(k)} + \cdots \\ & + h_{2,2} f^{(4)} + h_{2,1} f''' + h_{2,0} f'' = 0, \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} h_{2,0} &= B''_2 + 2G' + R - B_2\frac{R''}{R} - \frac{\phi(z)}{\psi(z)}\left(B'_2 + G - B_2\frac{R'}{R}\right), \\ h_{2,1} &= E_1 + G - B_3\frac{R''}{R} - \frac{\phi(z)}{\psi(z)}\left(F_1 - B_3\frac{R'}{R}\right), \\ h_{2,i} &= E_i - B_{i+2}\frac{R''}{R} - \frac{\phi(z)}{\psi(z)}\left(F_i - B_{i+2}\frac{R'}{R}\right), \quad (i = 2, \ldots, k-3), \end{aligned}$$

$$h_{2,k-2} = E_{k-2} - \frac{R''}{R} - \frac{\phi(z)}{\psi(z)}\left(B_{k-1} - \frac{R'}{R}\right),$$
$$h_{2,k-1} = B_{k-1} - \frac{\phi(z)}{\psi(z)}.$$

Substituting $f^{(j+2)} = g_2^{(j)} + \varphi^{(j)}$ $(j = 0, \ldots, k)$ into (4.21) we get

$$g_2^{(k)} + h_{2,k-1}g_2^{(k-1)} + h_{2,k-2}g_2^{(k-2)} + \cdots + h_{2,1}g_2' + h_{2,0}g_2 = h_2, \tag{4.22}$$

where

$$h_2 = -\left[\varphi^{(k)} + h_{2,k-1}\varphi^{(k-1)} + h_{2,k-2}\varphi^{(k-2)} + \cdots + h_{2,2}\varphi'' + h_{2,1}\varphi' + h_{2,0}\varphi\right].$$

We can get

$$h_{2,i} = \frac{L_i(z)}{\psi(z)}, \quad (i = 0, 1, \ldots, k-1), \tag{4.23}$$

where

$$\begin{aligned} L_0(z) = {} & B_2''G'R + B_2''R^2 - B_2''GR' + 2G'^2R + 3G'R^2 - 2GG'R' + R^3 \\ & - 3GR'R - B_2G'R'' - B_2R''R - B_2'G''R - G''GR + B_2G''R' \\ & - 2B_2'R'R + 2B_2R'^2 + B_2'GR'' + G^2R'', \end{aligned} \tag{4.24}$$

$$\begin{aligned} L_1(z) = {} & E_1G'R + E_1R^2 - E_1GR' + G'GR + GR^2 - G^2R' - B_3G'R'' \\ & - B_3R''R - F_1G''R + B_3G''R' - 2F_1R'R + 2B_3R'^2 + F_1GR'', \end{aligned} \tag{4.25}$$

$$\begin{aligned} L_i = {} & E_iG'R + E_iR^2 - E_iGR' - B_{i+2}G'R'' - B_{i+2}R''R - F_iG''R \\ & + B_{i+2}G''R' - 2F_iR'R + 2B_{i+2}R'^2 + F_iGR'', \quad (i = 2, \ldots, k-3), \end{aligned} \tag{4.26}$$

$$\begin{aligned} L_{k-2} = {} & E_{k-2}G'R + E_{k-2}R^2 - E_{k-2}GR' - G'R'' - R''R - B_{k-1}G''R + G''R' \\ & - 2B_{k-1}R'R + 2R'^2 + B_{k-1}GR'', \end{aligned} \tag{4.27}$$

$$L_{k-1} = B_{k-1}G'R + B_{k-1}R^2 - B_{k-1}GR' - G''R - 2R'R + GR''. \tag{4.28}$$

Therefore

$$\frac{-h_2}{\varphi} = \frac{1}{\psi}\left[\frac{\varphi^{(k)}}{\varphi}\psi + \frac{\varphi^{(k-1)}}{\varphi}L_{k-1} + \cdots + \frac{\varphi''}{\varphi}L_2 + \frac{\varphi'}{\varphi}L_1 + L_0\right]. \tag{4.29}$$

Now we prove that $h_2 \not\equiv 0$. In fact, if $h_2 \equiv 0$, then $\frac{-h_2}{\varphi} \equiv 0$. Hence, by (4.29) we have

$$\frac{\varphi^{(k)}}{\varphi}\psi + \frac{\varphi^{(k-1)}}{\varphi}L_{k-1} + \frac{\varphi^{(k-2)}}{\varphi}L_{k-2} + \cdots + \frac{\varphi''}{\varphi}L_2 + \frac{\varphi'}{\varphi}L_1 + L_0 = 0. \tag{4.30}$$

Obviously, $\frac{\varphi^{(j)}}{\varphi}$ $(j = 1, \ldots, k)$ are meromorphic functions with $\sigma\left(\frac{\varphi^{(j)}}{\varphi}\right) < 1$. By (4.18) and (4.24)–(4.28), we can rewrite (4.30) in the form

$$\begin{aligned} & f_1e^{(a_1+a_0)z} + f_2e^{(a_2+a_0)z} + f_3e^{(a_1+2a_0)z} + f_4e^{(a_2+2a_0)z} \\ & + f_5e^{2a_1z} + f_6e^{2a_2z} + f_7e^{(a_1+a_2)z} \end{aligned}$$

$$+ f_8 e^{(2a_1+a_0)z} + f_9 e^{(2a_2+a_0)z} + f_{10} e^{(a_1+a_2+a_0)z} + A_1^3 e^{3a_1 z} + A_2^3 e^{3a_2 z}$$
$$+ 3A_1^2 A_2 e^{(2a_1+a_2)z} + 3A_1 A_2^2 e^{(a_1+2a_2)z} = 0, \quad (4.31)$$

where $f_j\ (j = 1, \ldots, 10)$ are meromorphic functions with $\sigma(f_j) < 1$. Set $J = \{a_1+a_0, a_2+a_0, a_1+2a_0, a_2+2a_0, 2a_1, 2a_2, a_1+a_2, 2a_1+a_0, 2a_2+a_0, a_1+a_2+a_0, 3a_1, 3a_2, 2a_1+a_2, a_1+2a_2\}$. If $\arg a_1 \neq \pi$ or $a_1 < a_0$, then $a_1 \neq ca_0\ (0 < c \leqslant 1)$. By the conditions of Theorem 1.1, it is clear that $3a_1 \neq a_1+a_0$, a_1+2a_0, $2a_1$, $2a_1+a_0$, $3a_2$, $2a_1+a_2$, a_1+2a_2 and $3a_2 \neq 2a_2$, $3a_1$, $2a_1+a_2$, a_1+2a_2.

(i) If $3a_1 \neq a_2+a_0$, a_2+2a_0, $2a_2$, a_1+a_2, $2a_2+a_0$, $a_1+a_2+a_0$, then we write (4.31) in the form

$$A_1^3 e^{3a_1 z} + \sum_{\beta\in\Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq J\setminus\{3a_1\}$. By Lemmas 2.8 and 2.9, we get $A_1 \equiv 0$, which is a contradiction.

(ii) If $3a_1 = \gamma$ such that $\gamma \in \{a_2+a_0, a_2+2a_0, 2a_2, a_1+a_2, 2a_2+a_0, a_1+a_2+a_0\}$, then $3a_2 \neq \beta$ for all $\beta \in J\setminus\{3a_2\}$. Hence, we write (4.31) in the form

$$A_2^3 e^{3a_2 z} + \sum_{\beta\in\Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq J\setminus\{3a_2\}$. By Lemmas 2.8 and 2.9, we get $A_2 \equiv 0$, it is a contradiction. Hence, $h_2 \not\equiv 0$ is proved. By Lemma 2.7 and (4.22), we have $\overline{\lambda}(g_2) = \overline{\lambda}(f''-\varphi) = \sigma(g_2) = \sigma(f) = \infty$. The proof of Theorem 1.2 is complete.

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