

Ground state solutions and least energy sign-changing solutions for a class of fourth order Kirchhoff-type equations in \mathbb{R}^N [☆]

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Abstract. In this article we study the problem

$$\Delta^2 u - \left(1 + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $\lambda > 0$ is a parameter, $p \in (2, 2_*)$, and $V(x) \in C(\mathbb{R}^N, \mathbb{R})$. Under appropriate assumptions on $V(x)$, the existence of ground state solutions and a least energy sign-changing solution is obtained by combining the variational methods and the Nehari method.

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1. INTRODUCTION AND MAIN RESULTS

Consider the following fourth order elliptic equation of Kirchhoff type

$$\Delta^2 u - \left(a + \lambda b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

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where $a, b > 0$, are constants, $\lambda > 0$ is a parameter, $2 < p < 2_*$ ($2_* = \frac{2N}{N-4}$ if $N > 4$ and $2_* = +\infty$ if $N \leq 4$ is the critical Sobolev exponent), and V is a nonnegative potential function.

Problem (1.1) is a nonlocal problem because of the so-called nonlocal term $(\int_{\mathbb{R}^N} |\nabla u|^2 dx) \Delta u$ involved in Eq. (1.1). The appearance of a nonlocal term in the equation causes some mathematical difficulties. This makes the study of problem (1.1) particularly interesting. If $V(x) = 0$, replace \mathbb{R}^N by a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and $|u|^{p-2}u$ by a generalized nonlinearity $f(x, u)$ and set $u = \Delta u = 0$ on $\partial\Omega$ and $\lambda = 1$, then problem (1.1) is reduced to the following fourth order elliptic equation of Kirchhoff type

$$\begin{cases} \Delta^2 u - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) is related to the stationary analogue of the following Kirchhoff equation

$$\Delta^2 u + u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u). \tag{1.3}$$

In one and two dimensions, (1.3) is used to describe some phenomena in different physical and engineering fields because it is regarded as a good approximation for describing nonlinear vibrations of beams or plates (see [3,6,1]). In [13,14], Ma applied the variational methods to study the existence and multiplicity of solutions for a nonlocal fourth order equation of Kirchhoff type:

$$\begin{cases} u^{(4)} - M \left(\int_0^1 |u'|^2 dx \right) u'' = h(x)f(x, u), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.4}$$

Replacing $h(x)f(x, u)$ by $h(x)f(x, u, u')$ in (1.4), Ma [15] studied the existence of positive solutions by using the fixed point theorems in cones of ordered Banach spaces. Recently, Wang et al. [18] studied the existence of nontrivial solutions for the fourth order elliptic equation

$$\begin{cases} \Delta^2 u - \lambda \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where λ is a positive parameter, and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz continuous. The authors show that there exists a λ_* such that the fourth order elliptic equation has nontrivial solutions for $0 < \lambda < \lambda_*$ by using the mountain pass techniques and the truncation method. More recently, Avci et al. [2], studied the following fourth order elliptic equation of Kirchhoff type

$$\Delta^2 u - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + cu = f(u), \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where $c > 0$ is a constant and $N > 4$. By using variational methods and truncation, they proved the existence of positive solutions for (1.6). Replacing $|u|^{p-2}u$ by the generalized

form $f(x, u)$, Xu and Chen [21] obtained infinitely many negative nontrivial solutions for (1.1) in \mathbb{R}^3 with $\lambda = 1$ by using genus theory. In [22] Xu and Chen have established the existence and multiplicity of solutions for (1.1) in \mathbb{R}^3 by using variational methods.

Inspired by the above facts, more precisely by [12], the aim of this paper is to study the existence of nontrivial solutions and least energy sign-changing solutions of problem (1.1). To the best of our knowledge, there has been no work concerning this case up to now. For the sake of simplicity, we consider the problem (1.1) with $a = b = 1$, that is,

$$\Delta^2 u - \left(1 + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \quad (1.7)$$

Before stating our main result, we introduce the following notations. Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$ and

$$L^p(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} |u|^p dx < +\infty \right\},$$

with the norm

$$\|u\|_{L^p} := \|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}.$$

Then $L^p(\mathbb{R}^N)$ is a reflexive separable Banach space.

Let $C_0^\infty(\mathbb{R}^N)$ be the collection of smooth functions with compact support in \mathbb{R}^N . For $m = 1, 2$, and a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}^N$ with $|\alpha| = \sum_{i=1}^N \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \dots \partial^{\alpha_N} x_N}$, let

$$H^m(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) \mid D^\alpha u \in L^2(\mathbb{R}^N), |\alpha| \leq m\}.$$

The space $H := H^2(\mathbb{R}^N)$ equipped with the inner product and norm

$$\langle u, v \rangle_H = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + uv) dx, \quad \|u\|_H = \langle u, u \rangle_H^{\frac{1}{2}},$$

is a Hilbert space. Now, let the following assumptions hold:

(V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$, where V_0 is a constant. Moreover, for every $M > 0$, $meas\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where $meas(\cdot)$ denotes the Lebesgue measure in \mathbb{R}^N .

Set

$$E = \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\},$$

with the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv) dx, \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}},$$

where $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_H$. Then, E is a Hilbert space. Furthermore, E is continuously embedded in $L^p(\mathbb{R}^N)$ for $2 \leq p \leq 2_*$ under the condition (V), that is, there exists $\gamma_p > 0$ such that

$$\|u\|_p \leq \gamma_p \|u\|, \quad \forall u \in E.$$

Moreover, we have the following compactness results.

Lemma 1.1 ([7], Lemma 2.1). *Under assumption (V) the continuous embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \leq s < 2_*$.*

Remark 1.2. Since the problem (1.1) is defined in \mathbb{R}^N which is unbounded, the lack of compactness of the Sobolev embedding becomes more delicate by using variational techniques. To overcome the lack of compactness, the condition (V), which was first introduced by Bartsch and Wang in [4], is always assumed to preserve the compactness of embedding of the working space. Furthermore, it is well known that assumption (V) implies a coercive condition on the potential $V(x)$, which was first introduced by Rabinowitz in [16].

Set

$$\tilde{E} = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\},$$

with the inner product and norm

$$\langle u, v \rangle_{\tilde{E}} = \int_{\mathbb{R}^N} \nabla u \nabla v dx, \quad \|u\|_{\tilde{E}} = \langle u, u \rangle_{\tilde{E}}^{\frac{1}{2}}.$$

Then, the embedding $E \hookrightarrow \tilde{E}$ is continuous, furthermore, the functional $\Upsilon : E \rightarrow \mathbb{R}$, defined by $\Upsilon(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx$, is weakly lower semicontinuous on E (see [20, Lemma 2]).

We say that $u \in E$ is a weak solution of problem (1.1) if

$$\langle u, \varphi \rangle + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx = \int_{\mathbb{R}^N} |u|^{p-2} u \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \quad (1.8)$$

where $\langle u, \varphi \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta \varphi + \nabla u \nabla \varphi + V(x)u\varphi) dx$. Define the energy functional $I_\lambda : E \rightarrow \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2} \|u\|^2 + \frac{\lambda}{4} \|u\|_{\tilde{E}}^4 - \frac{1}{p} \|u\|_p^p. \quad (1.9)$$

Then, I_λ is well defined on E , moreover, the functional $\Phi : E \rightarrow \mathbb{R}$, defined by

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx,$$

belongs to $C^1(E, \mathbb{R})$ (see [19], Chapter 1), and

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^N} |u|^{p-2} u v dx, \quad \forall u, v \in E.$$

Therefore, $I_\lambda \in C^1(E, \mathbb{R})$ and

$$\langle I'_\lambda(u), v \rangle = \langle u, v \rangle + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx - \int_{\mathbb{R}^N} |u|^{p-2} u v dx, \quad (1.10)$$

where $\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv) dx$. Consequently, seeking a weak solution of problem (1.7) is equivalent to finding a critical point of the functional I_λ .

Throughout this paper, we denote $u^+ = \max\{u(x), 0\}$ and $u^- = \min\{u(x), 0\}$ then $u = u^+ + u^-$. C, C_i denote positive constants, and $\rightarrow (\rightharpoonup)$ denotes strong (weak) convergence.

- Definition 1.3.** (i) If $u \in E$ is a weak solution of (1.1) and $I(u) = \inf\{I(v) : v \text{ is a nontrivial solution of (1.1)}\}$, we call u the ground state solution of (1.1).
- (ii) If $u \in E$ is a weak solution of (1.1) with $u^\pm \neq 0$, then we call u a sign-changing solution of (1.1). Furthermore if u is a sign-changing solution of (1.1) with $I(u) = \inf\{I(v) : v \text{ is a sign-changing solution of (1.1)}\}$, then we call u the least energy sign-changing solution of (1.1).

The principle of the Nehari method is to seek a minimizer of the energy functional I over the Nehari manifold \mathcal{N} defined by

$$\mathcal{N} = \{u \in E \mid u \neq 0, \langle I'(u), u \rangle = 0\}. \quad (1.11)$$

Let

$$c = \inf_{u \in \mathcal{N}} I(u). \quad (1.12)$$

Now, we are ready to state the main results of this paper.

Theorem 1.4. Assume that (V) holds and $p \in (2, 4]$. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, problem (1.7) has a positive ground state solution $u \in \mathcal{N}$.

Theorem 1.5. Let $\overline{\mathcal{N}}_\pm$ be given by (3.1). Suppose that $p \in (4, 2_*)$, $\lambda > 0$ and condition (V) holds. Then the problem (1.7) has a least energy sign-changing solution $u \in \overline{\mathcal{N}}_\pm$, which has exactly two nodal domains.

Theorem 1.6. Under the assumptions of Theorem 1.5, $c > 0$ is achieved and

$$I(u) > c,$$

where u is the least energy sign-changing solution obtained in Theorem 1.5.

Remark 1.7. In fact our results still hold for $a > 0$ and $b > 0$ (i.e. problem (1.1)).

Remark 1.8. Theorem 1.6 indicates that the energy of any sign-changing solution of (1.1) is strictly larger than the ground state energy.

Remark 1.9. Under the assumptions of Theorem 1.5, by using almost the same procedure in [12] (or in [10,8,17]), we can prove that the problem (1.1) has a ground state solution v with $I(v) = c$, when $p \in (4, 2_*)$. Therefore, Theorem 1.5 not only includes but also improves this result.

2. EXISTENCE OF GROUND STATE SOLUTION

To prove Theorem 1.4, we state the following mountain pass theorem (see [19, Theorem 1.17]).

Proposition 2.1 ([19]). *Let X be a Banach space, $I \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} I(u) > I(0) \geq I(e).$$

If I satisfies the Palais–Smale condition at the level $c \in \mathbb{R}$ ($(PS)_c$ -condition for short), then c is a critical value of I .

Recall that a sequence $\{u_n\} \subset E$ is said to be a Palais–Smale sequence at the level $c \in \mathbb{R}$ ($(PS)_c$ -sequence for short) if $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$. I is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ -sequence has a convergent subsequence.

Lemma 2.2. *If $\{u_n\} \subset E$ is a bounded sequence with $I'_\lambda(u_n) \rightarrow 0$, then $\{u_n\} \subset E$ has a convergent subsequence.*

Proof. Since $\{u_n\} \subset E$ is bounded, passing to a subsequence we may assume that $u_n \rightharpoonup u$ in E , then Lemma 1.1 implies that $u_n \rightarrow u$ in $L^p(\mathbb{R}^N)$ for $p \in [2, 2_*)$. Note that

$$\begin{aligned} & \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u \rangle \\ &= \int_{\mathbb{R}^N} |\Delta(u_n - u)|^2 dx + \left(1 + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx \\ & \quad - \lambda \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right) \int_{\mathbb{R}^N} \nabla u \nabla(u_n - u) dx \\ & \quad - \int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) dx + \int_{\mathbb{R}^N} V(x) |u_n - u|^2 dx \\ & \geq \|u_n - u\|^2 - \lambda \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right) \int_{\mathbb{R}^N} \nabla u \nabla(u_n - u) dx \\ & \quad - \int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) dx. \end{aligned}$$

We then get

$$\begin{aligned} \|u_n - u\|^2 & \leq \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u \rangle \\ & \quad + \lambda \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right) \int_{\mathbb{R}^N} \nabla u \nabla(u_n - u) dx \\ & \quad + \int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) dx. \end{aligned} \tag{2.1}$$

By the Hölder inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) dx & \leq \int_{\mathbb{R}^N} (|u_n|^{p-1} + |u|^{p-1})(u_n - u) dx \\ & \leq (\|u_n\|_p^{p-1} + \|u\|_p^{p-1}) (\|u_n - u\|_p) \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{2.2}$$

On the other hand, the continuity of the embedding $E \hookrightarrow \tilde{E}$ and the boundedness of u_n implies that

$$\lambda \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right) \int_{\mathbb{R}^N} \nabla u \nabla (u_n - u) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.3)$$

It follows from (2.1)–(2.3) that $\|u_n - u\| \rightarrow 0$. This completes the proof. \square

Proof of Theorem 1.4. First, for $u \in E \setminus \{0\}$ with $\|u\| = \rho$ small enough and $p \in (2, 4]$, one has

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \|u\|^2 + \frac{\lambda}{4} \|u\|_{\tilde{E}}^4 - \frac{1}{p} \|u\|_p^p \\ &\geq \frac{1}{2} \|u\|^2 - \frac{\gamma_p}{p} \|u\|^p \\ &= \rho^2 \left(\frac{1}{2} - \frac{\gamma_p}{p} \rho^{p-2} \right) = \alpha > 0. \end{aligned}$$

On the other hand, we have $I_0(tu) \rightarrow -\infty$ as $t \rightarrow \infty$, since $p \in (2, 4]$, which implies that there exist $\lambda_0 > 0$ and $e \in E \setminus \{0\}$ such that $I_\lambda(e) < 0$ for all $\lambda \in (0, \lambda_0)$. Therefore, I_λ satisfies the mountain pass geometry. Moreover, by Lemma 2.2, I_λ satisfies the (PS)-condition, then, by applying Proposition 2.1, problem (1.7) has a ground state solution provided $\lambda \in (0, \lambda_0)$ and $p \in (2, 4]$. Next, to obtain the positive solution, we may consider the following functional

$$I_\lambda^+(u) = \frac{1}{2} \|u\|^2 + \frac{\lambda}{4} \|u\|_{\tilde{E}}^4 - \frac{1}{p} \int_{\mathbb{R}^N} |u^+|_p^p dx,$$

and repeat the above steps to conclude that problem (1.7) has a nontrivial nonnegative solution provided $\lambda \in (0, \lambda_0)$ and $p \in (2, 4]$. Then it follows from the Maximum Principle that this nonnegative solution is positive. \square

3. EXISTENCE OF LEAST ENERGY SIGN-CHANGING SOLUTION

In this section, without loss of generality, we may assume that $\lambda = 1$ and denote $I_1 := I$. Motivated by [12], in order to get a least energy sign-changing solution of problem (1.1), we shall seek a minimizer of the energy functional I under the following constraint:

$$\bar{\mathcal{N}}_\pm = \{u \in E, u^\pm \neq 0 \text{ and } \langle I'(u), u^+ \rangle = 0 = \langle I'(u), u^- \rangle\}, \quad (3.1)$$

and then we show that the minimizer is a least energy sign-changing solution of (1.7).

For each $u \in \bar{\mathcal{N}}_\pm$ and $p \in (4, 2_*)$ we have the following decompositions

$$I(u) = I(u^+) + I(u^-) + \frac{1}{2} \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2, \quad (3.2)$$

$$\langle I'(u), u^+ \rangle = \langle I'(u^+), u^+ \rangle + \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2, \quad (3.3)$$

$$\langle I'(u), u^- \rangle = \langle I'(u^-), u^- \rangle + \|u^-\|_{\tilde{E}}^2 \|u^+\|_{\tilde{E}}^2, \quad (3.4)$$

$$I(u) = I(u) - \frac{1}{4} \langle I'(u), u \rangle = \frac{1}{4} \|u\|^2 + \left(\frac{1}{4} - \frac{1}{p} \right) \|u\|_p^p. \quad (3.5)$$

For $u \in E$ with $u^\pm \neq 0$, we define the function $\beta_u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ by $\beta_u(t, s) = I(tu^+ + su^-)$. Then, we prove the following lemma.

Lemma 3.1. *For each $u \in E$ with $u^\pm \neq 0$, there exists a unique $(t_u, s_u) \in \mathbb{R} \times \mathbb{R}$ with $t_u, s_u > 0$ such that $t_u u^+ + s_u u^- \in \overline{N}_\pm$, moreover*

$$I(t_u u^+ + s_u u^-) = \max\{I(tu^+ + su^-) : t, s \geq 0\}.$$

Proof. For $u \in E$ with $u^\pm \neq 0$, by definition of $\beta_u(t, s)$ we have

$$\begin{aligned} \beta_u(t, s) &= I(tu^+ + su^-) \\ &= I(tu^+) + I(su^-) + \frac{t^2 s^2}{2} \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 \\ &= \frac{t^2}{2} \|u^+\|^2 + \frac{t^4}{4} \|u^+\|_{\tilde{E}}^4 - \frac{|t|^p}{p} \|u^+\|_p^p + \frac{s^2}{2} \|u^-\|^2 \\ &\quad + \frac{s^4}{4} \|u^-\|_{\tilde{E}}^4 - \frac{|s|^p}{p} \|u^-\|_p^p + \frac{t^2 s^2}{2} \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2. \end{aligned}$$

By a simple computation we get

$$\begin{aligned} \nabla \beta_u(t, s) &= \langle I'(tu^+ + su^-), u^+ \rangle, \langle I'(tu^+ + su^-), u^- \rangle \\ &= \left(\frac{1}{t} \langle I'(tu^+ + su^-), tu^+ \rangle, \frac{1}{s} \langle I'(tu^+ + su^-), su^- \rangle \right) \\ &:= (th_u(t, s), sk_u(t, s)), \end{aligned}$$

where

$$h_u(t, s) = \|u^+\|^2 + t^2 \|u^+\|_{\tilde{E}}^4 + s^2 \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 - |t|^{p-2} \|u^+\|_p^p, \quad (3.6)$$

$$k_u(t, s) = \|u^-\|^2 + s^2 \|u^-\|_{\tilde{E}}^4 + t^2 \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 - |s|^{p-2} \|u^-\|_p^p. \quad (3.7)$$

Then, $tu^+ + su^- \in \overline{N}_\pm$ if and only if the pair (t, s) is a critical point of β_u with $t, s > 0$. So, the problem is reduced to investigating the existence of a unique solution of the following system

$$\begin{cases} \|u^+\|^2 + t^2 \|u^+\|_{\tilde{E}}^4 + s^2 \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 - t^{p-2} \|u^+\|_p^p = 0, \\ \|u^-\|^2 + s^2 \|u^-\|_{\tilde{E}}^4 + t^2 \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 - s^{p-2} \|u^-\|_p^p = 0. \end{cases} \quad (3.8)$$

Let $u \in E$ with $u^\pm \neq 0$, and $\bar{s} \geq 0$ fixed. We have

$$h_u(t, \bar{s}) = \|u^+\|^2 + t^2 \|u^+\|_{\tilde{E}}^4 + \bar{s}^2 \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 - t^{p-2} \|u^+\|_p^p,$$

which implies that $h_u(t, \bar{s}) > 0$ for $t \geq 0$ sufficiently small and $h_u(t, \bar{s}) \rightarrow -\infty$ as $t \rightarrow +\infty$, then there exists a $t_{\bar{s}} > 0$ such that $h_u(t_{\bar{s}}, \bar{s}) = 0$. We claim $t_{\bar{s}}$ is unique. Suppose to the contrary that there exist $0 < \bar{t}_1 < \bar{t}_2$ such that $h_u(\bar{t}_1, \bar{s}) = h_u(\bar{t}_2, \bar{s}) = 0$. Then

$$\frac{1}{\bar{t}_1^2} \|u^+\|^2 + \|u^+\|_{\tilde{E}}^4 + \frac{\bar{s}^2}{\bar{t}_1^2} \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 = \bar{t}_1^{p-4} \|u^+\|_p^p,$$

and

$$\frac{1}{\bar{t}_2^2} \|u^+\|^2 + \|u^+\|_{\bar{E}}^4 + \frac{\bar{s}^2}{\bar{t}_2^2} \|u^+\|_{\bar{E}}^2 \|u^-\|_{\bar{E}}^2 = \bar{t}_2^{p-4} \|u^+\|_p^p.$$

Since $p > 4$ and $0 < \bar{t}_1 < \bar{t}_2$ we get

$$0 < \left(\frac{1}{\bar{t}_1^2} - \frac{1}{\bar{t}_2^2} \right) \left(\|u^+\|^2 + \bar{s}^2 \|u^+\|_{\bar{E}}^2 \|u^-\|_{\bar{E}}^2 \right) = \left(\bar{t}_1^{p-4} - \bar{t}_2^{p-4} \right) \|u^+\|_p^p < 0, \quad (3.9)$$

which is absurd. Therefore, there exists a unique $t_{\bar{s}} > 0$ such that $h_u(t_{\bar{s}}, \bar{s}) = 0$. We define the map $\eta_t(s) = t_s$, where t_s satisfies the properties as mentioned before with s instead of \bar{s} . Then, by the above argument $\eta_t : \mathbb{R}_+ \rightarrow (0, +\infty)$ is well defined and $\eta_t(s) > 0$ for all $s \in \mathbb{R}_+$. Furthermore, we have

$$\frac{\partial \beta_u}{\partial t}(\eta_t(s), s) = \eta_t(s) h_u(\eta_t(s), s) = 0,$$

that is,

$$\|u^+\|^2 + \eta_t^2(s) \|u^+\|_{\bar{E}}^4 + s^2 \|u^+\|_{\bar{E}}^2 \|u^-\|_{\bar{E}}^2 = \eta_t^{p-2}(s) \|u^+\|_p^p. \quad (3.10)$$

The function η_t has the following properties:

(a) η_t is continuous. In fact, if $s_n \rightarrow \bar{s}$ as $n \rightarrow +\infty$, we prove that $\{\eta_t(s_n)\}$ is bounded. Arguing by contradiction, suppose that there exists a subsequence (still denoted by s_n), such that $\eta_t(s_n) \rightarrow +\infty$ as $n \rightarrow +\infty$. Then, for some n large enough, we have $\eta_t(s_n) \geq s_n$. From (3.10) we get,

$$\frac{1}{\eta_t^2(s_n)} \|u^+\|^2 + \|u^+\|_{\bar{E}}^4 + \frac{s_n^2}{\eta_t^2(s_n)} \|u^+\|_{\bar{E}}^2 \|u^-\|_{\bar{E}}^2 = \eta_t^{p-4}(s_n) \|u^+\|_p^p. \quad (3.11)$$

Passing to the limit as $n \rightarrow +\infty$ with $p > 4$, we obtain $\|u^+\|_{\bar{E}}^4 = +\infty$, which is absurd. So $\{\eta_t(s_n)\}$ is bounded. Therefore, there exists a $\bar{t} > 0$ such that, up to a subsequence, one has

$$\eta_t(s_n) \rightarrow \bar{t}.$$

Moreover, by passing to the limit as $n \rightarrow +\infty$ in (3.10) with s_n instead of s we get

$$\|u^+\|^2 + \bar{t}^2 \|u^+\|_{\bar{E}}^4 + \bar{s}^2 \|u^+\|_{\bar{E}}^2 \|u^-\|_{\bar{E}}^2 = \bar{t}^{p-2} \|u^+\|_p^p,$$

which implies that

$$\frac{\partial \beta_u}{\partial t}(\bar{t}, \bar{s}) = 0.$$

As a result, $\bar{t} = \eta_t(\bar{s})$ implies that η_t is continuous.

(b) There exists $C_1 > 0$ large enough such that $\eta_t(s) < s$ for all $s \geq C_1$. In fact, suppose by contradiction that there exists a sequence $\{s_n\}$ such that $\eta_t(s_n) \geq s_n$ for all $n \in \mathbb{N}$. Then,

from (3.10) we have

$$\frac{1}{\eta_t^2(s_n)} \|u^+\|^2 + \|u^+\|_{\bar{E}}^4 + \frac{s_n^2}{\eta_t^2(s_n)} \|u^+\|_{\bar{E}}^2 \|u^-\|_{\bar{E}}^2 = \eta_t^{p-4}(s_n) \|u^+\|_p^p,$$

which implies that

$$\eta_t^{p-4}(s_n) \|u^+\|_p^p \leq \frac{1}{\eta_t^2(s_n)} \|u^+\|^2 + \|u\|_{\bar{E}}^2 \|u^+\|_{\bar{E}}^2.$$

Since $p - 4 > 0$, passing to the limit as $n \rightarrow +\infty$ we obtain $+\infty \leq C$ which is a contradiction. Hence, there exists $C_1 > 0$ large enough such that $\eta_t(s) < s$ for all $s \geq C_1$.

By (b) there exist $C_1 > 0$ such that $\eta_t(s) \leq s$ and $\mu_s(t) \leq t$ respectively when $t, s > C_1$. Let

$$C_2 = \max\left\{ \max_{s \in [0, C_1]} \eta_t(s), \max_{t \in [0, C_1]} \mu_s(t) \right\}.$$

Let $C = \max\{C_1, C_2\}$. We define $F : K \rightarrow \mathbb{R}_+^2$ by $F(t, s) = (\eta_t(s), \mu_s(t))$ where $K = [0, C] \times [0, C]$ is a bounded closed convex subset of \mathbb{R}_+^2 . It is clear that F is continuous and for all $(t, s) \in K$ we have

$$\begin{cases} \eta_t(s) \leq s \leq C, & s > C_1, \\ \eta_t(s) \leq C_2 \leq C, & s \leq C_1. \end{cases}$$

Thus, $\eta_t(s) \leq C$. Analogously, we have $\mu_s(t) \leq C$. Therefore, $F(K) \subset K$. Then, the Brouwer fixed point theorem implies that there exists $(t_u, s_u) \in [0, C] \times [0, C]$ such that

$$(\eta_t(s), \mu_s(t)) = (t_u, s_u).$$

Moreover, $t_u, s_u > 0$, because η_t and μ_s are positive by construction, and

$$\frac{\partial \beta_u}{\partial t}(t_u, s_u) = \frac{\partial \beta_u}{\partial s}(t_u, s_u) = 0.$$

It remains to show the uniqueness of (t_u, s_u) . Assume that $v \in \bar{\mathcal{N}}_{\pm}$, then

$$\begin{aligned} \nabla \beta_v(1, 1) &= \left(\frac{\partial \beta_v}{\partial t}(1, 1), \frac{\partial \beta_v}{\partial s}(1, 1) \right) \\ &= (\langle I'(v^+ + v^-), v^+ \rangle, \langle I'(v^+ + v^-), v^- \rangle) \\ &= (0, 0), \end{aligned}$$

which means that $(1, 1)$ is a critical point of β_v . Now, we shall show that $(1, 1)$ is the unique critical point of β_v with positive coordinates. Assume that (\bar{t}, \bar{s}) is a critical point of β_v . Without loss of generality, we assume that $0 < \bar{t} \leq \bar{s}$. Then

$$\|v^+\|^2 + \bar{t}^2 \|v^+\|_{\bar{E}}^4 + \bar{s}^2 \|v^+\|_{\bar{E}}^2 \|v^-\|_{\bar{E}}^2 = \bar{t}^{p-2} \|v^+\|_p^p, \quad (3.12)$$

and

$$\|v^-\|^2 + \bar{s}^2 \|v^-\|_{\bar{E}}^4 + \bar{t}^2 \|v^+\|_{\bar{E}}^2 \|v^-\|_{\bar{E}}^2 = \bar{s}^{p-2} \|v^-\|_p^p. \quad (3.13)$$

By (3.13) and $\frac{\bar{t}}{\bar{s}} \leq 1$ we get

$$\frac{1}{\bar{s}^2} \|v^-\|^2 + \|v\|_{\bar{E}}^2 \|v^+\|_{\bar{E}}^2 \geq \bar{s}^{p-4} \|v^+\|_p^p. \quad (3.14)$$

On the other hand, since $v \in \bar{\mathcal{N}}_{\pm}$ we have

$$\|v^-\|^2 + \|v\|_{\bar{E}}^2 \|v^+\|_{\bar{E}}^2 = \|v^-\|_p^p. \quad (3.15)$$

Combining (3.14) and (3.15) we get

$$\left(\frac{1}{\bar{s}^2} - 1\right) \|v^-\|^2 \geq (\bar{s}^{p-4} - 1) \|v^-\|_p^p. \quad (3.16)$$

If $\bar{s} > 1$ we get a contradiction in (3.16) by a similar argument as in (3.9). Therefore, $0 < \bar{t} \leq \bar{s} \leq 1$. Now we prove that $\bar{t} \geq 1$. In fact, from (3.12) and $1 \leq \frac{\bar{s}}{\bar{t}}$, we have

$$\frac{1}{\bar{t}^2} \|v^+\|^2 + \|v\|_{\bar{E}}^2 \|v^+\|_{\bar{E}}^2 \leq \bar{t}^{p-4} \|v^+\|_p^p. \quad (3.17)$$

On the other hand, since $v \in \bar{\mathcal{N}}_{\pm}$, we have

$$\|v^+\|^2 + \|v\|_{\bar{E}}^2 \|v^+\|_{\bar{E}}^2 = \|v^+\|_p^p. \quad (3.18)$$

Combining (3.17) and (3.18) we get

$$\left(\frac{1}{\bar{t}^2} - 1\right) \|v^+\|^2 \leq (\bar{t}^{p-4} - 1) \|v^+\|_p^p. \quad (3.19)$$

If $t < 1$ we get a contradiction in (3.19) by a similar argument as in (3.9), therefore $\bar{t} \geq 1$. Consequently, $\bar{t} = \bar{s} = 1$, which implies that $(1, 1)$ is the unique critical point of β_v with positive coordinates. Now, let $u \in E$ with $u^{\pm} \neq 0$, and $(t_u, s_u), (\bar{t}_u, \bar{s}_u)$ two critical points of β_u with $t_u, s_u, \bar{t}_u, \bar{s}_u > 0$. Then

$$u = t_u u^+ + s_u u^- \in \bar{\mathcal{N}}_{\pm}, \quad \bar{u} = \bar{t}_u u^+ + \bar{s}_u u^- \in \bar{\mathcal{N}}_{\pm}.$$

Let

$$v^+ = \bar{t}_u u^+, \quad v^- = \bar{s}_u u^-, \quad \tilde{t}_u = \frac{t_u}{\bar{t}_u}, \quad \tilde{s}_u = \frac{s_u}{\bar{s}_u}. \quad (3.20)$$

Then, $\tilde{t}_u v^+ + \tilde{s}_u v^- = t_u u^+ + s_u u^- \in \bar{\mathcal{N}}_{\pm}$ and $v = v^+ + v^- = \bar{t}_u u^+ + \bar{s}_u u^- \in \bar{\mathcal{N}}_{\pm}$. But we have proved above that if $v \in \bar{\mathcal{N}}_{\pm}$, then the unique critical point of β_v with positive coordinates is $(1, 1)$. Hence $\tilde{t}_u = \tilde{s}_u = 1$, which implies that $t_u = \bar{t}_u$ and $s_u = \bar{s}_u$, therefore, (t_u, s_u) is unique. Finally, we prove that the unique critical point (t_u, s_u) of β_u corresponds to the unique maximum point of β_u . In fact, since $p > 4$ for $(t, s) \in \mathbb{R}_+^2$ such that $|(t, s)| > 0$ small enough, $\beta_u(t, s) > 0$ and $\lim_{|(t, s)| \rightarrow +\infty} \beta_u(t, s) = -\infty$. Note that $\beta_u(t, s) = \beta_u(|t|, |s|)$, which implies that there exists $(t_u, s_u) \in \mathbb{R}_+^2$ such that $\beta_u(t_u, s_u) = \max_{(t, s) \in \mathbb{R}_+^2} \beta_u(t, s)$. So, to complete the proof we need to check that the

maximum of β_u cannot be achieved on the boundary of \mathbb{R}_+^2 . Without loss of generality, we may assume that $s_u = 0$. Then, for $s > 0$ sufficiently small, we have

$$\begin{aligned} \beta_u(t_u, 0) &\geq \beta_u(t_u, s) \\ &= \beta_u(t_u, 0) + \frac{s^2}{2} \|u^-\|^2 + \frac{s^4}{4} \|u^-\|_{\tilde{E}}^4 + \frac{t_u^2 s^2}{2} \|u^+\|_{\tilde{E}}^2 \|u^-\|_{\tilde{E}}^2 - \frac{|s|^p}{p} \|u^-\|_p^p \\ &\geq \beta_u(t_u, 0) + s^2 \|u^-\|^2 - \frac{|s|^p}{p} \|u^-\|_p^p \\ &> \beta_u(t_u, 0), \end{aligned}$$

which is a contradiction. Therefore, $s_u > 0$. Similarly, we prove that $t_u > 0$. Hence $\beta_u(t_u, s_u) = I(t_u u^+ + s_u u^-) = \max\{I(tu^+ + su^-) : t, s \geq 0\}$, which completes the proof. \square

Lemma 3.2. *For all $u \in \overline{\mathcal{N}}_{\pm}$ and $p \in (4, 2_*)$, there exists $C > 0$ such that $\|u\|_p^p \geq C$. Furthermore*

$$\bar{c} = \inf_{u \in \overline{\mathcal{N}}_{\pm}} I(u) > 0. \quad (3.21)$$

Proof. Arguing by contradiction, suppose that there exists a sequence $\{u_n\} \subset \overline{\mathcal{N}}_{\pm}$ such that $\|u_n\|_p^p \rightarrow 0$ as $n \rightarrow +\infty$. Then $\|u_n^+\|_p^p \leq \|u_n\|_p^p \rightarrow 0$ as $n \rightarrow +\infty$. It follows from $\langle I'(u_n), u_n^+ \rangle = 0$ that $\|u_n^+\| \rightarrow 0$ as $n \rightarrow +\infty$. On the other hand, by using the Sobolev embedding inequality and $\langle I'(u_n), u_n^+ \rangle = 0$ again, we have

$$\|u_n^+\|^2 + \|u_n\|_{\tilde{E}}^2 \|u_n^+\|_{\tilde{E}}^2 = \|u_n^+\|_p^p \leq C \|u_n^+\|^p.$$

Then

$$\|u_n^+\|^2 \leq \|u_n^+\|_p^p \leq C \|u_n^+\|^p,$$

which implies that there exists $C > 0$ such that $\|u_n^+\| \geq C$ since $p \in (4, 2_*)$. By passing to the limit as $n \rightarrow +\infty$, we obtain $0 < C \leq \lim_{n \rightarrow +\infty} \|u_n^+\| = 0$, which is absurd. Therefore, there exists $C > 0$ such that $\|u\|_p^p \geq C$ for all $u \in \overline{\mathcal{N}}_{\pm}$ and $p \in (4, 2_*)$. Now, for $u \in \overline{\mathcal{N}}_{\pm}$, we have from (3.5)

$$\begin{aligned} I(u) &= I(u) - \frac{1}{4} \langle I'(u), u \rangle \\ &= \frac{1}{4} \|u\|^2 + \left(\frac{1}{4} - \frac{1}{p} \right) \|u\|_p^p \\ &\geq \frac{1}{4} C + \left(\frac{1}{4} - \frac{1}{p} \right) C' = \alpha > 0. \end{aligned}$$

Hence $\bar{c} \geq \alpha > 0$. \square

Lemma 3.3. *For \bar{c} defined in (3.15), if there exists $u \in \overline{\mathcal{N}}_{\pm}$ such that $I(u) = \bar{c}$, then u is a weak solution of problem (1.7).*

Proof. The proof of this lemma is almost the same as that of Lemma 2.5 in [12] (see also [9, 5, 11]). So we omit it here. \square

Next, we shall prove that the minimizer u for (3.21) is achieved and it is indeed a least energy sign-changing solution of (1.7) using Lemmas 3.1 and 3.3.

Proof of Theorem 1.5. Let $\{u_n\} \subset \bar{\mathcal{N}}_{\pm}$ be a minimizing sequence of \bar{c} , i.e., $I(u_n) \rightarrow \bar{c}$ as $n \rightarrow +\infty$. Going if necessary to a subsequence, we may assume that $I(u_n) \leq 2\bar{c}$ for all $n \in \mathbb{N}$. Then, we have

$$2\bar{c} \geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n \rangle = \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\|u_n\|_p^p,$$

and

$$\|u_n\|^2 \leq 8\bar{c}; \quad \|u_n\|_p^p \leq \frac{8p}{p-4}\bar{c},$$

which implies that, $\{u_n\}$ is a bounded sequence of E . Therefore, by Lemma 1.1, there exists $u \in E$ such that $u_n \rightharpoonup u$ and $u_n^{\pm} \rightharpoonup u^{\pm}$ in E as $n \rightarrow +\infty$, $u_n^{\pm} \rightharpoonup u^{\pm}$ in \tilde{E} as $n \rightarrow +\infty$ and $u_n^{\pm} \rightarrow u^{\pm}$ in $L^s(\mathbb{R}^N)$ as $n \rightarrow +\infty$ for $2 \leq s < 2_*$. Moreover, by Lemma 3.2 there exists $C > 0$ such that $\|u_n^{\pm}\| \geq C$ and $\|u_n^{\pm}\|_p^p \geq C$, which implies that $u^{\pm} \neq 0$. Now, by Lemma 3.1 there exists $t_+, s_- > 0$ such that $\bar{u} = t_+u^+ + s_-u^- \in \bar{\mathcal{N}}_{\pm}$. Without loss of generality we may assume that $t_+ \geq s_- > 0$. Since $\{u_n\} \subset \bar{\mathcal{N}}_{\pm}$ we have

$$\|u_n^+\|^2 + \|u_n\|_{\tilde{E}}^2 \|u_n^+\|_{\tilde{E}}^2 = \|u_n^+\|_p^p,$$

and by the weak lower semicontinuity of the norm, we obtain

$$\|u^+\|^2 + \|u\|_{\tilde{E}}^2 \|u^+\|_{\tilde{E}}^2 \leq \|u^+\|_p^p. \quad (3.22)$$

On the other hand, since $t_+u^+ + s_-u^- \in \bar{\mathcal{N}}_{\pm}$, we have

$$\|u^+\|^2 + t_+^2 \|u^+\|_{\tilde{E}}^4 + s_-^2 \|u^-\|_{\tilde{E}}^2 \|u^+\|_{\tilde{E}}^2 = t_+^{p-2} \|u^+\|_p^p.$$

But $t_+ \geq s_-$, thus

$$\frac{1}{t_+^2} \|u^+\|^2 + \|u\|_{\tilde{E}}^2 \|u^+\|_{\tilde{E}}^2 \geq t_+^{p-4} \|u^+\|_p^p. \quad (3.23)$$

Combining (3.22) and (3.23), we get

$$\left(1 - \frac{1}{t_+^2}\right) \|u^+\|^2 \leq \left(1 - t_+^{p-4}\right) \|u^+\|_p^p,$$

which implies that $t_+ \leq 1$ since $p \in (4, 2_*)$. Therefore, $0 < s_- \leq t_+ \leq 1$. It follows from (3.5) and the weak lower semicontinuity of the norm that

$$\begin{aligned} \bar{c} &\leq I(\bar{u}) = I(t_+u^+ + s_-u^-) - \frac{1}{4}\langle I'(t_+u^+ + s_-u^-), t_+u^+ + s_-u^- \rangle \\ &= \frac{t_+^2}{4} \|u^+\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) t_+^p \|u^+\|_p^p + \frac{s_-^2}{4} \|u^-\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) s_-^p \|u^-\|_p^p \\ &\leq \frac{1}{4} \|u^+\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \|u^+\|_p^p + \frac{1}{4} \|u^-\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \|u^-\|_p^p \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4} \|u\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \|u\|_p^p \\ &\leq \liminf_{n \rightarrow +\infty} \left[\frac{1}{4} \|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \|u_n\|_p^p \right] \\ &= \liminf_{n \rightarrow +\infty} I(u_n) = \bar{c}. \end{aligned}$$

By the above inequality we deduce that $t_+ = s_- = 1$. Thus $\bar{u} = u$ and $I(u) = \bar{c}$. Then, by Lemma 3.3 we conclude that $u = u^+ + u^- \in \bar{\mathcal{N}}_{\pm}$ is a weak solution of (1.7). That is, a least energy sign-changing solution of problem (1.7).

Now, we show that u has exactly two nodal domains. Assume by contradiction that

$$u = u_1 + u_2 + u_3,$$

with

$$u_i \neq 0, \quad u_1(x) \geq 0, \quad u_2(x) \leq 0, \quad \text{and} \quad \text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset,$$

for $i \neq j, i, j = 1, 2, 3$, and

$$\langle I'(u), u_i \rangle = 0, \quad \text{for } i = 1, 2, 3.$$

Set $v = u_1 + u_2$, then $v^+ = u_1$ and $v^- = u_2$, i.e., $v^{\pm} \neq 0$. So, Lemma 3.1 implies that there is a unique pair (t_u, s_u) of positive numbers such that $t_u v^+ + s_u v^- \in \bar{\mathcal{N}}_{\pm}$, which means that $t_u u_1 + s_u u_2 \in \bar{\mathcal{N}}_{\pm}$. Noting that $\langle I'(u), u_i \rangle = 0$, for $i = 1, 2, 3$, we have $t_u, s_u \in (0, 1]$, therefore

$$\bar{c} \leq I(t_u u_1 + s_u u_2) \leq I(u) - \frac{1}{4} \|u_3\|^2 - \left(\frac{1}{4} - \frac{1}{p}\right) \|u_3\|_p^p < \bar{c}.$$

This is a contradiction, hence u has exactly two nodal domains. \square

Proof of Theorem 1.6. Let \mathcal{N} and c be given by (1.11) and (1.12) respectively, then, by using almost the same procedure in [12] (or in [10,8,17]), we can prove that, for each $v \in \tilde{E}$ with $v \neq 0$, there exists a unique $t_v > 0$ such that $t_v v \in \mathcal{N}$, $c > 0$ and there exists $v \in \mathcal{N}$ such that $I(v) = c$. Then Lemma 2.5 in [12] implies that v is a weak solution of (1.7), that is, a ground state solution of problem (1.7). From Theorem 1.4, we know that the problem (1.7) has a least energy sign-changing solution u . Suppose that $u = u^+ + u^-$. Since $u^+ \neq 0$ there exists $t_+ > 0$ such that $t_+ u^+ \in \mathcal{N}$, then by Proposition 2.1, we get

$$c \leq I(t_+ u^+) = I(t_+ u^+ + 0u^-) < I(u^+ + u^-) = \bar{c}.$$

That is $I(u) > c$, which implies that c cannot be achieved by a sign-changing function. This completes the proof. \square

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