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Generalized derivations as homomorphisms or anti-homomorphisms on Lie ideals

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Abstract. Let R be a prime ring of $char(R) \neq 2$, Z the center of R, and L a nonzero Lie ideal of R. If R admits a generalized derivation F associated with a derivation d which acts as a homomorphism or as anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z$. This result generalizes a theorem of Wang and You.

Keywords: Generalized derivations; Martindale ring of quotients; Prime ring; Lie ideal; Homomorphisms and anti-homomorphisms

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1. INTRODUCTION

Throughout this paper, unless specifically stated, R will be an associative ring, Z the center of R, Q its two-sided Martindale quotient ring and U its right Utumi quotient ring (some times, as in [2], U is called the maximal right ring of quotients). The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [2], for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. Recall that a ring R is prime if $xRy = 0$ implies either $x = 0$ or $y = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$, if $d(x) = [q, x]$ holds for all $x \in R$. By a generalized inner derivation on R, one usually means an additive mapping $F : R \to R$ if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that $F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y)$,

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where I_b is an inner derivation determined by b. This observation leads to the definition given in [5]: an additive mapping $F : R \to R$ is called generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Obviously any derivation is a generalized derivation. Other basic examples of generalized derivations are the following: (i) $F(x) = ax + xb$ for $a, b \in R$; (ii) $F(x) = ax$ for some $a \in R$. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$ is a generalized derivation, where c is a fixed element of R and d is a derivation of R. In [16], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation on U, and thus all generalized derivations of R will be implicitly assumed to be defined on dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U (see Theorem 3, in [16]).

In [3, Theorem 3], Bell and Kappe proved that if d is a derivation of a prime ring R which acts as homomorphisms or anti-homomorphisms on a nonzero right ideal of R then $d = 0$ on R. Further Asma et al. [1], extend this result to Lie ideals of 2-torsion free prime rings. More precisely they prove that if L is a noncentral Lie ideal of R such that $u^2 \in L$, for all $u \in L$ and d acts as a homomorphism or anti-homomorphism on L, then $d = 0$. In 2007 Wang and You [19], eliminate the hypothesis $u^2 \in L$, for all $u \in L$ and prove the same result as Asma et al. [1]. To be more specific, the statement of Wang and You theorem is the following:

Theorem 1.1 (*[19, Theorem 1.2]*). *Let R be a 2-torsion free prime ring and L a nonzero Lie ideal of R. If d is a derivation of R which acts as a homomorphism or an anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z$.*

In [18], First author studies the case when the derivation d is replaced by a generalized derivation F and obtain the following: if R is a 2-torsion free prime ring and F acts as a homomorphism or an anti-homomorphism on a nonzero ideal of R, then R must be commutative. For more details related results we refer the reader to [7,8,10,11]. Our work is then motivated by the previous results. The aim of the present paper is to generalize Theorem 1.1, for generalized derivation F by using the same technique as Wang and You [19] with necessary variations.

Explicitly we shall prove the following theorem.

Theorem 1.2. *Let R be a prime ring with $char(R) \neq 2$, L a nonzero Lie ideal of R. If R admits a generalized derivation F associated with a derivation d which acts as a homomorphism or an anti-homomorphism on L, then $d = 0$ or $L \subseteq Z$.*

2. Main result

We will make frequent use of the following result due to Kharchenko [14] (see also [15]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two sided ideal of R. Let $f(x_1, \ldots, x_n, d(x_1, \ldots, x_n))$ be a differential identity in I, that is

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0 \quad \text{for all } r_1, \ldots, r_n \in I.$$

Then one of the following holds:

(1) either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)(x) = [q, x]$, for all $x \in R$ and I satisfies the generalized polynomial identity

$$f(r_1, \ldots, r_n, [q, r_1], \ldots, [q, r_n]) = 0;$$

(2) if it is not inner then d is called Q-outer and I satisfies the generalized polynomial identity

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0.$$

Remark 2.1. If I is a nonzero ideal of the prime ring R, then

(i) I, R and U satisfy the same generalized polynomial identities with coefficient in U [6, Theorem 2].
(ii) I, R and U satisfy the same differential identities [15, Theorem 2].

Now, we are in a position to prove the main result of the paper.

Proof of the Theorem 1.2. Assume on contrary that both $d \neq 0$ and $L \nsubseteq Z$. Since R is a prime ring and F is a generalized derivation associated with derivation d of R, by Lee [16, Theorem 3], $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Also by a result of Herstein [12, Lemma 1.3.], there exist a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. In particular, $[I, I] \subseteq L$, hence without loss of generality we may assume that $L = [I, I] \subseteq L$. We divide the proof into two cases:

Case 1. If F acts as a homomorphism on L, then we have

$$F([x, y])F([x_1, y_1]) = F([x, y][x_1, y_1]) = F([x, y])[x_1, y_1] + [x, y]d([x_1, y_1]),$$

for all $x, y, x_1, y_1 \in I$. Thus for all $x, y, x_1, y_1 \in I$, I satisfies the differential identity

$$\begin{aligned} a[x, y]a[x_1, y_1] &+ [d(x), y]a[x_1, y_1] + [x, d(y)]a[x_1, y_1] \\ &+ a[x, y][d(x_1), y_1] + [d(x), y][d(x_1), y_1] + [x, d(y)][d(x_1), y_1] \\ &+ a[x, y][x_1, d(y_1)] + [d(x), y][x_1, d(y_1)] + [x, d(y)][x_1, d(y_1)] \\ &= a[x, y][x_1, y_1] + [d(x), y][x_1, y_1] + [x, d(y)][x_1, y_1] \\ &+ [x, y][d(x_1), y_1] + [x, y][x_1, d(y_1)]. \end{aligned}$$

In the light of Kharchenko's theory [14], we divide the proof into two cases:

If the derivation d is Q-outer, by Kharchenko's theorem [14], I satisfies the polynomial identity

$$\begin{aligned} a[x, y]a[x_1, y_1] &+ [s, y]a[x_1, y_1] + [x, t]a[x_1, y_1] \\ &+ a[x, y][s_1, y_1] + [s, y][s_1, y_1] + [x, t][s_1, y_1] \\ &+ a[x, y][x_1, t_1] + [s, y][x_1, t_1] + [x, t][x_1, t_1] \\ &= a[x, y][x_1, y_1] + [s, y][x_1, y_1] + [x, t][x_1, y_1] \\ &+ [x, y][s_1, y_1] + [x, y][x_1, t_1], \quad \text{for all } x, y, x_1, y_1, s, t, s_1,\ t_1 \in I. \end{aligned}$$

In particular, for $x = x_1 = t_1 = 0$, I satisfies the blended component $[s, y][s_1, y_1] = 0$ for all $s, y, s_1, t_1 \in I$. In other words, $[I, I]^2 = 0$ i.e., $L^2 = 0$. By [4, Lemma 4], $L = 0$, a contradiction.

Let now d be an inner derivation induced by an element $q \in Q$, that is, $d(x) = [q, x]$ for all $x \in R$. Then, for any $x, y, x_1, y_1, s, t, s_1, t_1 \in I$,

$$\begin{aligned} a[x,y]a[x_1,y_1] &+ [[q,x],y]a[x_1,y_1] + [x,[q,y]]a[x_1,y_1] \\ &+ a[x,y][[q,x_1],y_1] + [[q,x],y][[q,x_1],y_1] + [x,[q,y]][[q,x_1],y_1] \\ &+ a[x,y][x_1,[q,y_1]] + [[q,x],y][x_1,[q,y_1]] + [x,[q,y]][x_1,[q,y_1]] \\ = a[x,y]&[x_1,y_1] + [[q,x],y][x_1,y_1] + [x,[q,y]][x_1,y_1] \\ &+ [x,y][[q,x_1],y_1] + [x,y][x_1,[q,y_1]]. \end{aligned}$$

By Chuang [6, Theorem 1], I and Q satisfy same generalized polynomial identities (GPIs), we have

$$\begin{aligned} a[x,y]a[x_1,y_1] &+ [[q,x],y]a[x_1,y_1] + [x,[q,y]]a[x_1,y_1] \\ &+ a[x,y][[q,x_1],y_1] + [[q,x],y][[q,x_1],y_1] + [x,[q,y]][[q,x_1],y_1] \\ &+ a[x,y][x_1,[q,y_1]] + [[q,x],y][x_1,[q,y_1]] + [x,[q,y]][x_1,[q,y_1]] \\ = a[x,y]&[x_1,y_1] + [[q,x],y][x_1,y_1] + [x,[q,y]][x_1,y_1] \\ &+ [x,y][[q,x_1],y_1] + [x,y][x_1,[q,y_1]], \quad \text{for all } x, y, x_1, y_1, s, t, s_1, t_1 \in Q. \end{aligned}$$

In case the center C of Q is infinite, we have

$$\begin{aligned} a[x,y]a[x_1,y_1] &+ [[q,x],y]a[x_1,y_1] + [x,[q,y]]a[x_1,y_1] \\ &+ a[x,y][[q,x_1],y_1] + [[q,x],y][[q,x_1],y_1] + [x,[q,y]][[q,x_1],y_1] \\ &+ a[x,y][x_1,[q,y_1]] + [[q,x],y][x_1,[q,y_1]] + [x,[q,y]][x_1,[q,y_1]] \\ = a[x,y]&[x_1,y_1] + [[q,x],y][x_1,y_1] + [x,[q,y]][x_1,y_1] \\ &+ [x,y][[q,x_1],y_1] + [x,y][x_1,[q,y_1]], \end{aligned}$$

for all $x, y, x_1, y_1, s, t, s_1, t_1 \in Q \otimes_C \overline{C}$, where $\overline{C}$ is algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C $(i.e., RC = R)$ which is either finite or algebraically closed and

$$\begin{aligned} a[x,y]a[x_1,y_1] &+ [[q,x],y]a[x_1,y_1] + [x,[q,y]]a[x_1,y_1] \\ &+ a[x,y][[q,x_1],y_1] + [[q,x],y][[q,x_1],y_1] + [x,[q,y]][[q,x_1],y_1] \\ &+ a[x,y][x_1,[q,y_1]] + [[q,x],y][x_1,[q,y_1]] + [x,[q,y]][x_1,[q,y_1]] \\ = a[x,y]&[x_1,y_1] + [[q,x],y][x_1,y_1] + [x,[q,y]][x_1,y_1] \\ &+ [x,y][[q,x_1],y_1] + [x,y][x_1,[q,y_1]], \end{aligned}$$

for all $x, y, x_1, y_1, s, t, s_1, t_1 \in R$. By Martindale [17, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H and the commuting division ring D is a finite dimensional central division algebra over Z. Since Z is either finite or algebraically closed, D must coincide with Z. Hence by Jacobson's theorem [13, p-75], R is isomorphic to a dense ring of linear transformations of some vector space V over Z i.e., $R \cong End(V_Z)$ and H consists of the finite rank linear transformations in R.

Step 1. we want to show that, for any $v \in V$, v and qv are linearly Z-dependent. Since if $qv = 0$ then $\{v, qv\}$ is linearly Z-dependent, suppose that $qv \neq 0$. If v and qv are linearly

Z-independent. By the density of R, there exist $x, y, x_1, y_1 \in R$ such that:

$$\begin{array}{llll} xv = 0, & xqv = qv, & yv = 0, & yqv = v; \\ x_1v = 0, & x_1qv = v, & y_1v = 0, & y_1qv = qv. \end{array}$$

These imply that

$$\begin{aligned} v =& \big(a[x,y]a[x_1,y_1] + [[q,x],y]a[x_1,y_1] + [x,[q,y]]a[x_1,y_1] \\ &+ a[x,y][[q,x_1],y_1] + [[q,x],y][[q,x_1],y_1] + [x,[q,y]][[q,x_1],y_1] \\ &+ a[x,y][x_1,[q,y_1]] + [[q,x],y][x_1,[q,y_1]] + [x,[q,y]][x_1,[q,y_1]]\big)v \\ =& \big(a[x,y][x_1,y_1] + [[q,x],y][x_1,y_1] + [x,[q,y]][x_1,y_1] \\ &+ [x,y][[q,x_1],y_1] + [x,y][x_1,[q,y_1]]\big)v \\ =& \ 0, \quad \text{a contradiction.} \end{aligned}$$

So we conclude that $\{v, qv\}$ are linearly Z-dependent, for all $v \in V$.

Step 2. We show here that there exists $\lambda \in Z$ such that $qv = v\lambda$, for any $v \in V$. Now choose $v, w \in V$ linearly Z-independent. By Step 1 there exist $\lambda_v, \lambda_w, \lambda_{v+w} \in Z$ such that $qv = v\lambda_v, qw = w\lambda_w, q(v+w) = (v+w)\lambda_{v+w}$, moreover $v\lambda_v + w\lambda_w = (v+w)\lambda_v + w$. Hence $v(\lambda_v - \lambda_{v+w}) + w(\lambda_w - \lambda_{v+w}) = 0$. Since v, w are linearly Z-independent, we have $\lambda_v = \lambda_w = \lambda_{v+w}$. This completes the proof of Step 2.

Let now for $r \in R, v \in V$. By Step 2, $qv = v\lambda$, $r(qv) = r(v\lambda)$, and also $q(rv) = (rv)\lambda$. Thus $0 = [q,r]v$, for any $v \in V$, that is $[q,r]V = 0$. Since V is a left faithful irreducible R-module, hence $[q,r] = 0$, for all $r \in R$, i.e., $q \in Z$ and $d = 0$, which contradicts our hypothesis.

Case 2. Now assume that F acts as an anti-homomorphism on L, so that for all $x, y, x_1, y_1 \in I$

$$F([x,y])[x_1,y_1] + [x,y]d([x_1,y_1]) = F([x,y][x_1,y_1]) = F([x_1,y_1])F([x,y]).$$

Thus I satisfies the differential identity

$$\begin{aligned} &a[x,y][x_1,y_1] + [d(x),y][x_1,y_1] + [x,d(y)][x_1,y_1] \\ &\quad + [x,y][d(x_1),y_1] + [x,y][x_1,d(y_1)] \\ &= a[x_1,y_1]a[x,y] + [d(x_1),y_1]a[x,y] + [x_1,d(y_1)]a[x,y] \\ &\quad + a[x_1,y_1][d(x),y] + [d(x_1),y_1][d(x),y] + [x_1,d(y_1)][d(x),y] \\ &\quad + a[x_1,y_1][x,d(y)] + [d(x_1),y_1][x,d(y)] + [x_1,d(y_1)][x,d(y)], \end{aligned}$$

for all $x, y, x_1, y_1 \in I$.

If d is not inner derivation, by Kharchenko's theorem [14], I satisfies the polynomial identity for all $x, y, x_1, y_1, s, t, s_1, t_1 \in I$

$$\begin{aligned} &a[x,y][x_1,y_1] + [s,y][x_1,y_1] + [x,t][x_1,y_1] + [x,y][s_1,y_1] + [x,y][x_1,t_1] \\ &= a[x_1,y_1]a[x,y] + [s_1,y_1]a[x,y] + [x_1,t_1]a[x,y] \\ &\quad + a[x_1,y_1][s,y] + [s_1,y_1][s,y] + [x_1,t_1][s,y] \\ &\quad + a[x_1,y_1][x,t] + [s_1,y_1][x,t] + [x_1,t_1][x,t]. \end{aligned}$$

In particular, for $x = x_1 = t_1 = 0$, I satisfies the blended component $[s, y][s_1, y_1] = 0$ for all $s, y, s_1, t_1 \in I$. It follows from Case 1 that $L = 0$, a contradiction.

Let now d be an inner derivation induced by an element $q \in Q$, that is, $d(x) = [q, x]$ for all $x \in R$. Since by Chuang [6, Theorem 1], I and Q satisfy same generalized polynomial identities (GPIs), we have

$$\begin{aligned}
&a[x,y][x_1,y_1] + [[q,x],y][x_1,y_1] + [x,[q,y]][x_1,y_1]\\
&\quad + [x,y][[q,x_1],y_1] + [x,y][x_1,[q,y_1]]\\
&= a[x_1,y_1]a[x,y] + [[q,x_1,y_1]]a[x,y] + [x_1,[q,y_1]]a[x,y]\\
&\quad + a[x_1,y_1][[q,x],y] + [[q,x_1],y_1][[q,x],y] + [x_1,[q,y_1]][[q,x],y]\\
&\quad + a[x_1,y_1][x,[q,y]] + [[q,x_1],y_1][x,[q,y]] + [x_1,[q,y_1]][x,[q,y]],
\end{aligned}$$

for all $x, y, x_1, y_1, s, t, s_1, t_1 \in Q$. In the view the above situation same as Case 1, now finally we claim that v and qv are Z-independent. Suppose to the contrary that v and qv are Z-independent. By the density of R, there exist $x, y, x_1, y_1 \in R$ such that

$$\begin{aligned}
&xv = 0, \qquad xqv = qv, \qquad yv = 0, \qquad yqv = v;\\
&x_1v = 0, \qquad x_1qv = v, \qquad y_1v = 0, \qquad y_1qv = qv.
\end{aligned}$$

We have,

$$\begin{aligned}
0 &= \big(a[x,y][x_1,y_1] + [[q,x],y][x_1,y_1] + [x,[q,y]][x_1,y_1]\\
&\quad + [x,y][[q,x_1],y_1] + [x,y][x_1,[q,y_1]]\big)v\\
&= \big(a[x_1,y_1]a[x,y] + [[q,x_1,y_1]]a[x,y] + [x_1,[q,y_1]]a[x,y]\\
&\quad + a[x_1,y_1][[q,x],y] + [[q,x_1],y_1][[q,x],y] + [x_1,[q,y_1]][[q,x],y]\\
&\quad + a[x_1,y_1][x,[q,y]] + [[q,x_1],y_1][x,[q,y]] + [x_1,[q,y_1]][x,[q,y]]\big)v\\
&= v, \quad \text{a contradiction.}
\end{aligned}$$

Thus, v and qv are Z-dependent. In the same way as Case 1 we can get $d = 0$, contradiction. With this the proof is complete.

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