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Hermite–Hadamard–Fejér type inequalities for p-convex functions

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Abstract. In this paper, firstly, Hermite–Hadamard–Fejér type inequalities for p-convex functions are built. Secondly, an integral identity and some Hermite–Hadamard–Fejér type integral inequalities for p-convex functions are obtained. Finally, some Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for convex, harmonically convex and p-convex functions are given. Some results presented here for p-convex functions provide extensions of others given in earlier works for convex and harmonically convex and p-convex functions.

Keywords: Hermite–Hadamard inequalities; Hermite–Hadamard–Fejér inequalities; Convex functions; Harmonically convex functions; p-Convex functions

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite–Hadamard's inequality [5,6].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite–Hadamard inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities.

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Definition 1. A function $w : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be symmetric with respect to $\frac{a+b}{2}$, if $w(x) = w(a+b-x)$ holds for all $x \in [a,b]$.

Example 1. Assume that $w_1, w_2 : [a,b] \subset \mathbb{R} \to \mathbb{R}$, $w_1(x) = c$ for $c \in \mathbb{R}$, $w_2(x) = \left(x - \frac{a+b}{2}\right)^2$, then w_1, w_2 are symmetric functions with respect to $\frac{a+b}{2}$.

In [4], Fejér established the following Hermite–Hadamard–Fejér inequality which is the weighted generalization of the Hermite–Hadamard inequality (1.1):

Theorem 1 *([4]). Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.2}$$

holds, where $w : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [1–3,7,9–12,16–19].

In [9], İşcan gave the definition of a harmonically convex function and established the following Hermite–Hadamard inequality for harmonically convex functions:

Definition 2 *([9]).* Let $I \subset \mathbb{R} \backslash \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.3) is reversed, then f is said to be harmonically concave.

We assume that $L[a,b]$ is the set of all Riemann integrable functions defined on the interval $[a,b]$.

Theorem 2 *([9]). Let $f : I \subset \mathbb{R} \backslash \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.4}$$

Definition 3 *([15]).* A function $w : [a,b] \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$, if $w(x) = w\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$ holds for all $x \in [a,b]$.

Example 2. Assume that $w_1, w_2 : [a,b] \subset \mathbb{R} \to \mathbb{R}$, $w_1(x) = c$ for $c \in \mathbb{R}$, $w_2(x) = \left(\frac{1}{x} - \frac{a+b}{2ab}\right)^2$, then w_1, w_2 are harmonically symmetric functions with respect to $\frac{2ab}{a+b}$.

In [2], Chan and Wu presented Hermite–Hadamard–Fejér inequality for harmonically convex functions:

Theorem 3 *([2]). Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ and $w : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}dx \le \int_a^b \frac{f(x)w(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}dx. \tag{1.5}$$

In [20], Zhang and Wan gave the definition of a p-convex function on $I \subset \mathbb{R}$, in [11], İşcan gave a different definition of a p-convex function on $I \subset (0,\infty)$:

Definition 4 *([11]).* Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be p-convex, if

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \le tf(x) + (1-t)f(y) \tag{1.6}$$

for all $x, y \in I$ and $t \in [0,1]$.

It can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Example 3. Let $f : I \subset (0,\infty) \to \mathbb{R}$, $f(x) = x^p + c$ for $p \ne 0$ and $c \in (0,\infty)$, then f is a p-convex function.

In [3, Theorem 5], if we take $I \subset (0,\infty)$, $p \in \mathbb{R}\backslash\{0\}$ and $h(t) = t$, then we have the following theorem.

Theorem 4 *([12]). Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \le \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \le \frac{f(a)+f(b)}{2}. \tag{1.7}$$

For some results related to p-convex functions and its generalizations, we refer the reader to see [3,11–13,16,17,20].

In this paper, we built Hermite–Hadamard–Fejér inequality for p-convex functions. We obtain an integral identity and some Hermite–Hadamard–Fejér type integral inequalities for p-convex functions. We give some Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for convex, harmonically convex and p-convex functions.

2. Main results

Throughout this section, $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$, for the continuous function $w : [a,b] \to \mathbb{R}$.

Definition 5. Let $p \in \mathbb{R}\backslash\{0\}$. A function $w : [a,b] \subset (0,\infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, if $w(x) = w\left([a^p + b^p - x^p]^{\frac{1}{p}}\right)$ holds for all $x \in [a,b]$.

Remark 1. In Definition 5, one can see the following.

(1) If one takes $p = 1$, one has Definition 1 for a function defined on $(0,\infty)$ (become symmetric with respect to $\frac{a+b}{2}$),

(2) If one takes $p = -1$, one has Definition 3 for a defined function on $(0,\infty)$ (become harmonically symmetric with respect to $\frac{2ab}{a+b}$).

Example 4. Let $p \in \mathbb{R}\backslash\{0\}$. Assume that $w_1, w_2 : [a,b] \subset (0,\infty) \to \mathbb{R}$, $w_1(x) = c$ for $c \in \mathbb{R}$, $w_2(x) = \left(x^p - \frac{a^p+b^p}{2}\right)^2$, then w_1, w_2 are p-symmetric functions with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$.

Theorem 5. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, $a,b \in I$ with $a < b$. If $f \in L[a,b]$ and $w : [a,b] \to \mathbb{R}$ is nonnegative, integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx \le \int_a^b \frac{f(x)w(x)}{x^{1-p}}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}dx. \tag{2.1}$$

Proof. Let $p > 0$. Since $f : I \subset (0,\infty) \to \mathbb{R}$ is a p-convex function, we have, for all $x, y \in I$ (with $t = \frac{1}{2}$ in the inequality (1.6))

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right) \le \frac{f(x)+f(y)}{2}.$$

Choosing $x = [ta^p + (1-t)b^p]^{1/p}$ and $y = [tb^p + (1-t)a^p]^{1/p}$, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \le \frac{f\left([ta^p+(1-t)b^p]^{1/p}\right) + f\left([tb^p+(1-t)a^p]^{1/p}\right)}{2}. \tag{2.2}$$

Since w is nonnegative, integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then

$$w\left([ta^p+(1-t)b^p]^{1/p}\right) = w\left([tb^p+(1-t)a^p]^{1/p}\right).$$

Multiplying both sides of (2.2) by $w\left([ta^p+(1-t)b^p]^{1/p}\right)$, then integrating with respect to t over $[0,1]$, and then changing variables we get

$$\begin{aligned} & f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\frac{p}{b^p-a^p}\int_a^b \frac{w(x)}{x^{1-p}}dx \\ &\quad = f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_0^1 w\left([ta^p+(1-t)b^p]^{1/p}\right)dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) dt \\
&\le \int_0^1 \frac{\begin{array}{c} f\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) \\ +f\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right) w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right)\end{array}}{2} dt \\
&= \frac{\begin{array}{c}\int_0^1 f\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) dt \\ +\int_0^1 f\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right) w\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right) dt\end{array}}{2} \\
&= \frac{\frac{p}{b^p-a^p}\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx + \frac{p}{b^p-a^p}\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx}{2} \\
&= \frac{p}{b^p-a^p}\int_a^b \frac{f\left(x\right) w\left(x\right)}{x^{1-p}}dx. \qquad (2.3)
\end{aligned}
$$

Multiplying both sides of (2.3) by $\frac{b^p-a^p}{p}$, we get

$$
f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w\left(x\right)}{x^{1-p}}dx \le \int_a^b \frac{f\left(x\right) w\left(x\right)}{x^{1-p}}dx
$$

the left hand side of (2.1).

For the proof of the second inequality in (2.1), we first note that if f is a p-convex function, then for all $t \in [0,1]$, it yields

$$
\frac{f\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right)+f\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right)}{2} \le \frac{f\left(a\right)+f\left(b\right)}{2}. \qquad (2.4)
$$

Since w is nonnegative, integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, multiplying both sides of (2.4) by $w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right)$, then integrating with respect to t over $[0,1]$, and then changing variables we get

$$
\begin{aligned}
\frac{p}{b^p-a^p}\int_a^b \frac{f\left(x\right) w\left(x\right)}{x^{1-p}}dx &= \frac{\frac{p}{b^p-a^p}\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx + \frac{p}{b^p-a^p}\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx}{2} \\
&= \frac{\begin{array}{c}\int_0^1 f\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) dt \\ +\int_0^1 f\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right) w\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right) dt\end{array}}{2} \\
&= \int_0^1 \frac{\begin{array}{c} f\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right) \\ +f\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right) w\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right)\end{array}}{2} dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^1 \frac{f(a)+f(b)}{2} w\left(\left[ta^p+(1-t)b^p\right]^{1/p}\right)dt \\
&= \frac{f(a)+f(b)}{2} \int_0^1 w\left(\left[ta^p+(1-t)b^p\right]^{1/p}\right)dt \\
&= \frac{f(a)+f(b)}{2} \frac{p}{b^p-a^p} \int_a^b \frac{w(x)}{x^{1-p}}dx. \qquad (2.5)
\end{aligned}
$$

Multiplying both sides of (2.5) by $\frac{b^p-a^p}{p}$, we get

$$
\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2} \int_a^b \frac{w(x)}{x^{1-p}}dx
$$

the right hand side of (2.1). This completes the proof. □

Remark 2. In Theorem 5, one can see the following.
(1) If one takes $p=1$ and $w(x)=1$, one has (1.1),
(2) If one takes $p=1$, one has (1.2),
(3) If one takes $p=-1$ and $w(x)=1$, one has (1.4),
(4) If one takes $p=-1$, one has (1.5),
(5) If one takes $w(x)=1$, one has (1.7).

Lemma 1. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° (the interior of I) and $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}\setminus\{0\}$. If $f'\in L[a,b]$ and $w:[a,b]\to\mathbb{R}$ is integrable, then the following equality holds:*

$$
\begin{aligned}
&\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx \\
&= \left(\frac{b^p-a^p}{p}\right)^2 \int_0^1 \frac{k(t)}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}} f'\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)dt, \qquad (2.6)
\end{aligned}
$$

where

$$
k(t)=\begin{cases} \displaystyle\int_0^t w\left(\left[sa^p+(1-s)b^p\right]^{\frac{1}{p}}\right)ds, & t\in\left[0,\frac{1}{2}\right) \\ \displaystyle -\int_t^1 w\left(\left[sa^p+(1-s)b^p\right]^{\frac{1}{p}}\right)ds, & t\in\left[\frac{1}{2},1\right]. \end{cases}
$$

Proof. It suffices to note that

$$
\begin{aligned}
J &= \left(\frac{b^p-a^p}{p}\right)^2 \int_0^1 \frac{k(t)}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}} f'\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)dt \\
&= \left(\frac{b^p-a^p}{p}\right)^2 \int_0^{\frac{1}{2}} \frac{\int_0^t w\left(\left[sa^p+(1-s)b^p\right]^{\frac{1}{p}}\right)ds}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}} f'\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)dt
\end{aligned}
$$

$$-\left(\frac{b^p-a^p}{p}\right)^2\int_{\frac{1}{2}}^{1}\frac{\int_t^1 w\left(\left[sa^p+(1-s)\,b^p\right]^{\frac{1}{p}}\right)ds}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}f'\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)dt$$

$$=J_1-J_2. \tag{2.7}$$

By integration by parts, we have

$$J_1=f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\left(\int_0^t w\left(\left[sa^p+(1-s)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}ds\right)\Bigg|_0^{\frac{1}{2}}$$

$$-\int_0^{\frac{1}{2}} f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)w\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}dt$$

$$=f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_0^{\frac{1}{2}} w\left(\left[sa^p+(1-s)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}ds$$

$$-\int_0^{\frac{1}{2}} f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)w\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}dt$$

$$=f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_b^{\left[\frac{a^p+b^p}{2}\right]^{1/p}}\frac{w(x)}{x^{1-p}}dx-\int_b^{\left[\frac{a^p+b^p}{2}\right]^{1/p}}\frac{f(x)\,w(x)}{x^{1-p}}dx \tag{2.8}$$

and similarly

$$J_2=f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\left(\int_t^1 w\left(\left[sa^p+(1-s)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}ds\right)\Bigg|_{\frac{1}{2}}^{1}$$

$$+\int_{\frac{1}{2}}^{1} f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)w\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}dt$$

$$=-f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_{\frac{1}{2}}^{1} w\left(\left[sa^p+(1-s)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}ds$$

$$+\int_{\frac{1}{2}}^{1} f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)w\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\frac{a^p-b^p}{p}dt$$

$$=-f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_{\left[\frac{a^p+b^p}{2}\right]^{1/p}}^{a}\frac{w(x)}{x^{1-p}}ds+\int_{\left[\frac{a^p+b^p}{2}\right]^{1/p}}^{a}\frac{f(x)\,w(x)}{x^{1-p}}ds. \tag{2.9}$$

A combination of (2.7)–(2.9) we have (2.6). This completes the proof. □

Remark 3. In Lemma 1, one can see the following.

(1) If one takes $p=1$ and $w(x)=1$, one has [14, Lemma 2.1],

(2) If one takes $p=1$, one has [18, Lemma 2.1],

(3) If one takes $w(x)=1$, one has [17, Lemma 2.7],

(4) If one takes $p=-1$, $w(x)=1$, one has [8, Lemma 6(for $\lambda=0$)].

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ and $a<b$. If $|f'|$ is p-convex function on $[a,b]$ for $p \in \mathbb{R}\backslash\{0\}$, $w:[a,b]\to\mathbb{R}$ is continuous, then the following inequality holds:*

$$\left|\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx\right| \le \left(\frac{b^p-a^p}{p}\right)^2 \|w\|_\infty \left[C_1(p)|f'(a)| + C_2(p)|f'(b)|\right]$$

where

$$C_1(p) = \left[\int_0^{\frac{1}{2}} \frac{t^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{t-t^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right],$$

$$C_2(p) = \left[\int_0^{\frac{1}{2}} \frac{t-t^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{\frac{1}{2}}^1 \frac{(1-t)^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right].$$

Proof. Using Lemma 1, it follows that

$$\begin{aligned}\left|\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx\right| &\le \left(\frac{b^p-a^p}{p}\right)^2 \int_0^1 \frac{|k(t)|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt \\ &\le \left(\frac{b^p-a^p}{p}\right)^2 \|w\|_\infty \\ &\times \left[\begin{array}{l}\int_0^{\frac{1}{2}} \frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt \\ +\int_{\frac{1}{2}}^1 \frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt\end{array}\right]. \end{aligned} \quad (2.10)$$

Since $|f'|$ is a p-convex function on $[a,b]$, we have

$$\left|f'\left([ta^p+(1-t)b^p]^{1/p}\right)\right| \le t|f'(a)| + (1-t)|f'(b)|. \quad (2.11)$$

A combination of (2.10) and (2.11), we have

$$\left|\int_a^b \frac{f(x)w(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx\right| \le \left(\frac{b^p-a^p}{p}\right)^2 \|w\|_\infty$$

$$\times \left[\begin{array}{l} \int_0^{\frac{1}{2}} \frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} \left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right] dt \\ +\int_{\frac{1}{2}}^1 \frac{1-t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} \left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right] dt \end{array}\right]$$

$$\leq \left(\frac{b^p-a^p}{p}\right)^2 \|w\|_\infty$$

$$\times \left[\begin{array}{l} \left[\int_0^{\frac{1}{2}} \frac{t^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} dt + \int_{\frac{1}{2}}^1 \frac{t-t^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} dt\right] \left|f'(a)\right| \\ +\left[\int_0^{\frac{1}{2}} \frac{t-t^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} dt + \int_{\frac{1}{2}}^1 \frac{(1-t)^2}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} dt\right] \left|f'(b)\right| \end{array}\right]$$

$$= \left(\frac{b^p-a^p}{p}\right)^2 \|w\|_\infty \left[C_1(p)\left|f'(a)\right|+C_2(p)\left|f'(b)\right|\right].$$

This completes the proof. □

Remark 4. In Theorem 6, one can see the following.

(1) If one takes $p=1$ and $w(x)=1$, one has [14, Theorem 2.2],

(2) If one takes $w(x)=1$, one has [17, Theorem 3.3].

Corollary 1. *In Theorem 6, one can see the following.*

(1) *If one takes $p=1$, one has the following Hermite–Hadamard–Fejér type inequality for convex functions:*

$$\left|\int_a^b f(x)w(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \leq \frac{(b-a)^2}{8}\|w\|_\infty \left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

(2) *If one takes $p=-1$, one has the following Hermite–Hadamard–Fejér type inequality for harmonically convex functions:*

$$\left|\int_a^b \frac{f(x)w(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}dx\right|$$

$$\leq \left(\frac{b-a}{ab}\right)^2 \|w\|_\infty \left[C_1(-1)\left|f'(a)\right|+C_2(-1)\left|f'(b)\right|\right].$$

(3) *If one takes $p=-1$, $w(x)=1$, one has the following Hermite–Hadamard type inequality for harmonically convex functions:*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\right| \leq \frac{b-a}{ab}\left[C_1(-1)\left|f'(a)\right|+C_2(-1)\left|f'(b)\right|\right].$$

Theorem 7. *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ and $a<b$. If $|f'|^q$, $q \geq 1$, is p-convex function on $[a,b]$*

for $p\in\mathbb{R}\backslash\{0\}$, $w:[a,b]\to\mathbb{R}$ *is continuous, then the following inequality holds:*

$$\begin{aligned}&\left|\int_a^b \frac{f(x)\,w(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx\right|\\&\le\left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\left[(C_3(p))^{1-\frac{1}{q}}\left[C_4(p)\left|f'(a)\right|^q+C_5(p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\&\left.+\ (C_6(p))^{1-\frac{1}{q}}\left[C_7(p)\left|f'(a)\right|^q+C_8(p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right]\end{aligned}$$

where

$$\begin{aligned}C_3(p)&=\int_0^{\frac{1}{2}}\frac{t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt,\ \ C_4(p)=\int_0^{\frac{1}{2}}\frac{t^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt,\\C_5(p)&=\int_0^{\frac{1}{2}}\frac{t-t^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt,\ \ C_6(p)=\int_{\frac{1}{2}}^1\frac{1-t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt,\\C_7(p)&=\int_{\frac{1}{2}}^1\frac{t-t^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt,\ \ C_8(p)=\int_{\frac{1}{2}}^1\frac{(1-t)^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt.\end{aligned}$$

Proof. Using (2.10), power mean inequality and the p-convexity of $|f'|^q$ it follows that

$$\begin{aligned}&\left|\int_a^b \frac{f(x)\,w(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx\right|\\&\le\left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\left[\begin{array}{l}\displaystyle\int_0^{\frac{1}{2}}\frac{t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\right|dt\\\displaystyle+\int_{\frac{1}{2}}^1\frac{1-t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\right|dt\end{array}\right]\\&\le\left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\\&\times\left[\begin{array}{c}\displaystyle\left(\int_0^{\frac{1}{2}}\frac{t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\\displaystyle\times\left[\int_0^{\frac{1}{2}}\frac{t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\right|^q dt\right]^{\frac{1}{q}}\end{array}\right.\\&+\left.\begin{array}{c}\displaystyle\left(\int_{\frac{1}{2}}^1\frac{1-t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\\displaystyle\times\left[\int_{\frac{1}{2}}^1\frac{1-t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\right|^q dt\right]^{\frac{1}{q}}\end{array}\right]\\&\le\left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\end{aligned}$$

$$\times\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\frac{t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\ \times\left[\int_0^{\frac{1}{2}}\frac{t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right]^{\frac{1}{q}}\end{array}\right.$$

$$+\left.\begin{array}{l}\left(\int_{\frac{1}{2}}^{1}\frac{1-t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\ \times\left[\int_{\frac{1}{2}}^{1}\frac{1-t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right]^{\frac{1}{q}}\end{array}\right]$$

$$\leq\left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\frac{t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\ \times\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\frac{t^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)\left|f'(a)\right|^q\\ +\left(\int_0^{\frac{1}{2}}\frac{t-t^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)\left|f'(b)\right|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right.$$

$$+\left.\begin{array}{l}\left(\int_{\frac{1}{2}}^{1}\frac{1-t}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\ \times\left[\begin{array}{l}\left(\int_{\frac{1}{2}}^{1}\frac{t-t^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)\left|f'(a)\right|^q\\ +\left(\int_{\frac{1}{2}}^{1}\frac{(1-t)^2}{\left[ta^p+(1-t)\,b^p\right]^{1-\frac{1}{p}}}dt\right)\left|f'(b)\right|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right]$$

$$\leq\left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\left[\left(C_3(p)\right)^{1-\frac{1}{q}}\left[C_4(p)\left|f'(a)\right|^q+C_5(p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left(C_6(p)\right)^{1-\frac{1}{q}}\left[C_7(p)\left|f'(a)\right|^q+C_8(p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right].$$

This completes the proof. □

Corollary 2. *In Theorem 7, one can see the following.*

(1) *If one takes $p=1$ and $w(x)=1$, one has the following Hermite–Hadamard type inequality for convex functions:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\begin{array}{l}\left[\left|f'(a)\right|^q+2\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\ +\left[2\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right).$$

(2) *If one takes* $w(x)=1$, *one has the following Hermite–Hadamard type inequality for* p*-convex functions:*

$$\begin{aligned}&\left|\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right|\\&\le\left(\frac{b^p-a^p}{p}\right)\left[(C_3(p))^{1-\frac{1}{q}}\left[C_4(p)\left|f'(a)\right|^q+C_5(p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\&\quad\left.+\ (C_6(p))^{1-\frac{1}{q}}\left[C_7(p)\left|f'(a)\right|^q+C_8(p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right].\end{aligned}$$

(3) *If one takes* $p=1$, *one has the following Hermite–Hadamard–Fejér type inequality for convex functions:*

$$\begin{aligned}&\left|\int_a^b f(x)w(x)dx-f\left(\frac{a+b}{2}\right)\int_a^b f(x)w(x)dx\right|\\&\le\frac{(b-a)^2}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\|w\|_\infty\left(\left[\left|f'(a)\right|^q+2\left|f'(b)\right|^q\right]^{\frac{1}{q}}+\left[2\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right).\end{aligned}$$

(4) *If one takes* $p=-1$, *one has the following Hermite–Hadamard–Fejér type inequality for harmonically convex functions:*

$$\begin{aligned}&\left|\int_a^b\frac{f(x)w(x)}{x^2}dx-f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{w(x)}{x^2}dx\right|\\&\le\left(\frac{b-a}{ab}\right)^2\|w\|_\infty\left[(C_3(-1))^{1-\frac{1}{q}}\left[C_4(-1)\left|f'(a)\right|^q+C_5(-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\&\quad\left.+\ (C_6(-1))^{1-\frac{1}{q}}\left[C_7(-1)\left|f'(a)\right|^q+C_8(-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right].\end{aligned}$$

(5) *If one takes* $p=-1$, $w(x)=1$, *one has the following Hermite–Hadamard type inequality for harmonically convex functions:*

$$\begin{aligned}&\left|\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx-f\left(\frac{2ab}{a+b}\right)\right|\\&\le\left(\frac{b-a}{ab}\right)\left[(C_3(-1))^{1-\frac{1}{q}}\left[C_4(-1)\left|f'(a)\right|^q+C_5(-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\&\quad\left.+\ (C_6(-1))^{1-\frac{1}{q}}\left[C_7(-1)\left|f'(a)\right|^q+C_8(-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 8. *Let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I^\circ$ *and* $a<b$. *If* $\left|f'\right|^q$, $q>1$, *is* p*-convex function on* $[a,b]$

for $p \in \mathbb{R}\backslash\{0\}$, $w:[a,b]\to\mathbb{R}$ *is continuous, then the following inequality holds:*

$$\begin{aligned}&\left|\int_a^b \frac{f(x)\,w(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx\right|\\ &\quad\le \left(\frac{b^p-a^p}{p}\right)^2 \|w\|_\infty \left[C_9(p)\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right.\\ &\qquad\left. +\ C_{10}(p)\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right]\end{aligned}$$

where

$$C_9(p,r)=\left(\int_0^{\frac{1}{2}}\left(\frac{t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}},$$

$$C_{10}(p)=\left(\int_{\frac{1}{2}}^1\left(\frac{1-t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}}$$

with $\frac{1}{q}+\frac{1}{r}=1$.

Proof. Using (2.10), Hölder's inequality and the p-convexity of $|f'|^q$ it follows that

$$\begin{aligned}&\left|\int_a^b \frac{f(x)\,w(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx\right|\\
&\le \left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\left[\begin{array}{l}\displaystyle\int_0^{\frac{1}{2}}\frac{t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right|dt\\ \displaystyle+\int_{\frac{1}{2}}^1\frac{1-t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right|dt\end{array}\right]\\
&\le \left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\\
&\quad\times\left[\begin{array}{l}\displaystyle\left(\int_0^{\frac{1}{2}}\left(\frac{t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|f'\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\\ \displaystyle+\left(\int_{\frac{1}{2}}^1\left(\frac{1-t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1\left|f'\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]\\
&\le \left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\\
&\quad\times\left[\begin{array}{l}\displaystyle\left(\int_0^{\frac{1}{2}}\left(\frac{t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\ \displaystyle+\left(\int_{\frac{1}{2}}^1\left(\frac{1-t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\end{array}\right]\\
&= \left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\end{aligned}$$

$$\times\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\left(\frac{t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q+3\,|f'(b)|^q}{8}\right)^{\frac{1}{q}}\\+\left(\int_{\frac{1}{2}}^1\left(\frac{1-t}{[ta^p+(1-t)\,b^p]^{1-\frac{1}{p}}}\right)^r dt\right)^{\frac{1}{r}}\left(\frac{3\,|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\end{array}\right]$$

$$=\left(\frac{b^p-a^p}{p}\right)^2\|w\|_\infty\left[C_9(p)\left(\frac{|f'(a)|^q+3\,|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right.$$

$$\left.+\;C_{10}(p)\left(\frac{3\,|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Remark 5. In Theorem 7, if one takes $p=1$ and $w(x)=1$, one has [14, Theorem 2.3].

Corollary 3. *In Theorem 8, one can see the following.*

(1) *If one takes $w(x)=1$, one has the following Hermite–Hadamard type inequality for p-convex functions:*

$$\left|\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right|$$

$$\leq\left(\frac{b^p-a^p}{p}\right)\left[\begin{array}{l}C_9(p)\left(\frac{|f'(a)|^q+3\,|f'(b)|^q}{8}\right)^{\frac{1}{q}}\\+C_{10}(p)\left(\frac{3\,|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\end{array}\right].$$

(2) *If one takes $p=1$, one has the following Hermite–Hadamard–Fejér type inequality for convex functions:*

$$\left|\int_a^b f(x)\,w(x)\,dx-f\left(\frac{a+b}{2}\right)\int_a^b f(x)\,w(x)\,dx\right|$$

$$\leq\frac{(b-a)^2}{16}\left(\frac{4}{r+1}\right)^{\frac{1}{r}}\|w\|_\infty\left[\begin{array}{l}\left(|f'(a)|^q+3\,|f'(b)|^q\right)^{\frac{1}{q}}\\+\left(3\,|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

(3) *If one takes $p=-1$, one has the following Hermite–Hadamard–Fejér type inequality for harmonically convex functions:*

$$\left|\int_a^b\frac{f(x)\,w(x)}{x^2}dx-f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{w(x)}{x^2}dx\right|$$

$$\leq\left(\frac{b-a}{ab}\right)^2\|w\|_\infty\left[\begin{array}{l}C_9(-1)\left(\frac{|f'(a)|^q+3\,|f'(b)|^q}{8}\right)^{\frac{1}{q}}\\+C_{10}(-1)\left(\frac{3\,|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\end{array}\right].$$

(4) *If one takes $p=-1$, $w(x)=1$, one has the following Hermite–Hadamard type inequality for harmonically convex functions:*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\right| \leq \left(\frac{b-a}{ab}\right)\left[\begin{array}{c} C_9(-1)\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}} \\ +C_{10}(-1)\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}} \end{array}\right].$$

Remark 6. Theorem 6 is a special case of Theorem 7 (If one takes $q=1$ in Theorem 7, one has Theorem 6). In the literature, as much as we know, midpoint type estimates have not compared so far. Since, the coefficients of Theorem 7 and Theorem 8 are in the Riemann integral forms and Theorem 7 and Theorem 8 are examined via the midpoint type estimates for p-convex functions, it is considered that Theorem 7 and Theorem 8 are not comparable.

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