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Generating relations of multi-variable Tricomi functions of two indices using Lie algebra representation

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Abstract. This paper is an attempt to stress the usefulness of the multi-variable special functions. In this paper, we derive certain generating relations involving 2-indices 5-variables 5-parameters Tricomi functions (2I5V5PTF) by using a Lie-algebraic method. Further, we derive certain new and known generating relations involving other forms of Tricomi and Bessel functions as applications.

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1. Introduction

The function [1]

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\ x^r}{r!(n+r)!}, \tag{1.1}$$

is a Bessel like function known as Tricomi function and is related to the cylindrical Bessel function $J_n(x)$ by the following link [1]:

$$C_n(x) = x^{-n/2} J_n\left(2\sqrt{x}\right). \tag{1.2}$$

The Tricomi function $C_n(x)$ is defined by means of the generating function

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$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=-\infty}^{\infty}C_n(x)\ t^n, \tag{1.3}$$

which yields the recurrence relations

$$\frac{d}{dx}\ C_n(x)=-C_{n+1}(x), \tag{1.4a}$$

$$x\ C_{n+1}(x)-nC_n(x)+C_{n-1}(x)=0. \tag{1.4b}$$

On combining the above recurrences, we get the following differential equation satisfied by $C_n(x)$:

$$\left(x\ \frac{d^2}{dx^2}+(n+1)\frac{d}{dx}+1\right)C_n(x)=0. \tag{1.5}$$

The study of the properties of multi-variable generalized special functions has provided new means of analysis for the derivation of the solution of large classes of partial differential equations often encountered in physical problems. The relevance of the special functions in physics is well established. Most of the special functions of mathematical physics as well as their generalizations have been suggested by physical problems.

In order to further stress the usefulness of the generalized special functions, Dattoli et al. [3] have introduced the three variables two indices extension of Tricomi functions defined as:

$$C_{m,n}(x,y,z)=\sum_{s=0}^{\infty}C_{m+s}(x)\ C_{n+s}(y)\ \frac{z^s}{s!}. \tag{1.6}$$

The generating function for 2-indices 3-variables Tricomi functions (2I3VTF) $C_{m,n}(x,y,z)$ is given as:

$$\exp\left(\left(u-\frac{x}{u}\right)+\left(v-\frac{y}{v}\right)+\frac{z}{uv}\right)=\sum_{m,n=-\infty}^{\infty}C_{m,n}(x,y,z)u^m\ v^n. \tag{1.7}$$

We call $\boldsymbol{x}=(x_1,x_2,x_3,x_4,x_5)$ and $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)$, where $\xi_1,\ \xi_2,\ \xi_3,\ \xi_4$ and ξ_5 are arbitrary complex parameters.

For more further generalizations, along with the function one index-two variables Tricomi function defined by the generating function [2]

$$\exp\left(t-\frac{x}{t}+t^2-\frac{y}{t^2}\right)=\sum_{n=-\infty}^{\infty}C_n(x,y)\ t^n, \tag{1.8}$$

and the expansion series:

$$C_n(x,y)=\sum_{l=-\infty}^{\infty}C_{n-2l}(x)\ C_l(y), \tag{1.9}$$

we can introduce the five variables two indices extension of Tricomi functions defined as:

$$C_{m,n}(\boldsymbol{x})=\sum_{s=0}^{\infty}C_{m+s}(x_1,x_3)\ C_{n+s}(x_2,x_4)\ \frac{x_5^s}{s!}. \tag{1.10}$$

The relevant generating function can easily be derived

$$\exp\left(\left(u-\frac{x_1}{u}\right)+\left(v-\frac{x_2}{v}\right)+\left(u^2-\frac{x_3}{u^2}\right)+\left(v^2-\frac{x_4}{v^2}\right)+\frac{x_5}{uv}\right)=\sum_{m,n=-\infty}^{\infty}C_{m,n}(\boldsymbol{x})u^m v^n. \tag{1.11}$$

We consider 2-indices 5-variables 5-parameters Tricomi function (2I5V5PTF) defined as:

$$C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})=\sum_{s=0}^{\infty}C_{m+s}(x_1,x_3;\xi_1,\xi_3)\ C_{n+s}(x_2,x_4;\xi_2,\xi_4)\ \frac{(\xi_5 x_5)^s}{s!}, \tag{1.12}$$

where $C_n(x,y;\xi_1,\xi_2)$ denotes 2-variables 2-parameters Tricomi functions (2V2PTF), which satisfy each of the following identities [2]

$$\exp\left(t-\frac{\xi_1 x}{t}+\xi_2 t^2-\frac{y}{\xi_2 t^2}\right)=\sum_{n=-\infty}^{\infty}C_n(x,y;\xi_1,\xi_2)\ t^n, \tag{1.13}$$

$$C_n(x,y;\xi_1,\xi_2)=\sum_{l=-\infty}^{\infty}\xi_2^l\ C_{n-2l}(x;\xi_1)\ C_l(y), \tag{1.14}$$

$$C_n(x,y;\xi_1,\xi_2)=x^{-n/2}J_n\left(2\sqrt{x},2\sqrt{y};\xi_1,\xi_2\frac{x}{\sqrt{y}}\right), \tag{1.15}$$

and

$$J_n(x,y;\xi_1,\xi_2)=\left(\frac{x}{2}\right)^n C_n\left(\frac{x^2}{4},\frac{y^2}{4};\xi_1,\xi_2\frac{2y}{x^2}\right), \tag{1.16}$$

where $J_m(x,y;\xi_1,\xi_2)$ denoted 2V2PBF defined by Eq. (3)

$$\exp\left[\frac{x}{2}\left(t-\frac{\xi_1}{t}\right)+\frac{y}{2}\left(\xi_2 t^2-\frac{1}{\xi_2 t^2}\right)\right]=\sum_{n=-\infty}^{\infty}J_n(x,y;\xi_1,\xi_2)\ t^n, \tag{1.17}$$

$$J_n(x,y;\xi_1,\xi_2)=\sum_{l=-\infty}^{\infty}\xi_2^l\ J_{n-2l}(x;\xi_1)\ J_l(y). \tag{1.18}$$

The generating function for 2I5V5PTF is given by

$$\exp\left(\left(u-\frac{\xi_1 x_1}{u}\right)+\left(v-\frac{\xi_2 x_2}{v}\right)+\left(\xi_3 u^2-\frac{x_3}{\xi_3 u^2}\right)+\left(\xi_4 v^2-\frac{x_4}{\xi_4 v^2}\right)+\frac{\xi_5 x_5}{uv}\right)=\sum_{m,n=-\infty}^{\infty}C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})\ u^m\ v^n. \tag{1.19}$$

This is the most convenient form to find the generating relations by using a Lie-theoretic approach.

The theory of special functions from the group-theoretic point of view is a well established topic, providing a unifying formalism to deal with the immense aggregate of the special functions and a collection of formulae such as the relevant differential equations, integral representations, recurrence formulae, composition theorems, etc., see for example [10,11]. The first significant advance in the direction of obtaining generating relations by a Lie-theoretic approach is made by Weisner [12–14] and Miller [9]. Within the group-theoretic context, indeed a given class of special functions appears as a set of matrix elements of irreducible representation of a given Lie group. The algebraic properties of the group are then reflected in the functional and differential equations satisfied by a given family of special functions, whilst the geometry of the homogeneous space determines the nature of the integral representation associated with the family.

Recently some contributions related to Lie-theoretical representations of generalized Laguerre and Hermite polynomials, Bessel functions of two variables and multivariable Bessel and Tricomi functions have been given, see for example Khan [4], Khan and Pathan [7], Khan et al. [8], Khan and Hassan [5] and Khan and Khan [6].

The 3-dimensional complex local Lie group T_3 is the set of all 4×4 matrices of the form

$$g = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & c \\ 0 & 0 & e^{\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b, c, \tau \in \mathbb{C}. \tag{1.20}$$

A basis for the Lie algebra $\mathcal{T}_3 = L(T_3)$ is provided by the matrices

$$\mathcal{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.21}$$

with commutation relations

$$[\mathcal{J}^3,\ \mathcal{J}^+] = \mathcal{J}^+, \quad [\mathcal{J}^3,\ \mathcal{J}^-] = -\mathcal{J}^-, \quad [\mathcal{J}^+,\ \mathcal{J}^-] = 0. \tag{1.22}$$

Also, the Lie algebra $\mathcal{E}_3$ of the Euclidean group E_3 (real 3-parameters global Lie group) has basis elements

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.23}$$

with commutation relations

$$[\mathcal{J}_1,\ \mathcal{J}_2] = 0, \quad [\mathcal{J}_3,\ \mathcal{J}_1] = \mathcal{J}_2, \quad [\mathcal{J}_3,\ \mathcal{J}_2] = -\mathcal{J}_1. \tag{1.24}$$

Further, we observe that the complex matrices

$$\mathcal{J}^{+'} = -\mathcal{J}_2 + i\mathcal{J}_1, \quad \mathcal{J}^{-'} = \mathcal{J}_2 + i\mathcal{J}_1, \quad \mathcal{J}^{3'} = i\mathcal{J}_3, \quad i = \sqrt{-1} \tag{1.25}$$

satisfy the commutation relations identical to (1.22). Thus we say that $\mathcal{T}_3$ is the *complexification* of $\mathcal{E}_3$, and $\mathcal{E}_3$ is a *real form* of $\mathcal{T}_3$ [9]. Due to this relationship between $\mathcal{T}_3$ and $\mathcal{E}_3$, the abstract irreducible representation $Q(w, m_0)$ of $\mathcal{T}_3$ [10] induces an irreducible representation of $\mathcal{E}_3$.

In this paper, we derive certain generating relations involving 2-indices 5-variables 5-parameters Tricomi functions (2I5V5PTF) by using a Lie-algebraic method. In Section 2, we give a review of the basic properties of 2I5V5PTF $C_{m,n}(\boldsymbol{x}; \boldsymbol{\xi})$ and its special cases. In Section 3, we derive generating relations of 2I5V5PTF by using the representation $Q(w, m_0)$ of the Lie algebra $\mathcal{T}_3$. In Section 4, we obtain certain new and known generating relations involving various forms of Tricomi and Bessel functions. Finally, in Section 5, some concluding remarks are given.

2. Properties of 2I5V5PTF $C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})$

The 2I5V5PTF $C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})$ defined by Eqs. (1.6) and (1.7) satisfy the following differential and pure recurrence relations

$$\begin{aligned}
\frac{\partial}{\partial x_1}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= -\xi_1\, C_{m+1,n}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial x_2}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= -\xi_2\, C_{m,n+1}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial x_3}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= -\xi_3^{-1}\, C_{m+2,n}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial x_4}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= -\xi_4^{-1}\, C_{m,n+2}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial x_5}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= \xi_5\, C_{m+1,n+1}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial \xi_1}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= -x_1\, C_{m+1,n}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial \xi_2}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= -x_2\, C_{m,n+1}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial \xi_3}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= C_{m-2,n}(\boldsymbol{x};\boldsymbol{\xi}) + x_3\xi_3^{-2}\, C_{m+2,n}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial \xi_4}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= C_{m,n-2}(\boldsymbol{x};\boldsymbol{\xi}) + x_4\xi_4^{-2}\, C_{m,n+2}(\boldsymbol{x};\boldsymbol{\xi}),\\
\frac{\partial}{\partial \xi_5}\, C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= x_5\, C_{m+1,n+1}(\boldsymbol{x};\boldsymbol{\xi}),
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
mC_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= C_{m-1,n}(\boldsymbol{x};\boldsymbol{\xi}) + \xi_1 x_1 C_{m+1,n}(\boldsymbol{x};\boldsymbol{\xi}) + 2x_3\xi_3^{-1} C_{m+2,n}(\boldsymbol{x};\boldsymbol{\xi}) - \xi_5 x_5 C_{m+,n+1}(\boldsymbol{x};\boldsymbol{\xi}),\\
nC_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) &= C_{m,n-1}(\boldsymbol{x};\boldsymbol{\xi}) + \xi_2 x_2 C_{m,n+1}(\boldsymbol{x};\boldsymbol{\xi}) + 2x_4\xi_4^{-1} C_{m,n+2}(\boldsymbol{x};\boldsymbol{\xi}) - \xi_5 x_5 C_{m+1,n+1}(\boldsymbol{x};\boldsymbol{\xi}).
\end{aligned} \tag{2.2}$$

The differential equations satisfied by 2I5V5PTF $C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})$ are

$$\left(\frac{x_1}{\xi_1}\,\frac{\partial^2}{\partial x_1^2} + 2\frac{x_3}{\xi_1}\,\frac{\partial^2}{\partial x_1 \partial x_3} + \frac{x_5}{\xi_1}\,\frac{\partial^2}{\partial x_1 \partial x_5} + \frac{(m+1)}{\xi_1}\,\frac{\partial}{\partial x_1} + 1\right) C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = 0, \tag{2.3}$$

$$\left(\frac{x_2}{\xi_2}\,\frac{\partial^2}{\partial x_2^2} + 2\frac{x_4}{\xi_2}\,\frac{\partial^2}{\partial x_2 \partial x_4} + \frac{x_5}{\xi_2}\,\frac{\partial^2}{\partial x_2 \partial x_5} + \frac{(n+1)}{\xi_2}\,\frac{\partial}{\partial x_2} + 1\right) C_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = 0. \tag{2.4}$$

We note the following special cases of 2I5V5PTF $C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})$:

(1) $C_{m,n}(\boldsymbol{x};1,1,1,1,1) = C_{m,n}(\boldsymbol{x}),$ (2.5) where $C_{m,n}(\boldsymbol{x})$ denotes 2I5VTF defined by Eqs. (1.10) and (1.11).

(**2**) $C_{m,n}(x_1 \to x, x_2 \to y, x_3 \to z, x_4 \to w, 0; \xi_1, \xi_2, \xi_3, \xi_4, 0)$

$$= C_{m,n}(x, y, z, w; \xi_1, \xi_2, \xi_3, \xi_4) = C_m(x, z; \xi_1, \xi_3)\ C_n(y, w; \xi_2, \xi_4), \quad (2.6)$$

where $C_m(x, z; \xi_1, \xi_3)$ denotes 2V2PTF defined by Eqs. (1.13) and (1.14).

(**3**) $C_{m,n}(x_1 \to x, x_2 \to z, x_3 \to y, x_4 \to w, 0; 1, 1, \xi_3 \to \xi_1, \xi_4 \to \xi_2, 0)$

$$= C_m(x, y; \xi_1)\ C_n(z, w; \xi_2), \quad (2.7)$$

where $C_n(x, y; \xi)$ denotes 2V1PTF defined by Eq. (3)

$$\exp\left(t - \frac{x}{t} + \xi t^2 - \frac{y}{\xi t^2}\right) = \sum_{n=-\infty}^{\infty} C_n(x, y; \xi)\ t^n, \quad (2.8)$$

$$C_n(x, y; \xi) = \sum_{l=-\infty}^{\infty} \xi^l\ C_{n-2l}(x)\ C_l(y). \quad (2.9)$$

(**4**) $C_{m,n}(x_1 \to x, x_2 \to y, x_3 \to z, x_4 \to w, 0; 1, 1, 1, 1, 0)$

$$= C_{m,n}(x, y, z, w) = C_m(x, z)\ C_n(y, w), \quad (2.10)$$

where $C_{m,n}(x, y, z, w)$ denotes 2I4VTF defined by the generating functions

$$\exp\left(\left(u - \frac{x}{u}\right) + \left(v - \frac{y}{v}\right) + \left(u^2 - \frac{z}{u^2}\right) + \left(v^2 - \frac{w}{v^2}\right)\right) = \sum_{m,n=-\infty}^{\infty} C_{m,n}(x, y, z, w)\, u^m\, v^n. \quad (2.11)$$

(**5**) $C_{m,n}(x_1 \to x, x_2 \to y, x_3 \to \xi_3^2 x_3, x_4 \to \xi_4^2 x_4, x_5 \to z; 1, 1, \xi_3 \to 0, \xi_4 \to 0, 1) = C_{m,n}(x, y, z), \quad (2.12)$

where $C_{m,n}(x, y, z)$ denotes 2I3VTF defined by Eqs. (1.6) and (1.7).

(**6**) $C_{m,n}(x_1 \to (x - u), x_2 \to y, x_3 \to z, x_4 \to w, x_5 \to h; 1, 1, 1, 1, 1)$

$$= \sum_{s=0}^{\infty} \frac{u^s}{s!}\ C_{m+s,n}(x, y, z, w),$$

$C_{m,n}(x_1 \to x, x_2 \to (y - v), x_3 \to z, x_4 \to w, x_5 \to h; 1, 1, 1, 1, 1)$

$$= \sum_{s=0}^{\infty} \frac{v^s}{s!}\ C_{m,n+s}(x, y, z, w),$$

$C_{m,n}(x_1 \to x, x_2 \to y, x_3 \to z, x_4 \to w, x_5 \to h + \xi; 1, 1, 1, 1, 1)$

$$= \sum_{s=0}^{\infty} \frac{\xi^s}{s!}\ C_{m+s,n+s}(x, y, z, w, h). \quad (2.13)$$

(**7**) $$C_{m,n}(0, 0, 0, 0, h; 1, 1, 1, 1, 1) = \sum_{s=0}^{\infty} \sum_{l=0}^{\left[\frac{m+s}{2}\right]} \sum_{r=0}^{\left[\frac{n+s}{2}\right]} \frac{h^s}{s!\,l!\,r!\,(m + s - 2l)!\,(n + s - 2r)!}. \quad (2.14)$$

$$(\mathbf{8})\quad C_{m,n}(x_1 \to x, 0, x_3 \to y, 0, 0; 1, 1, 1, 1, 1) = \sum_{l=0}^{[\frac{n}{2}]} \frac{1}{l!(n-2l)!} C_m(x, y), \tag{2.15}$$

or

$$C_{m,n}(0, x_2 \to x, 0, x_4 \to y, 0; 1, 1, 1, 1, 1) = \sum_{l=0}^{[\frac{m}{2}]} \frac{1}{l!(m-2l)!} C_n(x, y), \tag{2.16}$$

where $C_m(x, y)$ denotes 2-variables Tricomi functions defined by Eqs. (1.8) and (1.9).

$$(\mathbf{9})\quad C_{m,n}(0, 0, 0, 0, 0; 1, 1, 1, 1, 1) = \sum_{l=0}^{[\frac{m}{2}]} \sum_{r=0}^{[\frac{n}{2}]} \frac{1}{l!r!(m-2l)!(n-2r)!}. \tag{2.17}$$

3. Generating relations involving 2I5V5PTF $C_{m,n}(\mathbf{x}; \boldsymbol{\xi})$

Miller [9] have determined realizations of the irreducible representation $Q(\omega, m_0)$ of $\mathcal{T}_3$ where $\omega, m_0 \in \mathbb{C}$ such that $\omega \neq 0$ and $0 \leqslant \text{Re } m_0 < 1$. The spectrum S of this representation is the set $\{m_0 + k : k \text{ an integer}\}$ and the representation space V has a basis $(f_m)_{m \in S}$, such that

$$\begin{aligned} &J^3 f_m = m f_m, \quad J^+ f_m = \omega f_{m+1}, \quad J^- f_m = \omega f_{m-1}, \\ &C_{0,0} f_m = (J^+ J^-) f_m = \omega^2 f_m, \quad \omega \neq 0. \end{aligned} \tag{3.1}$$

The commutation relations satisfied by the operators $J^3,\ J^\pm$ are

$$[J^3, J^+] = J^+, \quad [J^3, J^-] = -J^-, \quad [J^+, J^-] = 0. \tag{3.2}$$

In order to find the realizations of this representation on spaces of functions of two complex variables x and y, Miller [9; pp. 59–60]; has taken the functions $f_m(x, y) = Z_m(x) e^{my}$, such that relations (3.1) are satisfied for all $m \in S$, where the differential operators $J^3,\ J^\pm$ are given by

$$\begin{aligned} J^3 &= \frac{\partial}{\partial y}, \\ J^+ &= e^y \left[\frac{\partial}{\partial x} - \frac{1}{x} \frac{\partial}{\partial y} \right], \\ J^- &= e^{-y} \left[-\frac{\partial}{\partial x} - \frac{1}{x} \frac{\partial}{\partial y} \right]. \end{aligned} \tag{3.3}$$

In particular, we look for the functions

$$f_{m,n}(\mathbf{x}, u, v; \boldsymbol{\xi}) = Z_{m,n}(\mathbf{x}; \boldsymbol{\xi})\ u^m v^n, \tag{3.4}$$

such that

$$\begin{aligned} &K^3 f_{m,n} = m f_{m,n}, \quad K^+ f_{m,n} = \omega f_{m+1,n}, \quad K^- f_{m,n} = \omega f_{m-1,n}, \\ &C_{0,0} f_{m,n} = (K^+ K^-) f_{m,n} = \omega^2 f_{m,n}, \quad (\omega \neq 0;\ m \in S) \end{aligned} \tag{3.5}$$

and also

$$K^{3'}f_{m,n} = nf_{m,n}, \quad K^{+'}f_{m,n} = \omega f_{m,n+1}, \quad K^{-'}f_{m,n} = \omega f_{m,n-1},$$
$$C_{0,0}f_{m,n} = (K^{+'}K^{-'})f_{m,n} = \omega^2 f_{m,n}, \quad (\omega \neq 0;\ n \in S). \tag{3.6}$$

The sets of operators $\{K^3, K^+, K^-\}$ and $\{K^{3'}, K^{+'}, K^{-'}\}$ satisfy the commutation relations identical to (3.2).

There are numerous possible solutions of Eq. (3.2). We assume that the sets of linear differential operators $\{K^3, K^+, K^-\}$ and $\{K^{3'}, K^{+'}, K^{-'}\}$ take the forms

$$\begin{aligned} K^3 &= u\,\frac{\partial}{\partial u}, \\ K^+ &= \frac{u}{\xi_1}\,\frac{\partial}{\partial x_1}, \\ K^- &= -\frac{x_1}{u}\,\frac{\partial}{\partial x_1} - \frac{2x_3}{u}\,\frac{\partial}{\partial x_3} - \frac{x_5}{u}\,\frac{\partial}{\partial x_5} - \frac{\partial}{\partial u}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} K^{3'} &= v\frac{\partial}{\partial v}, \\ K^{+'} &= \frac{v}{\xi_2}\,\frac{\partial}{\partial x_2}, \\ K^{-'} &= -\frac{x_2}{v}\,\frac{\partial}{\partial x_2} - \frac{2x_4}{v}\,\frac{\partial}{\partial x_4} - \frac{x_5}{v}\,\frac{\partial}{\partial x_5} - \frac{\partial}{\partial v}, \end{aligned} \tag{3.8}$$

respectively. The operators in Eqs. (3.7) and (3.8) satisfy the commutation relations (3.2). In terms of the functions $Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi})$ and using operators (3.7) and (3.8), relations (3.5) and (3.6) reduce to

$$\begin{aligned} &\text{(i)}\ \ \frac{1}{\xi_1}\frac{\partial}{\partial x_1}Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = \omega Z_{m+1,n}(\boldsymbol{x};\boldsymbol{\xi}), \\ &\text{(ii)}\ \ \left[-x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3} - x_5\frac{\partial}{\partial x_5} - m\right]Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = \omega Z_{m-1,n}(\boldsymbol{x};\boldsymbol{\xi}), \\ &\text{(iii)}\ \ \left[\frac{-x_1}{\xi_1}\frac{\partial^2}{\partial x_1^2} - 2\frac{x_3}{\xi_1}\frac{\partial^2}{\partial x_1\partial x_3} - \frac{x_5}{\xi_1}\frac{\partial^2}{\partial x_1\partial x_5} - \frac{(m+1)}{\xi_1}\frac{\partial}{\partial x_1}\right]Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = \omega^2 Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\text{(i)}\ \ \frac{1}{\xi_2}\frac{\partial}{\partial x_2}Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = \omega Z_{m,n+1}(\boldsymbol{x};\boldsymbol{\xi}), \\ &\text{(ii)}\ \ \left[-x_2\frac{\partial}{\partial x_2} - 2x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5} - n\right]Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = \omega Z_{m,n-1}(\boldsymbol{x};\boldsymbol{\xi}), \\ &\text{(iii)}\ \ \left[\frac{-x_2}{\xi_2}\frac{\partial^2}{\partial x_2^2} - 2\frac{x_4}{\xi_2}\frac{\partial^2}{\partial x_2\partial x_4} - \frac{x_5}{\xi_2}\frac{\partial^2}{\partial x_2\partial x_5} - \frac{(n+1)}{\xi_2}\frac{\partial}{\partial x_2}\right]Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = \omega^2 Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}), \end{aligned} \tag{3.10}$$

respectively. We can take $\omega = -1$, without any loss of generality. For this choice of ω and in terms of the functions $Z_m(x)$, relations (3.1) become [9; p. 60 (3.25)]

$$\begin{aligned}
&\text{(i)} \quad \left[\frac{d}{dx}-\frac{m}{x}\right]Z_m(x) = -Z_{m+1}(x),\\
&\text{(ii)} \quad \left[\frac{d}{dx}+\frac{m}{x}\right]Z_m(x) = Z_{m-1}(x),\\
&\text{(iii)} \quad \left[-\frac{d^2}{dx^2}-\frac{1}{x}\frac{d}{dx}+\frac{m^2}{x^2}\right]Z_m(x) = Z_m(x).
\end{aligned} \tag{3.11}$$

We observe that (i) and (ii) of Eq. (3.11) agree with the conventional recurrence relations for Bessel functions $J_m(x)$ and (iii) coincides with the differential equation for $J_m(x)$. Thus we see that $Z_m(x) = J_m(x)$ is a solution of Eqs. (3.11) for all $m \in S$.

Similarly, we see that for $\omega = -1$, (iii) of Eqs. (3.9) and (3.10) coincide with the differential Eqs. (1.12) and (1.19) respectively of 2I5V5PTF $C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})$. In fact, for all $m, n \in S$ the choice for $Z_{m,n}(\boldsymbol{x};\boldsymbol{\xi}) = C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})$ satisfy Eqs. (3.9) and (3.10). Thus we conclude that the functions $f_{m,n}(\boldsymbol{x},u,v;\boldsymbol{\xi}) = C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})\, u^m v^n,\ m,n \in S$ form a basis for a realization of the representation $Q(-1,m_0)$ of $\mathcal{T}_3$. By using [9; p. 18 (Theorem 1.10)], this representation of $\mathcal{T}_3$ can be extended to a local multiplier representation [12; p. 17] of T_3. Using operators (3.7), the local multiplier representation $T(g),\ g \in T_3$ defined on $\mathcal{F}$, the space of all functions analytic in a neighborhood of the point $\left(x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, u^0, v^0; \xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0, \xi_5^0\right) = (1,1,1,1,0,1,1,0,0,1,1,1)$, takes the form

$$\begin{aligned}
&[T(\exp\tau\mathcal{J}^3)f](\boldsymbol{x},u,v;\boldsymbol{\xi}) = f(\boldsymbol{x},ue^{\tau},v;\boldsymbol{\xi}),\\
&[T(\exp b\mathcal{J}^+)f](\boldsymbol{x},u,v;\boldsymbol{\xi}) = f\left(x_1\left(1+\frac{bu}{\xi_1x_1}\right),x_2,x_3,x_4,x_5,u,v;\boldsymbol{\xi}\right), \quad \left|\frac{bu}{\xi_1x_1}\right|<1,\\
&[T(\exp c\mathcal{J}^-)f](\boldsymbol{x},u,v;\boldsymbol{\xi}) = f\left(x_1\left(1-\frac{c}{u}\right),x_2,x_3\left(1-\frac{c}{u}\right)^2,x_4,x_5\left(1-\frac{c}{u}\right),u\left(1-\frac{c}{u}\right),v;\boldsymbol{\xi}\right), \quad \left|\frac{c}{u}\right|<1.
\end{aligned} \tag{3.12}$$

If $g \in T_3$ is given by Eq. (1.20), we find

$$T(g) = T(\exp b\mathcal{J}^+)T(\exp c\mathcal{J}^-)T(\exp\tau\mathcal{J}^3),$$

and therefore we obtain

$$[T(g)f](\boldsymbol{x},u,v;\boldsymbol{\xi}) = f\left(x_1\left(1+\frac{bu}{\xi_1x_1}\right)\left(1-\frac{c}{u}\right),x_2,x_4,x_5\left(1-\frac{c}{u}\right),ue^{\tau}\left(1-\frac{c}{u}\right),v;\boldsymbol{\xi}\right),$$
$$\left|\frac{bu}{\xi_1x_1}\right|<1, \quad \left|\frac{c}{u}\right|<1. \tag{3.13}$$

The matrix elements of $T(g)$ with respect to the analytic basis $(f_{m,n})_{m,n\in S}$ are the functions $A_{lk}(g)$ uniquely determined by $Q(-1,m_0)$ of $\mathcal{T}_3$, and are defined by

$$[T(g)f_{m_0+k,n}](\boldsymbol{x},u,v;\boldsymbol{\xi}) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l,n}(\boldsymbol{x},u,v;\boldsymbol{\xi}),$$
$$k = 0, \pm1, \pm2, \pm3, \ldots \tag{3.14}$$

Therefore, we prove the following result

Theorem 3.1. *The following generating equation holds*

$$\left(1-\frac{c}{u}\right)^m C_{m,n}\left(x_1\left(1+\frac{bu}{\xi_1 x_1}\right)\left(1-\frac{c}{u}\right), x_2, x_3\left(1-\frac{c}{u}\right)^2, x_4, x_5\left(1-\frac{c}{u}\right); \boldsymbol{\xi}\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\, b^{(p+|p|)/2}\, c^{(-p+|p|)/2}\, {}_0F_1[-;|p|+1;bc]\, C_{m+p,n}(\boldsymbol{x};\boldsymbol{\xi})\, u^p,$$
$$\left|\frac{bu}{\xi_1 x_1}\right| < 1, \quad \left|\frac{c}{u}\right| < 1. \tag{3.15}$$

Proof. Using (3.13), we obtain

$$\exp(m\tau)\left(1-\frac{c}{u}\right)^m C_{m,n}\left(x_1\left(1+\frac{bu}{\xi_1 x_1}\right)\left(1-\frac{c}{u}\right), x_2, x_3\left(1-\frac{c}{u}\right)^2, x_4, x_5\left(1-\frac{c}{u}\right); \boldsymbol{\xi}\right)$$
$$= \sum_{l=-\infty}^{\infty} A_{l,m-m_0}(g) C_{m_0+l,n}(\boldsymbol{x};\boldsymbol{\xi})\, u^{m_0+l-m}, \tag{3.16}$$

and the matrix elements $A_{lk}(g)$ are given by ([9; p. 56], (3.12)′)

$$A_{lk}(g) = \exp((m_0+k)\tau)\, \frac{(-1)^{|k-l|}}{|k-l|!}\, b^{(l-k+|k-l|)/2}\, c^{(k-l+|k-l|)/2}\, {}_0F_1[-;|k-l| + 1; bc], \tag{3.17}$$

valid for all integral values of l, k and where ${}_0F_1$ denotes confluent hypergeometric function [1].

Substituting the value of $A_{lk}(g)$ given by (3.17) into (3.16) and simplifying, we obtain (3.15).

Similarly, for the operators (3.8), we have the following result □

Theorem 3.2. *The following generating equation holds*

$$\left(1-\frac{c'}{v}\right)^n C_{m,n}\left(x_1, x_2\left(1+\frac{b'v}{\xi_2 x_2}\right)\left(1-\frac{c'}{v}\right), x_3, x_4\left(1-\frac{c'}{v}\right)^2, x_5\left(1-\frac{c'}{v}\right); \boldsymbol{\xi}\right)$$
$$= \sum_{q=-\infty}^{\infty} \frac{(-1)^{|q|}}{|q|!}\, (b')^{(q+|q|)/2} (c')^{(-q+|q|)/2}\, {}_0F_1[-;|q|+1;b'c']\, C_{m,n+q}(\boldsymbol{x};\boldsymbol{\xi})\, v^q,$$
$$\left|\frac{b'v}{\xi_2 x_2}\right| < 1, \quad \left|\frac{c'}{v}\right| < 1. \tag{3.18}$$

The following corollaries are immediate consequences of Theorems 3.1 and 3.2.

Corollary 3.1. *The following generating equation holds*

$$\left(1+\frac{r}{2vu}\right)^m C_{m,n}\left(x_1\left(1+\frac{rvu}{2\xi_1 x_1}\right)\left(1+\frac{r}{2vu}\right), x_2, x_3\left(1+\frac{r}{2vu}\right)^2, x_4, x_5\left(1+\frac{r}{2vu}\right); \boldsymbol{\xi}\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p J_p(r)\, C_{m+p,n}(\boldsymbol{x};\boldsymbol{\xi})\, u^p, \qquad \left|\frac{rvu}{2\xi_1 x_1}\right| < 1, \quad \left|\frac{r}{2vu}\right| < 1. \tag{3.19}$$

Proof. If $bc \neq 0$, we can introduce the co-ordinates r, v such that $b = \frac{rv}{2}$ and $c = -\left(\frac{r}{2v}\right)$; with these new co-ordinates the matrix elements (3.17) can be expressed as

$$A_{lk}(g) = \exp((m_0 + k)\tau)\ (-v)^{l-k}\ J_{l-k}(r), \quad k = 0, \pm 1, \pm 2, \ldots \tag{3.20}$$

and generating relation (3.15) yields (3.19). □

Corollary 3.2. *The following generating equation holds*

$$\left(1+\frac{r'}{2v'v}\right)^n C_{m,n}\left(x_1, x_2\left(1+\frac{r'v'v}{2\xi_2 x_2}\right)\left(1+\frac{r'}{2v'v}\right), x_3, x_4\left(1+\frac{r'}{2v'v}\right)^2, x_5\left(1+\frac{r'}{2v'v}\right); \boldsymbol{\xi}\right)$$
$$= \sum_{q=-\infty}^{\infty} (-v')^q\ J_q(r')\ C_{m,n+q}(\boldsymbol{x};\boldsymbol{\xi})\ v^q, \qquad \left|\frac{r'v'v}{2\xi_2 x_2}\right| < 1, \quad \left|\frac{r'}{2v'v}\right| < 1. \tag{3.21}$$

4. Applications

We discuss some applications of the generating relations obtained in the preceding section.

I. Taking $c = 0$ and $u = 1$ in generating relation (3.15), we get

$$C_{m,n}\left(x_1\left(1+\frac{b}{\xi_1 x_1}\right), x_2, x_3, x_4, x_5; \boldsymbol{\xi}\right) = \sum_{p=0}^{\infty} \frac{(-b)^p}{p!}\ C_{m+p,n}(\boldsymbol{x};\boldsymbol{\xi}), \quad \left|\frac{b}{\xi_1 x_1}\right| < 1. \tag{4.1}$$

Taking $b = 0$ and $u = 1$ in generating relation (3.15), we get

$$(1-c)^m\ C_{m,n}(x_1(1-c), x_2, x_3(1-c)^2, x_4, x_5(1-c); \boldsymbol{\xi})$$
$$= \sum_{p=0}^{\infty} \frac{c^p}{p!}\ C_{m-p,n}(\boldsymbol{x};\boldsymbol{\xi}), \qquad |c| < 1. \tag{4.2}$$

Similarly, we can obtain results corresponding to generating relation (3.18).

II. Taking $\boldsymbol{\xi} = (1, 1, 1, 1, 1)$ in generating relations (3.15) and (3.19) and using Eq. (1.11), we get the following results:

$$\left(1-\frac{c}{u}\right)^m\ C_{m,n}\left(x_1\left(1+\frac{bu}{x_1}\right)\left(1-\frac{c}{u}\right), x_2, x_3\left(1-\frac{c}{u}\right)^2, x_4, x_5\left(1-\frac{c}{u}\right)\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\ b^{(p+|p|)/2}\ c^{(-p+|p|)/2}\ {}_0F_1[-; |p|+1; bc]\ C_{m+p,n}(\boldsymbol{x})\ u^p,$$
$$\left|\frac{bu}{x_1}\right| < 1, \left|\frac{c}{u}\right| < 1, \tag{4.3}$$

$$\left(1+\frac{r}{2vu}\right)^m\ C_{m,n}\left(x_1\left(1+\frac{rvu}{2x_1}\right)\left(1+\frac{r}{2vu}\right), x_2, x_3\left(1+\frac{r}{2vu}\right)^2, x_4, x_5\left(1+\frac{r}{2vu}\right)\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p\ J_p(r)\ C_{m+p,n}(\boldsymbol{x})\ u^p, \qquad \left|\frac{rvu}{2x_1}\right| < 1, \quad \left|\frac{r}{2vu}\right| < 1, \tag{4.4}$$

respectively, where $C_{m,n}(\boldsymbol{x})$ is given by Eqs. (1.10) and (1.11).
Similar results can be obtained from generating relations (3.18) and (3.21).

III. Taking $x_5 = 0$; $\boldsymbol{\xi} = (1,1,1,1,1)$ and replacing x_1 by x, x_2 by y, x_3 by z and x_4 by w in generating relations (3.15) and (3.19) and using Eq. (1.12), we get

$$\left(1-\frac{c}{u}\right)^m C_{m,n}\left(x\left(1+\frac{bu}{x}\right)\left(1-\frac{c}{u}\right), y, z\left(1-\frac{c}{u}\right)^2, w\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\ b^{(p+|p|)/2}\ c^{(-p+|p|)/2}\ {}_0F_1[-;|p|+1;bc]\ C_{m+p,n}(x,y,z,w)\ u^p,$$
$$\left|\frac{bu}{x}\right| < 1, \quad \left|\frac{c}{u}\right| < 1, \tag{4.5}$$

$$\left(1+\frac{r}{2vu}\right)^m C_{m,n}\left(x\left(1+\frac{rvu}{2x}\right)\left(1+\frac{r}{2vu}\right), y, z\left(1+\frac{r}{2vu}\right)^2, w\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p\ J_p(r)\ C_{m+p,n}(x,y,z,w)\ u^p, \quad \left|\frac{rvu}{2x}\right| < 1, \quad \left|\frac{r}{2vu}\right| < 1, \tag{4.6}$$

respectively, where $C_{m,n}(x,y,z,w)$ is given by Eq. (2.11).
Similar results can be obtained from generating relations (3.18) and (3.21).

IV. Taking $x_2 = x_4 = x_5 = 0$ and replacing x_1 by x, x_3 by z and ξ_3 by ξ_2 in generating relations (3.15) and (3.19) and using Eq. (1.12), we get

$$\left(1-\frac{c}{u}\right)^m C_m\left(x\left(1+\frac{bu}{x}\right)\left(1-\frac{c}{u}\right), z\left(1-\frac{c}{u}\right)^2; \xi_1, \xi_2\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\ b^{(p+|p|)/2}\ c^{(-p+|p|)/2}\ {}_0F_1[-;|p|+1;bc]\ C_{m+p}(x,z;\xi_1,\xi_2)\ u^p,$$
$$\left|\frac{bu}{x}\right| < 1, \quad \left|\frac{c}{u}\right| < 1, \tag{4.7}$$

$$\left(1+\frac{r}{2vu}\right)^m C_m\left(x\left(1+\frac{rvu}{2x}\right)\left(1+\frac{r}{2vu}\right), z\left(1+\frac{r}{2vu}\right)^2; \xi_1, \xi_2\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p\ J_p(r)\ C_{m+p}(x,z;\xi_1,\xi_2)\ u^p, \quad \left|\frac{rvu}{2x}\right| < 1, \quad \left|\frac{r}{2vu}\right| < 1, \tag{4.8}$$

respectively, where $C_m(x,z;\xi_1,\xi_2)$ is given by Eqs. (1.13) and (1.14).
For $\xi_1 = \xi_2 = 1$ and replacing z by y, the generating Eqs. (4.7) and (4.8) reduced to the results

$$\left(1-\frac{c}{u}\right)^m C_m\left(x\left(1+\frac{bu}{x}\right)\left(1-\frac{c}{u}\right), y\left(1-\frac{c}{u}\right)^2\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\ b^{(p+|p|)/2}\ c^{(-p+|p|)/2}\ {}_0F_1[-;|p|+1;bc]\ C_{m+p}(x,y)\ u^p,$$
$$\left|\frac{bu}{x}\right| < 1, \quad \left|\frac{c}{u}\right| < 1, \tag{4.9}$$

$$\left(1+\frac{r}{2vu}\right)^{m} C_m\left(x\left(1+\frac{rvu}{2x}\right)\left(1+\frac{r}{2vu}\right), y\left(1+\frac{r}{2vu}\right)^{2}\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p\ J_p(r)\ C_{m+p}(x,y)\ u^p, \qquad \left|\frac{rvu}{2x}\right| < 1, \quad \left|\frac{r}{2vu}\right| < 1, \tag{4.10}$$

respectively, where $C_m(x,y)$ is given by Eqs. (1.8) and (1.9).
Further, by replacing x by $x^2/4$, z by $z^2/4$, ξ_2 by $2\xi_2 y/x^2$ and u by $xu/2$ in Eqs. (4.7) and (4.8) and using Eqs. (1.15) and (1.16), we get the results:

$$\left(\frac{1-\frac{2c}{xu}}{1+\frac{2bu}{x}}\right)^{m/2} J_m\left(x\left(1+\frac{2bu}{x}\right)^{1/2}\left(1-\frac{2c}{xu}\right)^{1/2}, z\left(1-\frac{2c}{xu}\right); \xi_1, \frac{2\xi_2 z}{x}\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\ b^{(p+|p|)/2}\ c^{(-p+|p|)/2}\ {}_0F_1[-;|p|+1;bc]\ J_{m+p}(x,z;\xi_1,\xi_2)\ u^p,$$
$$\left|\frac{2bu}{x}\right| < 1, \quad \left|\frac{2c}{xu}\right| < 1, \tag{4.11}$$

$$\left(\frac{1+\frac{r}{vxu}}{1+\frac{rvu}{x}}\right)^{m/2} J_m\left(x\left(1+\frac{rvu}{x}\right)^{1/2}\left(1+\frac{r}{vxu}\right)^{1/2}, z\left(1+\frac{r}{vxu}\right); \xi_1, \frac{2\xi_2 z}{x}\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p\ J_p(r)\ J_{m+p}(x,z;\xi_1,\xi_2)\ u^p, \quad \left|\frac{r}{vxu}\right| < 1, \quad \left|\frac{rvu}{x}\right| < 1, \tag{4.12}$$

respectively, where $J_m(x,y;\xi_1,\xi_2)$ is defined by Eqs. (1.17) and (1.18).
Furthermore, by taking $\xi_1 = 1$ and replacing ξ_2 by β in generating relations (4.11) and (4.12), we get the results

$$\left(\frac{1-\frac{2c}{xu}}{1+\frac{2bu}{x}}\right)^{m/2} J_m\left(x\left(1+\frac{2bu}{x}\right)^{1/2}\left(1-\frac{2c}{xu}\right)^{1/2}, z\left(1-\frac{2c}{xu}\right); \frac{2\beta z}{x}\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\ b^{(p+|p|)/2}\ c^{(-p+|p|)/2}\ {}_0F_1[-;|p|+1;bc]\ J_{m+p}(x,z;\beta)\ u^p,$$
$$\left|\frac{2bu}{x}\right| < 1, \quad \left|\frac{2c}{xu}\right| < 1, \tag{4.13}$$

$$\left(\frac{1+\frac{r}{vxu}}{1+\frac{rvu}{x}}\right)^{m/2} J_m\left(x\left(1+\frac{rvu}{x}\right)^{1/2}\left(1+\frac{r}{vxu}\right)^{1/2}, z\left(1+\frac{r}{vxu}\right); \frac{2\beta z}{x}\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p\ J_p(r)\ J_{m+p}(x,z;\beta)\ u^p, \quad \left|\frac{r}{vxu}\right| < 1, \quad \left|\frac{rvu}{x}\right| < 1, \tag{4.14}$$

respectively, where $J_m(x,y;\beta)$ denotes a 2V1PBF defined by Eq. (3)

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(\beta t^2-\frac{1}{\beta t^2}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x,y;\beta)\ t^n, \tag{4.15}$$

$$J_n(x,y;\beta) = \sum_{l=-\infty}^{\infty} \beta^l\ J_{n-2l}(x)\ J_l(y). \tag{4.16}$$

Also, for $\beta = 1$ and replacing z by $x/2$, the generating relations (4.13) and (4.14) are reduced to the results

$$\left(\frac{1-\frac{2c}{xu}}{1+\frac{2bu}{x}}\right)^{m/2} J_m\left(x\left(1+\frac{2bu}{x}\right)^{1/2}\left(1-\frac{2c}{xu}\right)^{1/2}, \left(\frac{x}{2}-\frac{c}{u}\right)\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} \, b^{(p+|p|)/2} \, c^{(-p+|p|)/2} \, {}_0F_1[-;|p|+1;bc] \, J_{m+p}\left(x,\frac{x}{2}\right) u^p,$$
$$\left|\frac{2bu}{x}\right| < 1, \quad \left|\frac{2c}{xu}\right| < 1, \tag{4.17}$$

$$\left(\frac{1+\frac{r}{vxu}}{1+\frac{rvu}{x}}\right)^{m/2} J_m\left(x\left(1+\frac{rvu}{x}\right)^{1/2}\left(1+\frac{r}{vxu}\right)^{1/2}, \left(\frac{x}{2}+\frac{r}{2vu}\right)\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p \, J_p(r) \, J_{m+p}\left(x,\frac{x}{2}\right) u^p, \quad \left|\frac{r}{vxu}\right| < 1, \quad \left|\frac{rvu}{x}\right| < 1, \tag{4.18}$$

respectively, where $J_m(x,y)$ denotes a (2VBF) defined by the equations

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(x,y) \, t^n, \tag{4.19}$$

$$J_m(x,y) = \sum_{l=-\infty}^{\infty} J_{m-2l}(x) \, J_l(y). \tag{4.20}$$

Similar results can be obtained from generating relations (3.18) and (3.21).

V. Taking $\xi_1 = \xi_2 = \xi_5 = 1$, replacing x_1 by x, x_2 by y, x_5 by z, x_3 by $\xi_3^2 x_3$ and x_4 by $\xi_4^2 x_4$ and then taking $\xi_3, \xi_4 \to 0$ in Eqs. (3.15) and (3.19), we obtain ([5, p. 12 (3.15), (3.19)], for $Z_{m,n}(x,y,z) = C_{m,n}(x,y,z)$))

$$\left(1-\frac{c}{u}\right)^m C_{m,n}\left(x\left(1+\frac{bu}{x}\right)\left(1-\frac{c}{u}\right), y, z\left(1-\frac{c}{u}\right)\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} \, b^{(p+|p|)/2} \, c^{(-p+|p|)/2} \, {}_0F_1[-;|p|+1;bc] \, C_{m+p,n}(x,y,z) \, u^p,$$
$$\left|\frac{bu}{x}\right| < 1, \quad \left|\frac{c}{u}\right| < 1, \tag{4.21}$$

$$\left(1+\frac{r}{2vu}\right)^m C_{m,n}\left(x\left(1+\frac{rvu}{2x}\right)\left(1+\frac{r}{2vu}\right), y, z\left(1+\frac{r}{2vu}\right)\right)$$
$$= \sum_{p=-\infty}^{\infty} (-v)^p \, J_p(r) \, C_{m+p,n}(x,y,z) \, u^p, \quad \left|\frac{rvu}{2x}\right| < 1, \quad \left|\frac{r}{2vu}\right| < 1, \tag{4.22}$$

respectively, where $C_{m,n}(x,y,z)$ denotes a 2I3VTF given by Eqs. (1.6) and (1.7).

Further, taking $c = 0$ and $u = 1$ in generating relation (4.21), we get [5; p. 16 (4.1)]

$$C_{m,n}\left(x\left(1+\frac{b}{x}\right), y, z\right) = \sum_{p=0}^{\infty} \frac{(-b)^p}{p!} \, C_{m+p,n}(x,y,z), \quad \left|\frac{b}{x}\right| < 1. \tag{4.23}$$

Taking $b = 0$ and $u = 1$ in generating relation (4.21), we get [5; p. 16 (4.2)]

$$(1-c)^m \; C_{m,n}(x(1-c), y, z(1-c)) = \sum_{p=0}^{\infty} \frac{c^p}{p!} \; C_{m-p,n}(x, y, z), \quad |c| < 1. \tag{4.24}$$

Taking $y = z = 0$ in generating relation (4.21), we get [5; p. 16 (4.5) and (4.6)]

$$\left(1-\frac{c}{u}\right)^m \; C_m\left(x\left(1+\frac{bu}{x}\right)\left(1-\frac{c}{u}\right)\right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} \; b^{(p+|p|)/2} \; c^{(-p+|p|)/2}$$

$$\times \; {}_0F_1[-;|p|+1;bc] \; C_{m+p}(x) \; u^p, \qquad \left|\frac{bu}{x}\right| < 1, \quad \left|\frac{c}{u}\right| < 1, \tag{4.25}$$

$$\left(1+\frac{r}{2vu}\right)^m C_m\left(x\left(1+\frac{rvu}{2x}\right)\left(1+\frac{r}{2vu}\right)\right)$$

$$= \sum_{p=-\infty}^{\infty} (-v)^p J_p(r) \; C_{m+p}(x) \; u^p, \quad \left|\frac{rvu}{2x}\right| < 1, \quad \left|\frac{r}{2vu}\right| < 1, \tag{4.26}$$

respectively, where $C_m(x)$ is given by Eqs. (1.1) and (1.3).

Further, replacing x by $z^2/4$, u by $zu/2$ in relation (4.25) and using Eq. (1.2), we obtain ([9, p. 62 (3.29)], for $Z_m = J_m$)

$$\left(\frac{1-\frac{2c}{zu}}{1+\frac{2bu}{z}}\right)^{m/2} J_m\left(z\left(1+\frac{2bu}{z}\right)^{1/2}\left(1-\frac{2c}{zu}\right)^{1/2}\right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!} \; b^{(p+|p|)/2}$$

$$\times \; c^{(-p+|p|)/2} \; {}_0F_1[-;|p|+1;bc] \; J_{m+p}(z) \; u^p, \qquad \left|\frac{2bu}{z}\right| < 1, \quad \left|\frac{2c}{zu}\right| < 1. \tag{4.27}$$

Several of the fundamental identities for cylindrical functions are special cases of relation (4.27).

Also, for $c = 0$, $u = 1$ and $b = 0$, $u = 1$, relation (4.27) gives the formulas of Lommel ([9, p. 62 (3.30) and (3.31)], for $Z_m = J_m$).

Again, replacing x by $z^2/4$, u by $z/2$ in relation (4.26) and using Eq. (1.2), we obtain a generalization of Graf's addition theorem ([9, p. 63 (3.32)], for $Z_m = J_m$)

$$\left(\frac{1+\frac{r}{vz}}{1+\frac{rv}{z}}\right)^{m/2} J_m\left(z\left(1+\frac{rv}{z}\right)^{1/2}\left(1+\frac{r}{vz}\right)^{1/2}\right)$$

$$= \sum_{p=-\infty}^{\infty} (-v)^p \; J_p(r) \; J_{m+p}(z), \quad \left|\frac{rv}{z}\right| < 1, \quad \left|\frac{r}{vz}\right| < 1. \tag{4.28}$$

Similar results can be obtained from generating relations (3.18) and (3.21).

5. Concluding remarks

We note that the expressions (3.15) are valid only for group elements g in a sufficiently small neighbourhood of the identity element of the Lie group T_3. However, we can also use the operators (3.7) to derive generating relations for 2I5V5PTF and related functions with group elements bounded away from the identity.

If $f_{m,n}(\boldsymbol{x},u,v;\boldsymbol{\xi})$ is a solution of equation $C_{0,0}f=\omega^2 f$, i.e.,

$$\left[\frac{-x_1}{\xi_1}\frac{\partial^2}{\partial x_1^2}-2\frac{x_3}{\xi_1}\frac{\partial^2}{\partial x_1\partial x_3}-\frac{x_5}{\xi_1}\frac{\partial^2}{\partial x_1\partial x_5}-\frac{(m+1)}{\xi_1}\frac{\partial}{\partial x_1}\right]f_{m,n}(\boldsymbol{x},u,v;\boldsymbol{\xi}) = \omega^2 f_{m,n}(\boldsymbol{x},u,v;\boldsymbol{\xi}), \tag{5.1}$$

then the function $T(g)f$ given by (3.13) satisfies the equation

$$C_{0,0}(T(g)f)=\omega^2(T(g)f).$$

This follows from the fact that $C_{0,0}$ commutes with the operators K^+, K^- and K^3. Now, if f is a solution of the equation

$$(x_1K^+ + x_2K^- + x_3K^3)f_{m,n}(\boldsymbol{x},u,v;\boldsymbol{\xi}) = \lambda f_{m,n}(\boldsymbol{x},u,v;\boldsymbol{\xi}), \tag{5.2}$$

for constants x_1, x_2, x_3 and λ, then $T(g)f$ is a solution of the equation

$$[T(g)(x_1K^+ + x_2K^- + x_3K^3)T(g^{-1})][T(g)f] = \lambda[T(g)f]. \tag{5.3}$$

The inner automorphism μ_g of the Lie group T_3 defined by

$$\mu_g(h)=ghg^{-1}, \quad h\in T_3 \tag{5.4}$$

induces an automorphism $\mu_g^\star$ of the Lie algebra $\mathcal{T}_3$ where

$$\mu_g^\star(\alpha)=g\alpha g^{-1}, \quad \alpha\in\mathcal{T}_3.$$

If $\alpha = x_1\mathcal{J}^+ + x_2\mathcal{J}^- + x_3\mathcal{J}^3$ where $\mathcal{J}^+$, $\mathcal{J}^-$ and $\mathcal{J}^3$ are given by Eq. (1.20) and g is given by Eq. (1.21), then we have

$$\mu_g^\star(\alpha) = (x_1e^{\tau}-bx_3)\mathcal{J}^+ + (x_2e^{-\tau}+cx_3)\mathcal{J}^- + x_3\mathcal{J}^3. \tag{5.5}$$

Consequently, we can write

$$T(g)(x_1K^+ + x_2K^- + x_3K^3)T(g^{-1}) = (x_1e^{\tau}-bx_3)K^+ + (x_2e^{-\tau}+cx_3)K^- + x_3K^3. \tag{5.6}$$

To give an example of the application of these remarks, we consider the function $f_{m,n}(\boldsymbol{x},u,v;\boldsymbol{\xi}) = C_{m,n}(\boldsymbol{x};\boldsymbol{\xi})\ u^m\ v^n$, $m,n\in\mathbb{C}$.

Since $C_{0,0}f=f$ and $K^3f=mf$, so the function

$$[T(g)f](\boldsymbol{x};\boldsymbol{\xi}) = e^{m\tau}(u-c)^m v^n\, C_{m,n}\left((x_1+bu)\left(1-\frac{c}{u}\right),x_2,x_3\left(1-\frac{c}{u}\right)^2,x_4,x_5\left(1-\frac{c}{u}\right);\boldsymbol{\xi}\right) \tag{5.7}$$

satisfies the equations

$$C_{0,0}[T(g)f]=[T(g)f], \tag{5.8}$$
$$(-bK^+ + cK^- + K^3)[T(g)f] = m[T(g)f]. \tag{5.9}$$

For $\tau=b=0$ and $c=-1$, we can express the function (5.7) in the form

$$h(\boldsymbol{x};\boldsymbol{\xi}) = (u+1)^m\, v^n\, C_{m,n}\left(\left(x_1+\frac{x_1}{u}\right),x_2,x_3\left(1+\frac{1}{u}\right)^2,x_4,x_5\left(1+\frac{1}{u}\right);\boldsymbol{\xi}\right), \quad |u|<1. \tag{5.10}$$

Now, using the Laurent expansion

$$h(\boldsymbol{x}, u, v; \boldsymbol{\xi}) = \sum_{k=-\infty}^{\infty} h_{k,n}(\boldsymbol{x}; \boldsymbol{\xi})\ u^k\ v^n, \quad |u| < 1,$$

in Eq. (5.8), we note that $h_{k,n}(\boldsymbol{x}; \boldsymbol{\xi})$ is a solution of the differential Eq. (2.4) for each integer k. Since the function $h(\boldsymbol{x}, u, v; \boldsymbol{\xi})$ is bounded for $\boldsymbol{x} = (0, 0, 0, 0, 0)$, so we must have

$$h_{k,n}(\boldsymbol{x}; \boldsymbol{\xi}) = c_k\ C_{k,n}(\boldsymbol{x}; \boldsymbol{\xi}), \quad c_k \in \mathbb{C}.$$

Thus

$$h(\boldsymbol{x}, u, v; \boldsymbol{\xi}) = \sum_{k=-\infty}^{\infty} c_k\ C_{k,n}(\boldsymbol{x}; \boldsymbol{\xi})\ u^k\ v^n. \tag{5.11}$$

Now, from Eq. (5.9), we have

$$(-K^- + K^3)\ h(\boldsymbol{x}, u, v; \boldsymbol{\xi}) = m\ h(\boldsymbol{x}, u, v; \boldsymbol{\xi})$$

and therefore it follows that

$$c_{k+1} = (m - k)c_k.$$

Further, taking $\boldsymbol{x} = (0, 0, 0, 0, 0)$ in (5.10) and using (5.11), we get

$$c_0 = 1/\Gamma(m + 1)$$

and hence

$$c_k = 1/\Gamma(m - k + 1).$$

Thus, we obtain the following result:

$$(u+1)^m\ C_{m,n}\left(\left(x_1 + \frac{x_1}{u}\right), x_2, x_3\left(1 + \frac{1}{u}\right)^2, x_4, x_5\left(1 + \frac{1}{u}\right); \boldsymbol{\xi}\right) = \sum_{k=-\infty}^{\infty} \frac{C_{k,n}(\boldsymbol{x}; \boldsymbol{\xi})\ u^k}{\Gamma(m - k + 1)}, \qquad |u| < 1, \tag{5.12}$$

which is obviously not a special case of generating relation (3.15).

The result (5.12) was obtained by using operators (3.7). We can obtain another result by using operators (3.8). Several other examples of generating relations can be derived by this method see e.g. Weisner [14].

We have considered 2I5V5PTF $C_{m,n}(\boldsymbol{x}; \boldsymbol{\xi})$ within the group representation formalism. These functions appeared as basis functions for a realization of the representation $Q(-1, m_0)$ of the Lie algebra $\mathcal{T}_3$. The analysis presented in this paper confirms the possibility of extending this approach to other useful forms of generalized Tricomi functions as well as to their Bessel counter parts.

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