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Gravitational field of Schwarzschild soliton

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Abstract. The aim of this paper is to study the gravitational field of Schwarzschild soliton. Use of characteristic of λ-tensor is given to determine the kinds of gravitational fields. Through the cases of two and three dimension for Schwarzschild soliton, the Gaussian curvature is expressed in terms of eigen values of the characteristic equation.

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1. Introduction

In 1982, Hamilton [5] introduced the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1.1}$$

to study compact three-manifolds with positive Ricci curvature and he called Eq. (1.1) as evolution equation. Hamilton proved many important and remarkable theorems for the Ricci flow, and laid the foundation for the program to approach the Poincare's conjecture and Thurstons geometrization conjecture via the Ricci flow. Further the idea was extended to Ricci soliton by pulling back the solutions of Ricci flow along a λ-dependent diffeomorphism. The Ricci soliton is a manifold (M, g_{ij}) whose metric tensor for a vector field ξ on it satisfies the equation

$$R_{ij} - \frac{1}{2}\pounds_{\xi} g_{ij} = kg_{ij} \tag{1.2}$$

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Here k is a constant and R_{ij} is the Ricci tensor for metric g_{ij}. The soliton is gradient if $\xi = \nabla\phi$, for some function ϕ and steady if $k = 0$. If $k < 0$ the soliton is called an expander; if $k > 0$ it is a shrinker.

For the four dimensional case Akbar and Woolger [3] have given a local $k = 0$ soliton, named as Schwarzschild soliton. Further the Ricci soliton for Lorentzian signature has been studied by Ali and Ahsan [2] and they have explored the case of Riesner-Nordström metric as a soliton. The metric of the Schwarzschild soliton is obtained by deforming the original Schwarzschild metric for a proper substitution of functions and vector fields, for which the new metric tensor satisfies Eq. (1.2). The Schwarzschild soliton is given by the following equation

$$ds^2 = -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 + dr^2 + (r^2 - 2mr)(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.3}$$

Motivated by the all important role of Ricci soliton in differential geometry and relativity, we have studied this concept for the spacetime of general relativity. We have chosen the Schwarzschild metric and studied its soliton in detail. By using the 6-dimensional formalism, the characteristic values of λ-tensor (i.e. $R_{AB} - \lambda g_{AB}$) have been given in this paper and an example of canonical form of the system is shown. Further the cases of 2 and 3-dimension for Schwarzschild soliton are discussed, in which Gaussian curvature is calculated and its dependence on characteristic value of λ-tensor is shown. Finally the geometry of Schwarzschild metric and Schwarzschild soliton was discussed.

2. Schwarzschild soliton

Eq. (1.3) for signature (1, 1, 1, −1) can also be written in the following form

$$ds^2 = dr^2 + (r^2 - 2mr)(d\theta^2 + sin^2\theta d\phi^2) - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 \tag{2.1}$$

The components of the potential for the gravitation or the metric tensor for Schwarzschild metric (2.1) in spherical coordinates $x^\alpha \equiv (r, \theta, \phi, t)$ are given by

$$g_{ij}(x^\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 - 2mr & 0 & 0 \\ 0 & 0 & (r^2 - 2mr)\sin^2\theta & 0 \\ 0 & 0 & 0 & -\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \end{pmatrix} \tag{2.2}$$

or

$$g_{11} = 1,\ g_{22} = r^2 - 2mr,\ g_{33} = (r^2 - 2mr)\sin^2\theta,\ g_{44} = -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \tag{2.3}$$

The Christoffel symbols, can be calculated from the formula [1]

$$\Gamma^i_{jk} = g^{il}\Gamma_{ljk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j}\right) \tag{2.4}$$

Thus the non-zero components of the Christoffel symbols for metric (2.1), by using Eq. (2.3) are

$$\begin{aligned}
&\Gamma^1_{22} = (m-r),\ \Gamma^1_{33} = (m-r)\sin^2\theta \\
&\Gamma^1_{44} = \frac{\sqrt{2}m}{r^2-2mr}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}},\ \Gamma^2_{12} = \Gamma^2_{21} = \frac{r-m}{r^2-2mr} \\
&\Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{13} = \Gamma^3_{31} = \frac{r-m}{r^2-2mr} \\
&\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta,\ \Gamma^4_{14} = \Gamma^4_{41} = \frac{\sqrt{2}m}{r^2-2mr}
\end{aligned} \tag{2.5}$$

While Riemann tensor for the Schwarzschild soliton (2.1) can be calculated from the formula [1]

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) + g_{mn}\left(\Gamma^m_{jk}\Gamma^n_{il} - \Gamma^m_{jl}\Gamma^n_{ik}\right) \tag{2.6}$$

and the non-zero components of Riemann tensor, by using Eq. (2.3) are

$$\begin{aligned}
&R_{1212} = \frac{m^2}{r^2-2mr} \\
&R_{1414} = \frac{2m}{(r^2-2mr)^2}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}[m+\sqrt{2}(m-r)] \\
&R_{2323} = -m^2\sin^2\theta \\
&R_{2424} = \frac{-\sqrt{2}m(m-r)}{(r^2-2mr)}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \\
&R_{3131} = \frac{m^2\sin^2\theta}{r^2-2mr} \\
&R_{3434} = \frac{-\sqrt{2}m(m-r)\sin^2\theta}{(r^2-2mr)}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}
\end{aligned} \tag{2.7}$$

We now use the 6-dimensional formalism in the pseudo-Euclidean space $\mathbb{R}^6$ by making the identification [4]

$$\begin{array}{lcccccc}
ij: & 23 & 31 & 12 & 14 & 24 & 34 \\
A: & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{2.8}$$

We also make use of the identification as

$$g_{ik}g_{jl} - g_{il}g_{jk} = g_{ijkl} \rightarrow g_{AB} \tag{2.9}$$

where $A, B = 1, 2, 3, 4, 5, 6$ and g_{ij} are the components of the metric tensor at an arbitrary point (x^α) of the Schwarzschild soliton, whose metric is given by Eq. (2.1). The new metric tensor $g_{AB}(A, B = 1, 2, 3, 4, 5, 6)$ is symmetric and non-singular.

The non-zero components of the metric tensor g_{AB} for Eq. (2.1) in 6-dimensional formalism, by using formulation (2.9) are as

$$
\begin{aligned}
&g_{11}(x^\alpha) = (r^2 - 2mr)^2 \sin^2\theta,\ g_{22}(x^\alpha) = (r^2 - 2mr)\sin^2\theta \\
&g_{33}(x^\alpha) = (r^2 - 2mr),\ g_{44}(x^\alpha) = -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \\
&g_{55}(x^\alpha) = -(r^2 - 2mr)\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \\
&g_{66}(x^\alpha) = -(r^2 - 2mr)\sin^2\theta\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}
\end{aligned} \tag{2.10}
$$

Similarly, we can transform the components of the Riemann tensor as $R_{ijkl} \to R_{AB}$. Thus, for example R_{1212} can be written as R_{33} [using identification (2.8)]. The non-zero components of the tensor R_{AB} under the identification (2.8) are

$$
\begin{aligned}
&R_{11}(x^\alpha) = -m^2 \sin^2\theta \\
&R_{22}(x^\alpha) = \frac{m^2 \sin^2\theta}{r^2 - 2mr},\ R_{33}(x^\alpha) = \frac{m^2}{r^2 - 2mr} \\
&R_{44}(x^\alpha) = \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}\left[m + \sqrt{2}(m - r)\right] \\
&R_{55}(x^\alpha) = \frac{-\sqrt{2}m(m - r)}{(r^2 - 2mr)}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \\
&R_{66}(x^\alpha) = \frac{-\sqrt{2}m(m - r)\sin^2\theta}{(r^2 - 2mr)}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}
\end{aligned} \tag{2.11}
$$

Further we use all these values to find a canonical form of the λ-tensor $R_{AB} - \lambda g_{AB}$. Next, we will be interested in eigen values for the Schwarzschild soliton (2.1), that is the solutions of the characteristic equation $|R_{AB} - \lambda g_{AB}| = 0$. By using Eqs. (2.10) and (2.11) easily, we calculate these eigen values and those are given by

$$
\begin{aligned}
&\lambda_1(r) = \frac{-m^2}{(r^2 - 2mr)^2} \\
&\lambda_2(r) = \frac{m^2}{(r^2 - 2mr)^2} = \lambda_3(r) \\
&\lambda_4(r) = \frac{-2m}{(r^2 - 2mr)^2}\left[m + \sqrt{2}(m - r)\right] \\
&\lambda_5(r) = \frac{\sqrt{2}m(m - r)}{(r^2 - 2mr)^2} = \lambda_6(r)
\end{aligned} \tag{2.12}
$$

$\lambda_i,\ i = 1, 2, 3, 4, 5, 6$, are the solution of the character equation $|R_{AB} - \lambda g_{AB}| = 0$ which depend on m and r. In other words we can say that for λ_i [equation (2.12)], the determinant of λ-tensor $R_{AB} - \lambda g_{AB}$ is zero. Thus we can transform the system in canonical form for values of λ_i as

$$g_{A'B'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and (2.13)

$$R_{A'B'} = \begin{pmatrix} \lambda_1(r) & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2(r) & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_4(r) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_5(r) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_6(r) \end{pmatrix}$$

Thus in our case (for Schwarzschild soliton) the gravitational field determined by λ- tensor is of the type $G_1[(1)(1)(11)(11)]$ in Segre symbols. From Eq. (2.13), we note that even if mass $m = 0$, the Schwarzschild soliton is flat.

2.1. Case I – $\theta = 0$ or $\theta = \pi$

When taking $\theta = 0$ or $\theta = \pi$ that is $d\theta = 0$, the Schwarzschild soliton, given by Eq. (2.1), reduces to the form

$$^{*}ds^2 = dr^2 - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 \tag{2.14}$$

Eq. (2.14) is a 2-dimensional surface now. The metric tensor $^{*}g$ in coordinates $x^\beta \equiv (r, t)$ is given by

$$^{*}g_{ij}(x^\beta) = \begin{bmatrix} 1 & 0 \\ 0 & -\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \end{bmatrix} \tag{2.15}$$

here $i,\ j = 1, 4$. Thus the hypersurface for $\theta = 0$ or $\theta = \pi$ (i.e., $^{*}H_0$ or $^{*}H_\pi$) degenerates to two dimensional surface. The non-zero component of Riemann curvature tensor for Eq. (2.14) is unique and given by

$$^{*}R_{1414}(x^\beta) = \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}\left[m + \sqrt{2}(m - r)\right] \tag{2.16}$$

so the Gaussian curvature $^{*}K$ for surface $^{*}H_0$ or $^{*}H_\pi$ is

$$^{*}K(x^\beta) = \frac{2m}{r^2 - 2mr}\left[m + \sqrt{2}(m - r)\right] \tag{2.17}$$

Eqs. (2.12) and (2.17) show that curvature of the 2-dimensional surface of the Schwarzschild soliton is related to the eigen value $\lambda_4(r)$.

2.2. Case II – $2m < r < \infty,\ 0 < \theta < \pi$ and $\phi = 0$

For this case, Eq. (2.1) reduces to

$$ds^2 = dr^2 + (r^2 - 2mr)d\theta^2 - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 \tag{2.18}$$

The metric tensor ${}^{**}g_{ij}$ for Eq. (2.18) in coordinate $x^\gamma \equiv (r, \theta, t)$ is given by

$$ {}^{**}g_{ij}(x^\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (r^2 - 2mr) & 0 \\ 0 & 0 & -\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \end{bmatrix} \tag{2.19}$$

The non-zero components of the Riemann curvature tensor for the metric (2.18) are as following

$$\begin{aligned} {}^{**}R_{1212}(x^\gamma) &= \frac{m^2}{r^2 - 2mr} \\ {}^{**}R_{1414}(x^\gamma) &= \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}\left[m + \sqrt{2}(m - r)\right] \\ {}^{**}R_{2424}(x^\gamma) &= \frac{-\sqrt{2}m(m - r)}{(r^2 - 2mr)}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \end{aligned} \tag{2.20}$$

So for the 3-dimensional space (2.18), the Gaussian curvature at each point $x^\gamma \equiv (r, \theta, t)$ is given by the following three physical quantities

$$\begin{aligned} {}^{**}K_1(x^\gamma) &= \frac{{}^{**}R_{2424}(x^\gamma)}{|{}^{**}g_{24}|} = \frac{\sqrt{2}m(m - r)}{(r^2 - 2mr)^2} \\ {}^{**}K_2(x^\gamma) &= \frac{{}^{**}R_{1414}(x^\gamma)}{|{}^{**}g_{14}|} = \frac{-2m}{(r^2 - 2mr)^2}\left[m + \sqrt{2}(m - r)\right] \\ {}^{**}K_4(x^\gamma) &= \frac{{}^{**}R_{1212}(x^\gamma)}{|{}^{**}g_{12}|} = \frac{m^2}{(r^2 - 2mr)^2} \end{aligned} \tag{2.21}$$

Here ${}^{**}g_{24}$ denotes the sub-matrix of ${}^{**}g_{ij}$ corresponding to $x^1 = r$. It is clear from Eqs. (2.12) and (2.21) that the curvature of the 3-dimensional space of Schwarzschild soliton can be expressed in terms of a λ-tensor which happens to be the solutions (eigen-values) of the characteristic equation $|R_{AB} - \lambda g_{AB}| = 0$.

3. Discussion

In this paper we worked out on gravitational field of Schwarzschild soliton by using characteristic of λ-tensor $R_{AB} - \lambda g_{AB}$, we have also discussed 2 and 3-dimensional cases. It is seen that Schwarzschild soliton, given by Akbar and Woolger [3] has a different geometry than that of Schwarzschild metric which is studied by Borgiel [4]. We see that the gravitational field for Schwarzschild soliton is of type $G_1[(1)(1)(11)(11)]$ [Eq. (16)] in Segre symbols while Borgiel has given type

$G_1[(1111)(11)]$ for Schwarzschild metric. For Schwarzschild soliton, not only does the Gaussian curvature differ with that of Schwarzschild metric but also the dependence of curvature on eigen values of λ-tensor $R_{AB} - \lambda g_{AB}$ is not similar. Thus the deformation in metric (along a λ-dependent diffeomorphism) of a spacetime is cause for change in geometry or gravitational field.

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