Hochschild cohomology of Sullivan algebras and mapping spaces

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Received 3 May 2018; revised 6 July 2018; accepted 26 July 2018
Available online 2 August 2018

Abstract. Let $f : X \to Y$ be a map between simply connected spaces having the homotopy of finite type CW-complexes, where $H^*(Y, \mathbb{Q})$ is finite dimensional and $\phi : (\wedge V, d) \to (B, d)$ a Sullivan model of $f$. We consider $(B, d)$ as a module over $\wedge V$ via the mapping $\phi$. Let $\text{map}(X, Y; f)$ denote the component of $f$ in the space of mappings from $X$ to $Y$. In this paper we show that there is a canonical injection $\pi_\ast(\Omega \text{map}(X, Y; f)) \otimes \mathbb{Q} \to H^*(\wedge V; B)$.

Keywords: Hochschild cohomology; Mapping space; $L_\infty$ algebra

2010 Mathematics Subject Classification: primary 55P62; secondary 54C35

1. Introduction

We work in the rational homotopy setting for which the standard reference is [6]. In this section we fix notation and recall a few facts on the Hochschild cohomology of an algebra. All vector spaces and algebras are taken over a field $k$ of characteristic 0.

Definition 1. A lower graded vector space $V$ is a direct sum of vector spaces, that is, $V = \bigoplus_i V_i$, where $i \in \mathbb{Z}$. We say that element $a \in V_i$ is homogeneous of degree $i$ and we write $|a| = i$ and $V = V_\bullet$ is lower or homologically graded. If $V = \bigoplus_{i \geq 0} V_i$, then $V$ is said to be non negatively graded. In the same way $V^\ast = \bigoplus_i V^i$ is called cohomologically graded.
graded. We use the standard convention $V^i := V_{-i}$. Hence if $V = \oplus_{i \geq 0} V^i$, the dual space of $V$ is denoted $V^* = \prod_i \text{Hom}(V^i, \mathbb{k}) = \prod_i \text{Hom}(V_{-i}, \mathbb{k})$ has a lower non negative grading.

**Definition 2.** A morphism of graded vector spaces $f : V \to W$ of degree $r$, is a family of linear maps $f_n : V_n \to W_{n+r}$.

Let $(M, d)$ be a differential $(A, d)$-bimodule. The Hochschild cohomology of $A$ with coefficients in $M$ is defined as $\text{Ext}_{A^e}(A, M)$ where $A$ is an $A^e = A \otimes A^{op}$-module under the action $(a_1 \otimes a_2)a = (-1)^{|a_1||a_2|}a_1a_2$, where $a, a_1, a_2 \in A$.

Let $(P, d_P) \to (A, d)$ be a semifree resolution of $A$ as an $A^e$-module [5], and $(M, d_M)$ an $A^e$-differential module. Then $HH^*(A; M) := \text{Ext}_{A^e}(A, M)$ is the homology of the complex $(\text{Hom}_{A^e}(P, M), D)$, where the differential is defined by

$$(Df)(x) = d_M f(x) - (-1)^{|f|} f(d_P x).$$

In the sequel we work in the category of commutative differential graded algebras (cdga’s for short). This implies that left (or right) modules have a natural bimodule structure. Let $f : A \to B$ be a morphism of cdga’s. Then $B$ is considered as an $A$-module by the action induced by $f$.

Our aim is to study the structure of $HH^*(A; B)$. Let $(\wedge V, d)$ be a Sullivan algebra, and $m : (\wedge V \otimes \wedge V, d' = d \otimes 1 + 1 \otimes d) \to (\wedge V, d)$ the multiplication. Then there is a quasi isomorphism

$$(\wedge V \otimes \wedge V \otimes \tilde{V}, D) \to (\wedge V, d)$$

making the following diagram commutative.

Moreover $\tilde{V}^n = V^{n+1}$ and the differential $D$ is defined by

$$D(\tilde{v}) = v \otimes 1 - 1 \otimes v + \alpha, \alpha \in \wedge V \otimes \wedge V \otimes \tilde{V},$$

and $i$ is the canonical inclusion [6, §15]. The quasi isomorphism

$$(\wedge V \otimes \wedge V \otimes \tilde{V}, D) \xrightarrow{\eta} (\wedge V, d)$$

is a semifree resolution of $(\wedge V, d)$ as a $\wedge V \otimes \wedge V$-module [5,10]. Therefore, for any $\wedge V$-module $M$, $HH^*(\wedge V; M)$ is the homology of the complex

$$(\text{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \tilde{V}, M), D),$$

where the differential is defined by (1).

We consider the cdga $(\wedge V \otimes \tilde{V}, D)$ where $Dv = dv$, $D(\tilde{v}) = -S(dv)$ and $S$ is the unique derivation on $\wedge V \otimes \tilde{V}$ defined by $SV = \tilde{v}$ and $S\tilde{v} = 0$. It is obtained as a push out in the diagram below.

$$(\wedge V \otimes \wedge V, d') \xrightarrow{i} (\wedge V \otimes \wedge V \otimes \tilde{V}, D)$$
Moreover, the composition with $m'$ yields an isomorphism of complexes

$$\text{Hom}_{\wedge V}(\wedge V \otimes \wedge \tilde{V}, M) \xrightarrow{\sim} \text{Hom}_{\wedge V \otimes V}(\wedge V \otimes \wedge \tilde{V}, M).$$

As $\tilde{D}(\wedge V \otimes \wedge^n \tilde{V}) \subset \wedge V \otimes \wedge^n \tilde{V}$, hence each $(\text{Hom}_{\wedge V}(\wedge V \otimes \wedge^n \tilde{V}, M), \tilde{D})$ is a sub cochain complex [8]. This gives a Hodge type decomposition of the Hochschild cohomology

$$HH^*(\wedge V; M) = \bigoplus_{n \geq 0} HH^*_n(\wedge V; M)$$

for any $\wedge V$-differential module $(M, d)$ [11,7].

Let $f : X \to Y$ be a map between simply connected spaces having the homotopy of finite type CW-complexes and assume that $H^*(Y, \mathbb{Q})$ is finite dimensional. Let $\phi : (\wedge V, d) \to (B, d)$ be a cdga model of $f$. We consider $(B, d)$ as a module over $\wedge V$ via the mapping $\phi$. Denote by $\text{map}(X, Y; f)$ the component of $f$ in the space of mappings from $X$ to $Y$. In this paper we show the following result.

**Theorem 3.** There is a canonical injection

$$\pi_*(\Omega \text{map}(X, Y; f)) \otimes \mathbb{k} \to HH^*(\wedge V; B).$$

Moreover $\pi_*(\text{map}(X, Y; f)) \otimes \mathbb{k} \cong HH^*_1(\wedge V; B)$.

The result is a generalization of the inclusion $\pi_*(\Omega \text{map}(X, X; 1_X)) \otimes \mathbb{k} \to HH^*(\wedge V; \wedge V)$. See [7, Theorem 2] and [9, Theorem 1.1].

## 2. $L_\infty$-MODELS OF MAPPING SPACES

The notion of $L_\infty$ algebra was introduced by Lada [14] and $L_\infty$ models of mapping spaces were used by Félix et al. in [3,4]. We remind here their definition.

**Definition 4.** A permutation $\sigma \in S_k$ is called an $(i, k-i)$ shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$ where $i = 1, \ldots, n$. For graded objects $x_1, \ldots, x_k$, the Koszul sign $\varepsilon(\sigma)$ is determined by

$$x_1 \wedge \cdots \wedge x_k = \varepsilon(\sigma)x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}.$$

It depends not only of the permutation $\sigma$ but also on degrees of $x_1, \ldots, x_k$.

**Definition 5.** An $L_\infty$-algebra or a strongly homotopy Lie algebra is a graded vector space $L = \bigoplus_i L_i$ with maps $\ell_k := [\ldots, \ldots] : L^\otimes k \to L$ of degree $k-2$ such that

1. $\ell_k$ is graded skew symmetric, that is, for a $k$-permutation $\sigma$

   $$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \text{sgn}(\sigma) \varepsilon(\sigma) \ell_k(x_1, \ldots, x_k),$$

   where $\text{sgn}(\sigma)$ is the sign of $\sigma$,

2. There are generalized Jacobi identities

   $$\sum_i \sum_{\sigma} \varepsilon(\sigma)(-1)^{i(k-i)} \ell_j(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(k)}) = 0,$$

   where the second summation extends to all $(i, k-i)$ shuffles of the symmetric group $S_k$.
In particular if \( \ell_k = 0 \) for \( k \geq 3 \), one recovers the notion of differential graded Lie algebra \((L, d)\) where \([x, y] := \ell_2(x, y)\) and \( dx = \ell_1(x)\).

There is a 1-1 correspondence between \( L_\infty \) structures on \( L \) and codifferentials \( d_n : \wedge^m(sL) \to \wedge^{m-n+1}(sL) \) of degree \(-1\) on the coalgebra \( sL \), such that \( d^2 = 0 \), where \( d = d_1 + d_2 + \cdots + d_n + \cdots \) [14].

**Definition 6 ([12]).** Let \((A, \mu)\) be a commutative algebra and \( D : A \to A\) an operator.

Define multi-brackets on \( A \) with a bracket \( D \)

\[
\begin{align*}
F^1_D(a) &= DA \\
F^n_D(a_1, \ldots, a_n) &= \mu((D \otimes 1)(a_1 \otimes 1 - 1 \otimes a_1) \cdots (a_n \otimes 1 - 1 \otimes a_n)).
\end{align*}
\]

Then \( D \) is called an operator of order \( n \) if \( F^n_D = 0 \).

There is a generalization of multi-brackets to non commutative algebras that is due to Akman [1].

**Definition 7.** A Gerstenhaber algebra is a graded commutative algebra \( A = \oplus_i A_i \) together with a bracket

\[
A_i \otimes A_j \to A_{i+j+1}, \quad a \otimes b \mapsto \{a, b\},
\]

such that \( sL \) is a graded Lie algebra and the bracket acts like a derivation of algebras. That is, for \( a, b, c \in A \),

1. \( \{a, b\} = -(-1)^{|a||b|+1}\{b, a\} \),
2. \( \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a||b|+1}\{b, \{a, c\}\} \),
3. \( \{a, bc\} = \{a, b\}c + (-1)^{|b||a|+1}b\{a, c\} \).

**Definition 8.** A Batalin–Vilkovisky algebra (BV-algebra for short) is a graded commutative algebra \( A \), together with an operator \( \Delta : A_i \to A_{i+1} \) of order 2 and of square 0.

Any BV-algebra \((A, \Delta)\) is a Gerstenhaber algebra with the bracket defined by

\[
\{a, b\} = (-1)^{|a|}(\Delta(ab) - \Delta(a)b - (-1)^{|a|}a \Delta(b)).
\]

**Definition 9 ([13,2]).** A commutative BV\(_\infty\)-algebra is a graded commutative algebra \( A = \oplus_{i \in \mathbb{Z}} A_i \) together with an operator \( D = \sum_{i \geq 1} D_i \) such that \( D^2 = 0 \) and each \( D_n \) is an operator of order \( n \) and of degree \( 2n - 3 \).

From the relation \( D^2 = 0 \), one gets \( D_1^2 = 0 \), hence \( D_1 \) is a differential on the algebra \( A \). Moreover \( D_1D_2 + D_2D_1 = 0 \), therefore \( D_2 \) induces an action on the homology \( H_*(A, D_1) \) which induces a BV-algebra structure [13]. If \( D_1 = 0 \) for all \( i \geq 3 \), then \((A, D_1 + D_2)\) is called a differential BV-algebra.

**Definition 10.** Let \( \phi : (A, d) \to (B, d) \) be a morphism of cdga’s. A \( \phi \)-derivation of degree \( k \) is a linear mapping \( \theta : A^n \to B^{n-k} \) such that \( \theta(ab) = \theta(a)\phi(b) + (-1)^{|a|}\phi(a)\theta(b) \).

We denote by \( \text{Der}_n(A, B; \phi) \) the vector space of \( \phi \)-derivations of degree \( n \) and by \( \text{Der}(A, B; \phi) = \oplus_n \text{Der}_n(A, B; \phi) \) the \( \mathbb{Z} \)-graded vector space of all \( \phi \)-derivations.

The differential on \( \text{Der}(A, B; \phi) \) is defined by \( \delta \theta = d\theta - (-1)^k\theta d \).
If \( A = B \) and \( \phi = 1_A \), then we get the Lie algebra of derivations \( \text{Der} A \), where the Lie bracket is the commutator bracket. If \( V \) is finite, then \( \text{Der}(\wedge V) \cong \wedge V \otimes V^\phi \). We have the following result for \( \phi \)-derivations.

**Proposition 11.** Let \( \phi : (\wedge V, d) \to (B, d) \) be a surjective morphism between cdga’s where \( V \) is finite dimensional and \( I = \text{Ker} \phi \). Then \( \text{Der}(\wedge V, B; \phi) \cong \wedge V/I \otimes V^\phi \).

**Proof.** Let \( \{v_1, \ldots, v_k\} \) be a basis of \( V \). In \( \text{Der}(\wedge V, B; \phi) \), we denote by \( (v_i, 1) \) the \( \phi \)-derivation \( \theta_i \) such that \( \theta_i(v_i) = \delta_{ij} \). We observe that \( v_i^\phi \) corresponds to the derivation \( \theta_i = (v_i, 1) \). Let \( \theta \) be a \( \phi \)-derivation. Then \( \theta(v_i) = b_i \), where \( b_i \in B \). As \( \phi \) is surjective, there exist \( a_i \in \wedge V \) such that \( \phi(a_i) = b_i \). Hence \( \theta = \sum_i a_i \theta_i = a_iv_i^\phi \). By the first isomorphism theorem \( \text{Der}(\wedge V, B; \phi) \cong \wedge V/I \otimes V^\phi \). \( \square \)

Define \( \tilde{\text{Der}}(A, B; \phi) \) as follows.

\[
\tilde{\text{Der}}(A, B; \phi)_i = \begin{cases} \text{Der}(A, B; \phi)_i, & i > 1, \\ \{ \theta \in \text{Der}_1(A, B; \phi) : \delta \theta = 0 \}, & i = 1. \end{cases}
\]

Let \( A = \wedge V \) and \( \theta_1, \ldots, \theta_k \in \tilde{\text{Der}}(\wedge V, B; \phi) \) be \( \phi \)-derivations of respective degrees \( n_1, \ldots, n_k \), define

\[
[\theta_1, \ldots, \theta_k](v) = (-1)^{\epsilon(v)} \sum_{i_1, \ldots, i_k} \epsilon(v_{ij} \ldots v_{ik}) \theta_1(v_{ij}) \ldots \theta_k(v_{ik}),
\]

where \( dv = \sum v_{ij} \ldots v_{ik} \). We note that \([\theta_1, \ldots, \theta_k]\) is of degree \( n_1 + \cdots + n_k - 1 \). Now define linear maps \( \ell_k \) of degree \( k - 2 \) on \( s^{-1}\text{Der}(\wedge V, B; \phi) \) by

\[
\ell_1(s^{-1}\theta) = -s^{-1}\delta \theta, \quad \ell_k(s^{-1}\theta_1, \ldots, s^{-1}\theta_k) = (-1)^{\epsilon_k} s^{-1}[\theta_1, \ldots, \theta_k],
\]

where \( \epsilon_k = \frac{k(k-1)}{2} + \sum_{i=1}^{k-1} (k - i) |\theta_i| [4] \).

**Proposition 12 (Lemma 3.3,[4]).** If \( \phi : \wedge V \to B \) is a Sullivan model of a mapping \( f : X \to Y \) between simply connected spaces and \( V \) is finite dimensional, then \( (s^{-1}\text{Der}(\wedge V, B; \phi), \ell_k) \) is an \( L_\infty \) model of map(\( X, Y; f \)).

**Theorem 13.** Let \((\wedge V, d) \to (B, d)\) be a cdga model of map \( f : X \to Y \) between \( l \)-connected spaces of finite type where \( Y \) is finite dimensional.

1. Then there is a natural isomorphism

\[
\Gamma : \pi_*(\Omega \text{map}(X, Y; f)) \otimes \mathbb{Q} \to \text{HH}^*_\{1\}(\wedge V; B).
\]

2. Moreover the following diagram commutes:

\[
\pi_*(\text{aut}_1 Y) \otimes \mathbb{Q} \longrightarrow \pi_*(\text{map}(X, Y; f)) \otimes \mathbb{Q} \quad \text{and} \quad \text{HH}^*(\wedge V; \wedge V) \longrightarrow \text{HH}^*(\wedge V; B).
\]
Proof of the theorem. Before we prove the theorem, we need a generalization of derivations.

Definition 14. Let $A$ be a commutative cochain algebra and $M$ a differential $A$-module (considered here as an $A$-bimodule). A derivation $\theta$ from $A$ to $M$ of degree $k$ is a linear map $\theta : A^* \to M^{*-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{|a|}a\theta(b)$.

It is easily seen that if $\theta : A \to M$ is derivation and $f : M \to N$ a morphism of $A$-bimodules, then the composition $f \circ \theta : A \to N$ is a derivation.

Let $(\wedge V, d)$ be a Sullivan model of a simply connected space. Define $\tilde{V} = sV$, that is, $\tilde{V}^n = V^{n+1}$. A Sullivan model of the loop space map $(S^1, X)$ is given by $(\wedge(V \oplus \tilde{V}), \tilde{D})$, the cdga defined in Section 1. For recall, $\tilde{D}v = dv, \tilde{D}\tilde{v} = -S(dv)$ where $S$ is the unique derivation defined by $Sv = \tilde{v}$ and $S\tilde{v} = 0$ [6].

Consider the linear map $S : (\wedge V, d) \to (\wedge V \otimes \tilde{V}, D)$ defined $Sv = \tilde{v}$ and extended $S$ as a derivation in the sense of Definition 14. As $S(dv) = -D(Sv)$, then $Sd + DS = 0$, then $S$ is a morphism of differential modules of upper degree $-1$.

We define a map

$$\Phi : \text{Hom}_{\wedge}((\wedge V \otimes \tilde{V}), B) \to \text{Der}(\wedge V, B; \phi)$$

such that $\Phi(f)$ is the following composition mapping

$$\wedge V \xrightarrow{S} \wedge V \otimes \tilde{V} \xrightarrow{f} B,$$

that is, $\Phi(f)(v) = f(\tilde{v})$.

Lemma 15. The map $\Phi$ commutes with differentials.

Proof. Let $f \in \text{Hom}_{\wedge}((\wedge V \otimes \tilde{V}), \wedge V)$. Then

$$(Df)(\tilde{v}) = d(f(\tilde{v})) - (-1)^{|f|}f(D(\tilde{v}))$$

$$= d(f(\tilde{v})) + (-1)^{|f|}(f(sdv)),$$

hence $(\Phi(Df))(v) = d(f(\tilde{v})) + (-1)^{|f|}(f(sdv))$.

On the other hand

$$(D\Phi(f))(v) = d(\Phi(f)(v)) - (-1)^{|\Phi(f)|}\Phi(f)(dv)$$

$$= d(f(sv)) + (-1)^{|f|}(f(sdv)).$$

Hence $\Phi$ is a morphism of chain complexes. \qed

Moreover, there are isomorphisms of vector spaces $\text{Hom}_{\wedge}(\wedge V \otimes \tilde{V}, B) \cong \text{Hom}(\tilde{V}, B) \cong \text{Der}(\wedge V, B)$. Hence $\Phi$ is bijective. Therefore

$$H_{\ast}(s^{-1}\text{Der}(\wedge V, B)) \cong HH^*_1(\wedge V, B) \hookrightarrow HH^*(\wedge V; B).$$

Remark 16. It was shown that if $L$ is an $L_\infty$-algebra, then $\wedge s^{-1}\Lambda L$ is a $BV_\infty$-algebra [2]. It would be interesting to find a link between the $BV_\infty$-algebra $\wedge s^{-1}L$ and $HH^*(\wedge V; B)$.

Acknowledgement

Partially supported by the Max Planck Institute for Mathematics, Bonn, Germany.
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