

## General stability result in a memory-type porous thermoelasticity system of type III

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**Abstract.** In this paper, we consider a one-dimensional porous thermoelasticity system of type III with a viscoelastic damping acting on one of the equations. We establish a general decay result for the case of equal as well as different speeds of wave propagation.

**Keywords:** General decay; Porous; Thermoelasticity type III; Relaxation function; Viscoelastic damping; Equal and nonequal speed propagation

**Mathematics Subject Classification:** 35B37; 35L55; 74D05; 93D15; 93D20

### 1. INTRODUCTION

In this paper, we consider the following system

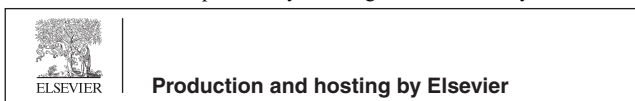
$$\begin{aligned} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \theta_x &= 0 \quad x \in (0, 1), \quad t > 0, \\ \rho_2 \psi_{tt} - \alpha \psi_{xx} + k(\varphi_x + \psi) - \theta + \int_0^t g(t-s) \psi_{xx}(x, s) ds &= 0, \quad x \in (0, 1), \quad t > 0 \\ \rho_3 \theta_{tt} - \kappa \theta_{xx} - \delta \theta_{xxt} + \beta \varphi_{xxt} + \beta \psi_{tt} &= 0, \quad x \in (0, 1), \quad t > 0 \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), & \quad x \in (0, 1) \\ \theta(x, 0) = \theta_0(x), \quad \theta_t(x, 0) = \theta_1(x), \quad & \quad x \in (0, 1) \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = \theta(0, t) = \theta(1, t) = 0, & \quad t \geq 0 \end{aligned} \tag{1.1}$$

where  $\varphi(x, t)$  is the longitudinal displacement,  $\psi(x, t)$  is the volume fraction,  $\theta(x, t)$  is the difference in temperature, the relaxation function  $g$  is positive and decreasing, the coefficients  $\rho_1, \rho_2, \rho_3, k, \alpha, \kappa, \delta, \beta$  are positive constants. This is a porous-thermoelastic system

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of type III with the presence of a viscoelastic damping supplemented by initial data  $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1$ .

The investigation of the asymptotic behavior of porous elastic problems has attracted lots of interest from researchers ever since Goodman and Cowin [4] first proposed an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter idea was introduced in the pioneered work of Nunziato and Cowin [20] in 1979 when they developed a nonlinear theory of elastic materials with voids. Later, Ieşan [5–8] added the temperature as well as the microtemperature elements to the theory. The importance of such materials could not be over-emphasized as it has resulted in the huge number of papers published in different fields of human endeavors most importantly, in petroleum industry, material science, biology and others.

The one-dimensional porous-elastic model has the form

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + b\varphi_x, \\ \rho \kappa \varphi_{tt} &= \alpha \varphi_{xx} - bu_x - \tau \varphi_t - a\varphi, \end{aligned} \quad (1.2)$$

where  $u$  is the longitudinal displacement,  $\varphi$  is the volume fraction,  $\rho > 0$  is the mass density,  $\kappa > 0$  is the equilibrated inertia and  $\mu, \alpha, \tau, a$  are the constitutive constants which are positive and satisfy  $\mu\alpha > b^2$ . Since this type of material has both macroscopic and microscopic structures, scientists have investigated the coupling and its strength as well as the long-time behavior of solution, using dissipation mechanisms at the microscopic and/or macroscopic levels. Many papers have appeared, where the authors tried to determine the type, as well as the rate of decay. The analysis of the time decay for this class of materials was started by Quintanilla [23] when he considered (1.2) with initial and mixed boundary conditions and showed that the damping in the porous equation ( $-\tau\varphi$ ) is not strong enough to obtain an exponential decay but only a slow decay. To improve this decay, several other damping mechanisms were considered, see for example, [13, 14, 19, 22].

For the porous case in classical thermoelasticity, we mention the following sample model which was developed by [1, 23],

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + bv_x - \beta\theta_x = 0, \quad x \in (0, L), \quad t > 0 \\ Jv_{tt} &= \alpha v_{xx} - bu_x - \zeta v + m\theta - \tau v_t, \quad x \in (0, L), \quad t > 0, \\ c\theta_t &= k\theta_{xx} - \beta u_{xt} - mv_t, \quad x \in (0, L), \quad t > 0 \\ u(0, t) &= u(L, t) = v_x(0, t) = v_x(L, t) = \theta_x(0, t) = \theta_x(L, t) = 0, \quad t > 0 \end{aligned} \quad (1.3)$$

where  $t$  denotes the time variable and  $x$  is the space variable, the functions  $u$  and  $v$  are the displacements of the solid elastic material, the function  $\theta$  is the temperature difference. The coefficients  $\rho, \mu, J, \alpha, \zeta, \tau, c$  and  $k$  are positive constants. Casas and Quintanilla [1] considered the above system and used the semigroup theory and the method developed by Liu and Zheng [10] to establish the exponential decay of the solutions. Later, with  $\tau = 0$  (absent of porous dissipation), the same author [2] showed that the heat effect alone is not strong enough to bring about an exponential decay but only a slow

decay could be established. However, the heat effect together with microtemperature produced exponential decay results. Similarly, when  $\tau = 0$  and  $\gamma u_{xxt}$  (viscoelastic dissipation) is added to the first equation in (1.3), Pamplona et al [21] proved that the system lacks exponential stability but by taking some regular initial data a polynomial stability is obtained. Also, for  $\tau = 0$ , Soufyane et al [25] considered (1.3) when the boundary conditions are replaced with

$$\begin{aligned} u(0, t) &= v(0, t) = \theta(0, t) = 0, \quad t \geq 0 \\ u(L, t) &= - \int_0^t g_1(t-s)[\mu u_x(L, s) + bv(L, s)]ds \\ v(L, t) &= - \int_0^t g_2(t-s)\alpha v_x(L, s)ds \end{aligned}$$

where  $g_1$  and  $g_2$  are positive nonincreasing functions. They obtained a general decay result, from which the usual exponential and polynomial decay rates are just special cases. Soufyane [24] considered

$$\begin{aligned} u_{tt} &= u_{xx} + v_x - \theta_x = 0, \quad x \in (0, L), \quad t > 0 \\ v_{tt} &= v_{xx} - u_x - v + \theta - \int_0^t g(t-s)v_{xx}(s)ds, \quad x \in (0, L), \quad t > 0 \\ \theta_t &= \theta_{xx} - u_{xt} - v_t, \quad x \in (0, L), \quad t > 0 \end{aligned} \tag{1.4}$$

with some initial and Dirichlet boundary conditions and  $g$  is a positive nonincreasing function. He used multiplier techniques to establish exponential and polynomial stability results depending on the relaxation function  $g$ . Recently, Messaoudi and Fareh [16,17] considered

$$\begin{aligned} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \theta_x &= 0, \quad x \in (0, 1), \quad t > 0 \\ \rho_2 \psi_{tt} - \alpha \psi_{xx} + k(\varphi_x + \psi) - \theta + \int_0^t g(t-s)\psi_{xx}(x, s)ds &= 0, \quad x \in (0, 1), \quad t > 0 \\ \rho_3 \theta_t - \kappa \theta_{xx} + \varphi_{xt} + \psi_t &= 0, \quad x \in (0, 1), \quad t > 0 \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), & \quad x \in (0, 1), \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, & \quad t \geq 0 \end{aligned} \tag{1.5}$$

where  $\rho_1, \rho_2, \rho_3, k, \alpha, \kappa$  are positive constants and  $g$  is a non-negative nonincreasing function. They established some general decay results for the solutions in the case of equal wave speeds  $\left(\frac{k}{\rho_1} = \frac{\alpha}{\rho_2}\right)$  as well as for different speeds of wave propagation  $\left(\frac{k}{\rho_1} \neq \frac{\alpha}{\rho_2}\right)$ .

For porous systems in nonclassical thermoelasticity, Magaña and Quintanilla [12] investigated the asymptotic behavior of the solutions of the following one-dimensional generalized porous-thermo-elasticity problem

$$\begin{aligned} \rho u_{tt} &= \mu u_{xx} + b\phi_x - \beta(\theta + \alpha\theta_t)_x, \quad x \in (0, \pi), \quad t > 0 \\ J\phi_{tt} &= \delta\phi_{xx} - bu_x - \zeta\phi + m(\theta + \alpha\theta_t) - \tau\phi_t, \quad x \in (0, \pi), \quad t > 0 \\ h\theta_{tt} &= k\theta_{xx} - \beta u_{xt} - m\phi_t - d\theta_t, \quad x \in (0, \pi), \quad t > 0 \end{aligned} \tag{1.6}$$

associated with some initial and mixed boundary conditions. They proved that, generically, the thermal damping ( $\tau = 0$ ) is not sufficiently strong to guarantee the exponential decay of solutions. But when the porous dissipation ( $\tau > 0$ ) is also present the

solutions decay exponentially. The arguments they used to prove the slow decay only work on a particular class of boundary conditions. However, for exponential decay of the solutions, the boundary conditions could be extended to other classes of boundary conditions.

Leseduarte et al [9] investigated the asymptotic behavior of solutions of the linear theory of thermo-porous-elasticity when the only dissipation mechanism is the porous dissipation. That is, they considered

$$\begin{aligned}
 \rho u_{tt} &= \mu u_{xx} + \gamma \phi_x - \beta \psi_{xt}, & x \in (0, \pi), \quad t > 0 \\
 J \phi_{tt} &= b \phi_{xx} + m \psi_{xx} - \xi \phi + d \psi_t - \tau \phi_t - \gamma u_x, & x \in (0, \pi), \quad t > 0 \\
 a \psi_{tt} &= k \psi_{xx} + m \phi_{xx} - d \phi_t - \beta u_{xt}, & x \in (0, \pi), \quad t > 0 \\
 u(0, t) = u(\pi, t) &= \phi_x(0, t) = \phi_x(\pi, t) = \psi_x(0, t) = \psi_x(\pi, t) = 0, & t > 0
 \end{aligned}
 \tag{1.7}$$

and showed that when the parameters  $m$  and  $\beta$  do not vanish, the decay of solutions is controlled by a negative exponential. However, the decay is not fast enough to allow a solution different from the null solution to vanish in a finite period of time. Whereas, if one of the parameters  $m$  or  $\beta$  vanishes, the decay of solutions is slow in the sense that it cannot be controlled by a negative exponential (generically).

In this paper, we investigate system (1.1) and establish a general decay result for the case of equal as well as different speeds of wave propagation. We should mention here that, to the best of our knowledge, there is no result concerning porous thermoelasticity systems of type III with the presence of a viscoelastic damping in the second equation. The rest of our paper is organized as follows. In Section 2, we introduce some transformations and assumptions needed in our work. We state and prove some technical lemmas in Section 3. The statements with proof for the case of equal and different speeds of wave propagation will be given in last section.

## 2. ASSUMPTIONS AND TRANSFORMATIONS

In this section, we present some materials needed in the proof of our results. In addition, we state without proof a global existence result. Throughout this paper,  $c$  is used to denote a generic positive constant. For the relaxation function  $g$ , we assume the following:

(A<sub>1</sub>)  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a  $C^1$  function satisfying

$$g(0) > 0, \alpha - \int_0^\infty g(s) ds = l > 0.$$

(A<sub>2</sub>) There exists a positive nonincreasing differentiable function  $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying

$$g'(t) \leq -\xi(t)g(t), \quad t \geq 0.$$

**Remark 2.1.** There are many functions satisfying (A<sub>1</sub>) and (A<sub>2</sub>). Examples of such functions are

$$g_1(t) = ae^{-bt}$$

$$g_2(t) = \frac{a}{(1+t)^b}$$

$$g_3(t) = \frac{a}{(e+t)[\ln(e+t)]^b}$$

for  $a, b > 1$  and  $a < \alpha(b - 1)$ . In fact, simple computations show that

$$g'_1(t) = -bg_1(t)$$

$$g'_2(t) = -\frac{b}{(1+t)}g_2(t)$$

$$g'_3(t) = -\left[\frac{1}{e+t} + \frac{b}{(e+t)\ln(e+t)}\right]g_3(t)$$

In order to exhibit the dissipative nature of system (1.1), we introduce the new variable

$$u(x, t) = \int_0^t \theta(x, s)ds + \chi(x), \tag{2.1}$$

where  $\chi(x)$  is the solution of

$$\begin{cases} -\kappa\chi'' = \delta\theta''_0 - \rho_3\theta_1 - \beta\varphi'_1 - \beta\psi_1, & \text{in } (0, 1), \\ \chi(0) = \chi(1) = 0. \end{cases} \tag{2.2}$$

A simple integration of the third equation in (1.1) with respect to  $t$  taking into account (2.1) and (2.2) transforms (1.1) into

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + u_{tx} &= 0, & x \in (0, 1), \quad t > 0 \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x + \psi) - u_t + \int_0^t g(t-s)\psi_{xx}(x, s)ds &= 0, & x \in (0, 1), \quad t > 0 \\ \rho_3u_{tt} - \kappa u_{xx} - \delta u_{txx} + \beta\varphi_{tx} + \beta\psi_t &= 0, & x \in (0, 1), \quad t > 0 \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), & x \in (0, 1), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in (0, 1), \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = u(0, t) = u(1, t) = 0, & t \geq 0. \end{aligned} \tag{2.3}$$

The well-posedness of (2.3) is stated in the following proposition.

**Proposition 2.2.** *Let  $((\varphi_0, \varphi_1), (\psi_0, \psi_1), (u_0, u_1)) \in (H_0^1(0, 1) \times L^2(0, 1))^3$  be given and assume that  $g$  satisfied  $(A_J)$  and  $(A_I)$ . Then, problem (2.3) has a unique global solution:*

$$(\varphi, \psi, u) \in (C(\mathbb{R}^+; H_0^1(0, 1)) \cap C^1(\mathbb{R}^+; L^2(0, 1)))^3$$

Moreover, if

$$(\varphi_0, \psi_0, u_0) \in (H^2(0, 1) \cap H_0^1(0, 1))^3 \text{ and } (\varphi_1, \psi_1, u_1) \in (H_0^1(0, 1))^3,$$

then the solution satisfies

$$(\varphi, \psi, u) \in (C(\mathbb{R}^+; H^2(0, 1) \cap H_0^1(0, 1)) \cap C^1(\mathbb{R}^+; H_0^1(0, 1)) \cap C^2(\mathbb{R}^+; L^2(0, 1)))^3$$

**Remark 2.3.** Proposition 2.2 can be established using standard methods such as the Galerkin method (see [3] for example).

The first-order energy associated with problem (2.3) is given as

$$\begin{aligned} E(t) &= E_1(\varphi, \psi, u) \\ &= \frac{\beta}{2} \int_0^1 \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k(\varphi_x + \psi)^2 + \left( \alpha - \int_0^t g(s) ds \right) \psi_x^2 \right] dx \\ &\quad + \frac{1}{2} \int_0^1 [\rho_3 u_t^2 + \kappa u_x^2] dx + \frac{\beta}{2} g \circ \psi_x, \end{aligned} \tag{2.4}$$

where

$$(g \circ v)(t) = \int_0^1 \int_0^t g(t-s)(v(x, t) - v(x, s))^2 ds dx, \quad \forall v \in L^2(0, 1).$$

### 3. TECHNICAL LEMMAS

In this section, we establish several lemmas needed to prove our main result.

**Lemma 3.1.** *Let  $(\varphi, \psi, u)$  be the solution of (2.3). Then the energy functional  $E$ , defined by (2.4) satisfies*

$$E'(t) = -\delta \int_0^1 u_{xt}^2 dx + \frac{\beta}{2} g' \circ \psi_x - \frac{\beta}{2} g(t) \int_0^1 \psi_x^2 dx \leq 0. \tag{3.1}$$

**Proof.** Multiplying the first equation of (2.3) by  $\beta \varphi_t$ , the second by  $\beta \psi_t$ , and the third by  $u_t$ , integrating over  $(0,1)$ , using integration by parts and the boundary conditions, then summing up, we obtain

$$\begin{aligned} &\frac{\beta}{2} \frac{d}{dt} \int_0^1 \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k(\varphi_x + \psi)^2 + \psi_x^2 \right] dx + \frac{1}{2} \frac{d}{dt} \int_0^1 [\rho_3 u_t^2 + \kappa u_x^2] dx \\ &\quad + \beta \int_0^1 \psi_t \int_0^t g(t-s) \psi_{xx}(s) ds dx = -\delta \int_0^1 u_{xt}^2 dx. \end{aligned} \tag{3.2}$$

The last term in the left-hand side of (3.2) gives

$$\begin{aligned} &\int_0^1 \psi_t \int_0^t g(t-s) \psi_{xx}(s) ds dx = \int_0^1 \psi_{xt} \int_0^t g(t-s) (\psi_x(t) - \psi_x(s)) ds dx \\ &\quad - \left( \int_0^t g(s) ds \right) \int_0^1 \psi_{xt} \psi_x dx = \frac{1}{2} \frac{d}{dt} \left[ g \circ \psi_x - \left( \int_0^t g(s) ds \right) \int_0^1 \psi_x^2 dx \right] \\ &\quad + \frac{1}{2} g(t) \int_0^1 \psi_x^2 dx - \frac{1}{2} g' \circ \psi_x. \end{aligned} \tag{3.3}$$

**Lemma 3.1** follows from the combination of 2.4, 3.2 and 3.3.  $\square$

**Lemma 3.2.** *Let  $(\varphi, \psi, u)$  be the solution of (2.3). Then the functional*

$$F_1(t) := -\rho_1 \int_0^1 \varphi_t \varphi dx - \rho_2 \int_0^1 \psi_t \psi dx$$

*satisfies, for any positive constant  $\varepsilon_1$ , the estimate*

$$\begin{aligned} F_1'(t) \leq & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + \frac{1}{\varepsilon_1} \int_0^1 u_t^2 dx + c(1 + \varepsilon_1) \int_0^1 \psi_x^2 dx \\ & + (k + \varepsilon_1) \int_0^1 (\varphi_x + \psi)^2 dx + cg \circ \psi_x. \end{aligned} \tag{3.4}$$

**Proof.** Direct computations, using the first and the second equations in (2.3), yields

$$\begin{aligned} F_1'(t) = & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx + \alpha \int_0^1 \psi_x^2 dx \\ & - \int_0^1 u_t \varphi_x dx - \int_0^1 u_t \psi dx - \int_0^1 \psi_x \int_0^t g(t-s) \psi_x(s) ds dx. \end{aligned}$$

By using Young’s and Poincaré’s inequalities, we get, for  $\varepsilon_1 > 0$ ,

$$\begin{aligned} F_1'(t) \leq & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \left(2\alpha + \frac{\varepsilon_1}{2}\right) \int_0^1 \psi_x^2 dx + \frac{\varepsilon_1}{2} \int_0^1 \varphi_x^2 dx + \frac{1}{\varepsilon_1} \int_0^1 u_t^2 dx \\ & + \frac{1}{4\alpha} \int_0^1 \left(\int_0^t g(t-s) \psi_x(s) ds\right)^2 dx. \end{aligned} \tag{3.5}$$

The fifth term in the right-hand side of (3.5) gives

$$\int_0^1 \varphi_x^2 dx = \int_0^1 [(\varphi_x + \psi) - \psi]^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi^2 dx.$$

By Poincaré’s inequality, we obtain

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi_x^2 dx. \tag{3.6}$$

By using the fact that  $(a + b)^2 \leq 2a^2 + 2b^2$  and the Cauchy–Schwarz inequality, we estimate the last term in the right-hand side of (3.5) as follows

$$\begin{aligned} & \int_0^1 \left(\int_0^t g(t-s) \psi_x(s) ds\right)^2 dx \\ & \leq 2 \int_0^1 \left(\int_0^t g(t-s) (\psi_x(s) - \psi_x(t)) ds\right)^2 dx + 2 \left(\int_0^t g(s) ds\right)^2 \int_0^1 \psi_x^2 dx \\ & \leq 2 \int_0^t g(s) ds \int_0^1 \int_0^t g(t-s) (\psi_x(s) - \psi_x(t))^2 ds dx + 2 \left(\int_0^t g(s) ds\right)^2 \int_0^1 \psi_x^2 dx. \end{aligned}$$

By using  $(A_1)$ , we obtain

$$\int_0^1 \left(\int_0^t g(t-s) \psi_x(s) ds\right)^2 dx \leq c \left(g \circ \psi_x + \int_0^1 \psi_x^2 dx\right). \tag{3.7}$$

By substituting (3.6) and (3.7) into (3.5) we get (3.4).  $\square$

**Lemma 3.3.** *Let  $(\varphi, \psi, u)$  be the solution of (2.3). Then the functional*

$$F_2(t) := -\rho_2 \int_0^1 \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx$$

satisfies, for any positive constant  $\varepsilon_2$ , the estimate

$$\begin{aligned} F_2'(t) \leq & -\left(\rho_2 \int_0^t g(s) ds - c\varepsilon_2\right) \int_0^1 \psi_t^2 dx + c\varepsilon_2 \int_0^1 \psi_x^2 dx \\ & + \varepsilon_2 \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_2 \int_0^1 u_t^2 dx + c\left(\varepsilon_2 + \frac{1}{\varepsilon_2}\right) g \circ \psi_x - \frac{c}{\varepsilon_2} g' \circ \psi_x. \end{aligned} \quad (3.8)$$

**Proof.** Taking the derivative of  $F_2(t)$  and using the second equation in (2.3), it easily follows that

$$\begin{aligned} F_2'(t) = & \alpha \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx - \rho_2 \left(\int_0^t g(t-s) ds\right) \int_0^1 \psi_t^2 dx \\ & + k \int_0^1 (\varphi_x + \psi) \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx \\ & - \int_0^1 u_t \int_0^t g(t-s)(\psi(t) - \psi(s)) ds dx \\ & - \rho_2 \int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s)) ds dx \\ & - \int_0^1 \int_0^t g(t-s)(\psi_x(s) ds) \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx, \end{aligned}$$

where we have used integration by parts and the boundary conditions in (2.3).

By using Young's inequality, we obtain for any  $\varepsilon_2 > 0$ ,

$$\begin{aligned} F_2'(t) \leq & -\left(\rho_2 \int_0^t g(s) ds - c\varepsilon_2\right) \int_0^1 \psi_t^2 dx + \varepsilon_2 \int_0^1 (\varphi_x + \psi) dx \\ & + \frac{c}{\varepsilon_2} \int_0^1 \left(\int_0^t g(t-s)(\psi(t) - \psi(s)) ds\right)^2 dx \\ & + c\varepsilon_2 \int_0^1 \psi_x^2 dx + \frac{\varepsilon_2}{2} \int_0^1 \left(\int_0^t g(t-s)\psi_x(s) ds\right)^2 dx \\ & + \frac{c}{\varepsilon_2} \int_0^1 \left(\int_0^t g'(t-s)(\psi(t) - \psi(s)) ds\right)^2 dx + \varepsilon_2 \int_0^1 u_t^2 dx \\ & + \frac{c}{\varepsilon_2} \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds\right)^2 dx. \end{aligned} \quad (3.9)$$

By exploiting the properties of  $g$ , Cauchy–Schwarz and Poincaré's inequalities, we get

$$\begin{aligned} & \int_0^1 \left(\int_0^t g(t-s)(\psi(t) - \psi(s)) ds\right)^2 dx \\ & \leq \int_0^t g(s) ds \int_0^1 \int_0^t g(t-s)(\psi(t) - \psi(s))^2 ds dx \\ & \leq (\alpha - l)g \circ \psi \leq cg \circ \psi_x. \end{aligned} \quad (3.10)$$

$$\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds\right)^2 dx \leq cg \circ \psi_x. \quad (3.11)$$



$$\begin{aligned}
 & \int_0^1 \left( \int_0^t g'(t-s)(\psi(t) - \psi(s)) ds \right)^2 dx \\
 & \leq \left( \int_0^t -g'(s) ds \right) \int_0^1 \int_0^t -g'(t-s)(\psi(t) - \psi(s))^2 ds dx \\
 & \leq -g(0)g' \circ \psi \leq -cg' \circ \psi_x.
 \end{aligned} \tag{3.12}$$

The substitution of (3.7), (3.10)–(3.12) into (3.9), gives (3.8).  $\square$

**Lemma 3.4.** *let  $(\varphi, \psi, u)$  be the solution of (2.3). Then the functional*

$$F_3(t) := \rho_3 \int_0^1 u_t u dx + \beta \int_0^1 \varphi_x u dx + \frac{\delta}{2} \int_0^1 u_x^2 dx,$$

*satisfies, for any positive constant  $\varepsilon_1$ , the estimate*

$$\begin{aligned}
 F_3' & \leq -\frac{\kappa}{2} \int_0^1 u_x^2 dx + c \left( 1 + \frac{1}{\varepsilon_1} \right) \int_0^1 u_t^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx + c \int_0^1 \psi_t^2 dx + \varepsilon_1 \\
 & \quad \times \int_0^1 (\varphi_x + \psi)^2 dx.
 \end{aligned} \tag{3.13}$$

**Proof.** By differentiating  $F_3(t)$  and using the third equation in (2.3), we obtain

$$F_3'(t) = -\kappa \int_0^1 u_x^2 dx + \rho_3 \int_0^1 u_t^2 dx + \beta \int_0^1 \varphi_x u_t dx - \beta \int_0^1 \psi_t u dx.$$

By using Young’s and Poincaré’s inequalities, we obtain for any  $\varepsilon_1 > 0$ ,

$$F_3'(t) \leq -\frac{\kappa}{2} \int_0^1 u_x^2 dx + \left( \rho_3 + \frac{\beta^2}{2\varepsilon_1} \right) \int_0^1 u_t^2 dx + \frac{\varepsilon_1}{2} \int_0^1 \varphi_x^2 dx + \frac{\beta^2}{2\kappa} \int_0^1 \psi_t^2 dx.$$

The conclusion of Lemma 3.4 follows courtesy (3.6).  $\square$

As in [19], we introduce the multiplier  $w$  which is the solution of

$$-w_{xx} = \psi_x, \quad w(0) = w(1) = 0. \tag{3.14}$$

**Lemma 3.5.** *The solution of (3.14) satisfies*

$$\begin{aligned}
 \int_0^1 w_x^2 dx & \leq \int_0^1 \psi^2 dx \leq \int_0^1 \psi_x^2 dx \\
 \int_0^1 w_t^2 dx & \leq \int_0^1 w_{tx}^2 dx \leq \int_0^1 \psi_t^2 dx.
 \end{aligned}$$

**Proof.** See [16].  $\square$

**Remark 3.6.** It can easily be shown that the solution of (3.14) is explicitly given as

$$w(x, t) = - \int_0^x \psi(s, t) ds + x \int_0^1 \psi(s, t) ds.$$

**Lemma 3.7.** *Let  $(\varphi, \psi, u)$  be the solution of (2.3). Then the functional*

$$F_4(t) := \rho_1 \int_0^1 \varphi_t w dx + \rho_2 \int_0^1 \psi_t \psi dx$$

*satisfies, for any positive constant  $\varepsilon_3$ , the estimate*

$$F_4'(t) \leq -\frac{l}{2} \int_0^1 \psi_x^2 dx + c \int_0^1 u_t^2 dx + c \left(1 + \frac{1}{\varepsilon_3}\right) \int_0^1 \psi_t^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + cg \circ \psi_x. \quad (3.15)$$

**Proof.** A simple differentiation of  $F_4(t)$ , then using the first and second equations in (2.3), leads to

$$F_4'(t) = k \int_0^1 w_x^2 dx + \int_0^1 u_t w_x dx + \rho_1 \int_0^1 \varphi_t w_t dx + \int_0^1 u_t \psi dx - \alpha \int_0^1 \psi_x^2 dx - k \int_0^1 \psi^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \int_0^1 \psi_x \int_0^t g(t-s) \psi_x(s) ds dx,$$

where we have used integration by parts, (3.14) and the boundary conditions in (2.3). By Young's and Poincaré's inequalities, we get for any  $\varepsilon_3, \varepsilon_4 > 0$ ,

$$F_4'(t) \leq (k + \varepsilon_4) \int_0^1 w_x^2 dx + \frac{1}{2\varepsilon_4} \int_0^1 u_t^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + \frac{\rho_2^2}{4\varepsilon_3} \int_0^1 w_t^2 dx - k \int_0^1 \psi^2 dx + (2\varepsilon_4 + \int_0^t g(s) ds - \alpha) \int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \int_0^1 \left(\int_0^t g(t-s)(\psi_x(s) - \psi_x(t)) ds\right)^2 dx.$$

By exploiting (A<sub>1</sub>), (3.11) and Lemma 3.5 we get

$$F_4'(t) \leq -(l - 3\varepsilon_4) \int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_4} \int_0^1 u_t^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + \left(\rho_2 + \frac{\rho_2^2}{4\varepsilon_3}\right) \int_0^1 \psi_t^2 dx + \frac{\alpha - l}{4\varepsilon_4} g \circ \psi_x. \quad (3.16)$$

By setting  $\varepsilon_4 = \frac{l}{6}$ , the conclusion of our proof follows.  $\square$

**Lemma 3.8.** *Let  $(\varphi, \psi, u)$  be the solution of (2.3). Then, the functional*

$$F_5(t) := \rho_2 \int_0^1 \psi_t (\varphi_x + \psi) dx + \frac{\alpha \rho_1}{k} \int_0^1 \varphi_t \psi_x dx - \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^t g(t-s) \psi_x(s) ds dx$$

*satisfies, for any positive constant  $\varepsilon_1$ , the estimate*

$$F_5'(t) \leq \left[ \varphi_x \left( \alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds \right) \right]_{x=0}^{x=1} + \rho_2 \int_0^1 \psi_t^2 dx + c \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \psi_x^2 dx - \frac{k}{2} \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_1 \int_0^1 \varphi_t^2 dx + c \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 u_{xt}^2 dx - \frac{c}{\varepsilon_1} g' \circ \psi_x + c\varepsilon_1 g \circ \psi_x + \left(\rho_2 - \frac{\alpha \rho_1}{k}\right) \int_0^1 \varphi_{xt} \psi_t dx. \quad (3.17)$$

**Proof.** By using Eq. (2.3) and integrating by parts, we get

$$\begin{aligned}
 F'_5(t) &= \left[ \varphi_x (\alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds) \right]_{x=0}^{x=1} + \rho_2 \int_0^1 \psi_t^2 dx \\
 &\quad - k \int_0^1 (\varphi_x + \psi)^2 dx + \frac{1}{k} \int_0^1 u_{xt} \int_0^t g(t-s) \psi_x(s) ds dx \\
 &\quad - \frac{\rho_1 g(0)}{k} \int_0^1 \varphi_t \psi_x dx - \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^t g'(t-s) \psi_x(s) ds dx \\
 &\quad + \left( \rho_2 - \frac{2\rho_1}{k} \right) \int_0^1 \varphi_{xt} \psi_t dx - \frac{z}{k} \int_0^1 u_{xt} \psi_x dx + \int_0^1 u_t (\varphi_x + \psi) dx \\
 &= \left[ \varphi_x (\alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds) \right]_{x=0}^{x=1} + \rho_2 \int_0^1 \psi_t^2 dx + \int_0^1 u_t (\varphi_x + \psi) dx \\
 &\quad - k \int_0^1 (\varphi_x + \psi)^2 dx + \frac{1}{k} \int_0^1 u_{xt} \int_0^t g(t-s) (\psi_x(s) - \psi_x(t)) ds dx \\
 &\quad - \frac{\rho_1 g(t)}{k} \int_0^1 \varphi_t \psi_x dx + \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^t g'(t-s) (\psi_x(t) - \psi_x(s)) ds dx \\
 &\quad + \left( \rho_2 - \frac{2\rho_1}{k} \right) \int_0^1 \varphi_{xt} \psi_t dx - \frac{z}{k} \int_0^1 u_{xt} \psi_x dx + \frac{1}{k} \int_0^1 g(s) ds \int_0^1 u_{xt} \psi_x dx.
 \end{aligned}$$

By using Young’s and Poincaré’s inequalities, (3.11), (3.12) and the properties of  $g$ , (3.17) is established.  $\square$

In consideration of the boundary terms that appear in (3.17), we define, as in [18], the function

$$m(x) = 2 - 4x, \quad x \in [0, 1].$$

Consequently, we have the following result:

**Lemma 3.9.** *Let  $(\varphi, \psi, u)$  be the solution of (2.3). Then, for any positive constant  $\varepsilon_1$ , the functional*

$$F_6(t) := \frac{\varepsilon_1 \rho_1}{k} \int_0^1 m(x) \varphi_t \varphi_x dx + \frac{\rho_2}{4\varepsilon_1} \int_0^1 m(x) \psi_t \left( \alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds \right) dx$$

satisfies the estimate

$$\begin{aligned}
 F'_6(t) &\leq - \left[ \varphi_x (\alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds) \right]_{x=0}^{x=1} + c \left( \varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^2} \right) \int_0^1 \psi_x^2 dx \\
 &\quad + c\varepsilon_1 \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{\varepsilon_1} \int_0^1 \psi_t^2 dx + c\varepsilon_1 \int_0^1 \varphi_t^2 dx + c\varepsilon_1 \int_0^1 u_{xt}^2 dx \\
 &\quad + c \left( \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^2} \right) g \circ \psi_x - \frac{c}{\varepsilon_1} g' \circ \psi_x.
 \end{aligned} \tag{3.18}$$

**Proof.** By using Eq. (2.3) and integrating by parts, we get

$$\begin{aligned}
 F'_6(t) &= - \left[ \varepsilon_1 \varphi_x^2(1) + \frac{1}{4\varepsilon_1} (\alpha \psi_x(1) - \int_0^t g(t-s) \psi_x(1, s) ds)^2 \right] - \frac{\varepsilon_1}{k} \int_0^1 m u_{xt} \varphi_x dx \\
 &\quad - \left[ \varepsilon_1 \varphi_x^2(0) + \frac{1}{4\varepsilon_1} (\alpha \psi_x(0) - \int_0^t g(t-s) \psi_x(0, s) ds)^2 \right] + 2\varepsilon_1 \int_0^1 \varphi_x^2 dx \\
 &\quad + \frac{2\rho_2}{2\varepsilon_1} \int_0^1 \psi_t^2 dx + \frac{2\varepsilon_1 \rho_1}{k} \int_0^1 \varphi_t^2 dx - \frac{\rho_2 g(t)}{4\varepsilon_1} \int_0^1 m \psi_t \psi_x dx \\
 &\quad + \frac{1}{2\varepsilon_1} \int_0^1 (\alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds)^2 dx + \varepsilon_1 \int_0^1 m \psi_x \varphi_x dx \\
 &\quad + \frac{1}{4\varepsilon_1} \int_0^1 m u_t (\alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds) dx \\
 &\quad - \frac{k}{4\varepsilon_1} \int_0^1 m (\varphi_x + \psi) (\alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds) dx \\
 &\quad + \frac{\rho_2}{4\varepsilon_1} \int_0^1 m \psi_t \left( \int_0^t g'(t-s) (\psi_x(t) - \psi_x(s)) ds \right) dx.
 \end{aligned} \tag{3.19}$$

In what follows, we use Young's and Poincaré's inequalities, (3.7), (3.12), properties of  $g$  and the fact that  $0 \leq m^2(x) \leq 4, \forall x \in [0,1]$  and  $(a + b)^2 \leq 2a^2 + 2b^2$ .

$$\begin{aligned} & \varphi_x(1)(\alpha\psi_x(1) - \int_0^1 g(t-s)\psi_x(1,s)ds) \\ & \leq \varepsilon_1 \varphi_x^2(1) + \frac{1}{4\varepsilon_1} (\alpha\psi_x(1) - \int_0^1 g(t-s)\psi_x(1,s)ds)^2. \end{aligned} \quad (3.20)$$

$$\begin{aligned} & -\varphi_x(0)(\alpha\psi_x(0) - \int_0^1 g(t-s)\psi_x(0,s)ds) \\ & \leq \varepsilon_1 \varphi_x^2(0) + \frac{1}{4\varepsilon_1} (\alpha\psi_x(1) - \int_0^1 g(t-s)\psi_x(0,s)ds)^2. \end{aligned} \quad (3.21)$$

$$\begin{aligned} -\frac{1}{k} \int_0^1 m u_{xt} \varphi_x dx & \leq \frac{1}{2k} \int_0^1 m^2 \varphi_x^2 dx + \frac{1}{2k} \int_0^1 u_{xt}^2 dx \\ & \leq c \int_0^1 \varphi_x^2 dx + c \int_0^1 u_{xt}^2 dx. \end{aligned} \quad (3.22)$$

$$\begin{aligned} -\frac{\rho_2 g(t)}{4} \int_0^1 m \psi_t \psi_x dx & \leq \frac{1}{2} \int_0^1 \psi_x^2 dx + \frac{\rho_2^2 g^2(t)}{32} \int_0^1 m^2 \psi_t^2 dx \\ & \leq c \int_0^1 \psi_x^2 dx + c \int_0^1 \psi_t^2 dx. \end{aligned} \quad (3.23)$$

$$\int_0^1 m \psi_x \varphi_x dx \leq c \int_0^1 \varphi_x^2 dx + c \int_0^1 \psi_x^2 dx. \quad (3.24)$$

$$\begin{aligned} & \frac{1}{2} \int_0^1 (\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds)^2 dx \\ & \leq \alpha^2 \int_0^1 \psi_x^2 dx + \int_0^1 (\int_0^t g(t-s)\psi_x(s)ds)^2 dx \\ & \leq c \int_0^1 \psi_x^2 dx + c g \circ \psi_x. \end{aligned} \quad (3.25)$$

$$\begin{aligned} & \frac{1}{4} \int_0^1 m u_t (\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds) dx \\ & \leq \frac{\varepsilon_5}{8} \int_0^1 m^2 u_t^2 dx + \frac{1}{8\varepsilon_5} \int_0^1 (\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds)^2 dx \\ & \leq c\varepsilon_5 \int_0^1 u_{xt}^2 dx + \frac{c}{\varepsilon_5} \int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_5} g \circ \psi_x. \end{aligned} \quad (3.26)$$

$$\begin{aligned} & -\frac{k}{4} \int_0^1 m(\varphi_x + \psi) (\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds) dx \\ & \leq \frac{\varepsilon_5}{8} \int_0^1 m^2 (\varphi_x + \psi)^2 dx + \frac{k^2}{8\varepsilon_5} \int_0^1 (\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds)^2 dx \\ & \leq c\varepsilon_5 \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{\varepsilon_5} \int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_5} g \circ \psi_x. \end{aligned} \quad (3.27)$$

$$\begin{aligned} & \frac{\rho_2}{4} \int_0^1 m \psi_t \left( \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))ds \right) dx \\ & \leq \frac{\rho_2^2}{32} \int_0^1 m^2 \psi_t^2 dx + \frac{1}{2} \int_0^1 \left( \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))ds \right)^2 dx \\ & \leq c \int_0^1 \psi_t^2 dx - c g' \circ \psi_x. \end{aligned} \quad (3.28)$$

By substituting (3.20)–(3.28) into (3.19), setting  $\varepsilon_5 = \varepsilon_1^2$  and then using (3.6), we obtain (3.18).  $\square$

### 4. GENERAL STABILITY RESULT

In this section, which is subdivided into two parts, we state and prove our main results.

#### 4.1. Equal speed of propagation $\left(\frac{k}{\rho_1} = \frac{\alpha}{\rho_2}\right)$

In this subsection, we state and prove a general stability result in the case of equal wave-speed propagation.

**Remark 4.1.** For  $\left(\frac{k}{\rho_1} = \frac{\alpha}{\rho_2}\right)$ , Eq. (3.17) takes the form

$$\begin{aligned}
 F'_5(t) \leq & \left[\varphi_x(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds)\right]_{x=0}^{x=1} + \rho_2 \int_0^1 \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \psi_x^2 dx \\
 & - \frac{k}{2} \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_1 \int_0^1 \varphi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 u_{x_t}^2 dx \\
 & - \frac{c}{\varepsilon_1} g' \circ \psi_x + c\varepsilon_1 g \circ \psi_x.
 \end{aligned}
 \tag{4.1}$$

Next, we define a Lyapunov functional  $\mathcal{L}$  equivalent to the first-order energy functional  $E$ . For positive constants;  $N, N_1, N_2$ , to be chosen appropriately later, we let

$$\mathcal{L}(t) := NE(t) + \frac{1}{8}F_1(t) + N_1F_2(t) + F_3(t) + N_2F_4(t) + F_5(t) + F_6(t).
 \tag{4.2}$$

**Lemma 4.2.** For  $N$  large enough, there exist two positive constants  $\alpha_1$  and  $\alpha_2$  such that

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad \forall t \geq 0.
 \tag{4.3}$$

**Proof.** Let

$$\mathcal{L}(t) = \frac{1}{8}F_1(t) + N_1F_2(t) + F_3(t) + N_2F_4(t) + F_5(t) + F_6(t).$$

$$\begin{aligned}
 |\mathcal{L}(t)| \leq & \frac{\rho_1}{8} \int_0^1 |\varphi_t \varphi| dx + \frac{\rho_2}{8} \int_0^1 |\psi_t \psi| dx + \frac{\rho_1}{k} \int_0^1 |\varphi_t \int_0^t g(t-s)\psi_x(s)ds| dx \\
 & + \rho_3 \int_0^1 |u_t u| dx + \beta \int_0^1 |\varphi_x u| dx + \frac{\varepsilon}{2} \int_0^1 u_x^2 dx \\
 & + \rho_2 N_2 \int_0^1 |\psi_t \psi| dx + \rho_2 \int_0^1 |\psi_t (\varphi_x + \psi)| dx + \frac{\alpha \rho_1}{k} \int_0^1 |\varphi_t \psi_x| dx \\
 & + \frac{\varepsilon_1 \rho_1}{k} \int_0^1 |m \varphi_t \varphi_x| dx + \rho_2 N_1 \int_0^1 |\psi_t \int_0^t g(t-s)(\psi(t) - \psi(s)ds)| dx \\
 & + \rho_1 N_2 \int_0^1 |\varphi_t w| dx + \frac{\rho_2}{4\varepsilon_1} \int_0^1 |m \psi_t (\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds)| dx
 \end{aligned}$$

By using Young’s and Poincaré’s inequalities, 3.6, 3.7, 3.10, 3.25 and Lemma 3.5, we obtain

$$|\mathcal{L}(t)| \leq c \int_0^1 \left(\varphi_t^2 + \psi_t^2 + u_t^2 + \psi_x^2 + u_x^2 + (\varphi_x + \psi)^2\right) dx + cg \circ \psi_x.$$

Consequently,

$$|\mathcal{L}(t) - NE(t)| \leq cE(t),$$

that is

$$(N - c)E(t) \leq \mathcal{L}(t) \leq (N + c)E(t).$$

By choosing  $N$  large enough, (4.3) follows.  $\square$

**Theorem 4.3.** *Let  $((\varphi_0, \varphi_1), (\psi_0, \psi_1), (u_0, u_1)) \in (H_0^1(0, 1) \times L^2(0, 1))^3$  be given and assume that  $g$  satisfies  $(A_1)$  and  $(A_2)$  and that*

$$\frac{k}{\rho_1} = \frac{\alpha}{\rho_2}.$$

Then, there exist two positive constants  $c_0$  and  $c_1$  such that

$$E(t) \leq c_0 e^{-c_1 \int_{t_0}^t \xi(s) ds}, \quad \forall t \geq t_0. \tag{4.4}$$

**Proof.** By differentiating (4.2) and using 3.1, 3.4, 3.8, 3.13, 3.15, 3.18 and 4.1, using Poincaré’s inequality and letting  $\varepsilon_2 = \frac{k}{4N_1}$ , we obtain

$$\begin{aligned} \mathcal{L}'(t) \leq & - \left[ \frac{\rho_1}{8} - c\varepsilon_1 - \varepsilon_3 N_2 \right] \int_0^1 \varphi_t^2 dx - \left[ \frac{k}{8} - c\varepsilon_1 \right] \int_0^1 (\varphi_x + \psi)^2 dx \\ & - \left[ N\delta - c \left( 1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + N_2 \right) \right] \int_0^1 u_{xt}^2 dx - \frac{\kappa}{2} \int_0^1 u_x^2 dx \\ & - \left[ \frac{IN_2}{2} - c \left( 1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3} \right) \right] \int_0^1 \psi_x^2 dx \\ & - \left[ N_1 \rho_2 \int_0^t g(s) ds - \frac{7\rho_2}{8} - c \left( 1 + \frac{1}{\varepsilon_1} + \left( 1 + \frac{1}{\varepsilon_3} \right) N_2 \right) \right] \int_0^1 \psi_t^2 dx \\ & + c \left[ 1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3} + N_2 N_1^2 \right] g \circ \psi_x + \left[ \frac{\beta N}{2} - c \left( \frac{1}{\varepsilon_1} + N_1^2 \right) \right] g' \circ \psi_x. \end{aligned}$$

Since  $g$  is continuous, positive and  $g(0) > 0$ , then for any  $t_0 > 0$ , we have

$$\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0, \quad \forall t \geq t_0. \tag{4.5}$$

Next, we choose  $\varepsilon_1$  small enough such that

$$\mu_1 = \frac{\rho_1}{8} - c\varepsilon_1 > 0 \quad \text{and} \quad \frac{k}{8} - c\varepsilon_1 > 0,$$

and then  $N_2$  large enough that

$$\frac{IN_2}{2} - c \left( 1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3} \right) > 0.$$

We then select  $\varepsilon_3$  so small that

$$\mu_1 - \varepsilon_3 N_2 > 0.$$

Next, we pick  $N_1$  large enough such that

$$N_1 \rho_2 g_0 - \frac{\rho_2}{2} - c \left( 1 + \frac{1}{\varepsilon_1} + \left( 1 + \frac{1}{\varepsilon_3} \right) N_2 \right) > 0.$$

Finally, we choose  $N$  large enough such that (4.3) remains valid and

$$N\delta - c \left( 1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + N_2 \right) > 0 \quad \text{and} \quad \frac{\beta N}{2} - c \left( \frac{1}{\varepsilon_1} + N_1^2 \right) > 0.$$

Consequently, by using Poincaré’s inequality and (2.4), we obtain

$$\mathcal{L}'(t) \leq -k_0 E(t) + cg \circ \psi_x, \quad \forall t \geq t_0, \tag{4.6}$$

where  $k_0$  positive constant. By multiplying (4.6) by  $\xi(t)$  and using (A<sub>2</sub>) and (3.1), we arrive at

$$\xi(t)\mathcal{L}'(t) \leq -k_0 \xi(t)E(t) - cE'(t), \quad \forall t \geq t_0,$$

which can be rewritten as

$$[\xi(t)\mathcal{L}(t) + cE(t)]' - \xi'(t)\mathcal{L}(t) \leq -k_0 \xi(t)E(t), \quad \forall t \geq t_0.$$

Using the fact that  $\xi'(t) \leq 0$ , we have

$$(\xi(t)\mathcal{L}(t) + cE(t))' \leq -k_0 \xi(t)E(t), \quad \forall t \geq t_0.$$

By exploiting (4.3), it can easily be shown that

$$\mathcal{R}(t) = \xi(t)\mathcal{L}(t) + cE(t) \sim E(t). \tag{4.7}$$

Consequently, for some positive constant  $c_1$ , we obtain

$$\mathcal{R}'(t) \leq -c_1 \xi(t)\mathcal{R}(t), \quad \forall t \geq t_0. \tag{4.8}$$

A simple integration of (4.8) over  $(t_0, t)$  leads to

$$\mathcal{R}(t) \leq \mathcal{R}(0) e^{-c_1 \int_{t_0}^t \xi(s) ds}, \quad \forall t \geq t_0. \tag{4.9}$$

Finally, (4.4) is established by combining (4.7) and (4.9).  $\square$

**Remark 4.4.** Estimate (4.4) also holds for  $t \in [0, t_0]$  by virtue of continuity and boundedness of  $E(t)$  and  $\xi(t)$ .

### 4.2. Nonequal speed propagation $\left( \frac{k}{\rho_1} \neq \frac{g}{\rho_2} \right)$

In this subsection, we treat the case of different wave-speed propagation. In this regard, we establish a general decay result which depends on the asymptotic behavior of  $g$  and the regularity of the initial data.

The main theorem in this subsection is:

**Theorem 4.5.** *Let  $(\varphi_0, \psi_0, u_0) \in (H^2(0, 1) \cap H_0^1(0, 1))^3$  and  $(\varphi_1, \psi_1, u_1) \in (H_0^1(0, 1))^3$  be given. Assume that  $g$  satisfies  $(A_1)$  and  $(A_2)$  and*

$$\frac{k}{\rho_1} \neq \frac{\alpha}{\rho_2}.$$

Then, there exists a positive constant  $c_2$  such that

$$E(t) \leq \frac{c_2}{\int_0^t \xi(s) ds}, \quad \forall t \geq t_0. \tag{4.10}$$

In order to establish this result, we need the second-order energy associated with problem (2.3). To this end, we differentiate (2.3) with respect to  $t$  and use the fact that

$$\begin{aligned} \frac{d}{dt} \int_0^t g(t-s)\psi_{xx}(x,s)ds &= \frac{d}{dt} \int_0^t g(s)\psi_{xx}(x,t-s)ds \\ &= g(t)\psi_{xx}(x,0) + \int_0^t g(s)\psi_{xxt}(x,t-s)ds \\ &= g(t)\psi_{0xx}(x) + \int_0^t g(t-s)\psi_{xxt}(x,s)ds. \end{aligned}$$

to get the system

$$\begin{aligned} \rho_1 \varphi_{ttt} - k(\varphi_{xt} + \psi_t)_x + u_{xtt} &= 0, \quad x \in (0, 1), \quad t > 0 \\ \rho_2 \psi_{ttt} - \alpha \psi_{xxt} + k(\varphi_{xt} + \psi_t) - u_{tt} + g(t)\psi_{0xx}(x) \\ &\quad + \int_0^t g(t-s)\psi_{xxt}(x,s)ds = 0, \quad x \in (0, 1), \quad t > 0 \\ \rho_3 u_{ttt} - \kappa u_{xxt} - \delta u_{xxtt} + \beta \varphi_{xtt} + \beta \psi_{tt} &= 0, \quad x \in (0, 1), \quad t > 0 \\ \varphi_t(0,t) = \varphi_t(1,t) = \psi_t(0,t) = \psi_t(1,t) = u_t(0,t) = u_t(1,t) &= 0, \quad t \geq 0. \end{aligned} \tag{4.11}$$

The second-order energy is defined by

$$\mathcal{E}(t) = E_1(\varphi_t, u_t, \psi_t), \tag{4.12}$$

where  $E_1$  is given in (2.4)

**Lemma 4.6.** *Let  $(\varphi, \psi, u)$  be the strong solution of (2.3). Then the energy functional  $\mathcal{E}(t)$ , defined by (4.12), satisfies*

$$\mathcal{E}'(t) = -\delta \int_0^1 u_{xxt}^2 dx + \frac{\beta}{2} g' \circ \psi_{xt} - \frac{\beta}{2} g(t) \int_0^1 \psi_{xt}^2 dx - \beta g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x) dx \tag{4.13}$$

and

$$\mathcal{E}(t) \leq c, \quad \forall t \geq 0. \tag{4.14}$$



**Proof.** Multiplying the first equation of (4.11) by  $\beta\varphi_{tt}$ , the second by  $\beta\psi_{tt}$ , and the third by  $u_{tt}$ , integrating over  $(0,1)$ , and summing up, as in Lemma 3.1, we obtain (4.13).

To prove (4.14), we observe that

$$\mathcal{E}'(t) \leq \frac{\beta}{2} g' \circ \psi_{xt} - \beta g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x) dx \leq -\beta g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x) dx, \quad \forall t \geq 0. \tag{4.15}$$

Then, using the fact that

$$\frac{\beta}{2} g(t) \int_0^1 \left( \sqrt{\rho_2} \psi_{tt} + \frac{1}{\sqrt{\rho_2}} \psi_{0xx} \right)^2 dx \geq 0, \quad \forall t \geq 0, \tag{4.16}$$

and following the same approach in [17], we obtain (4.14)

**Lemma 4.7.** *Let  $(\varphi, \psi, u)$  be the strong solution of (2.3), then  $\forall t \geq t_0$ , we have*

$$\left( \rho_2 - \frac{\alpha\rho_1}{k} \right) \int_0^1 \varphi_{xt} \psi_t dx \leq \varepsilon_1 \int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon_1} (g(t) - g' \circ \psi_x + g \circ \psi_{xt}). \tag{4.17}$$

**Proof.** This easily follows by repeating the same steps as in [17] using 2.4, 3.11, 3.12, properties of  $g$  and the nonincreasingness of  $E$ .  $\square$

**Remark 4.8.** For  $\left(\frac{k}{\rho_1} \neq \frac{\alpha}{\rho_2}\right)$ , taking Lemma 4.7 into account, Eq. (3.17) takes the form

$$\begin{aligned} F_5'(t) &\leq \left[ \varphi_x \left( \alpha \psi_x - \int_0^t g(t-s) \psi_x(s) ds \right) \right]_{x=0}^{x=1} + \rho_2 \int_0^1 \psi_t^2 dx \\ &\quad + 2\varepsilon_1 \int_0^1 \varphi_t^2 dx - \frac{k}{2} \int_0^1 (\varphi_x + \psi)^2 dx + c \left( 1 + \frac{1}{\varepsilon_1} \right) \int_0^1 u_{xt}^2 dx \\ &\quad + c \left( 1 + \frac{1}{\varepsilon_1} \right) \int_0^1 \psi_x^2 dx \\ &\quad - \frac{c}{\varepsilon_1} g' \circ \psi_x + c\varepsilon_1 g \circ \psi_x + \frac{c}{\varepsilon_1} g \circ \psi_{xt} + \frac{c}{\varepsilon_1} g(t). \end{aligned} \tag{4.18}$$

**Proof of Theorem 4.5.** To finalize the proof of Theorem 4.5, we use the same Lyapunov functional  $\mathcal{L}(t)$  defined in (4.2). That is

$$\mathcal{L}(t) := NE(t) + \frac{1}{8} F_1(t) + N_1 F_2(t) + F_3(t) + N_2 F_4(t) + F_5(t) + F_6(t),$$

but we use (4.18) instead of (4.1). Following the same steps with the same choice of the constants (up to (4.6)) as in the proof of Theorem 4.3, we obtain

$$\begin{aligned} \mathcal{L}'(t) &\leq -k_0 E(t) + cg \circ \psi_x + cg \circ \psi_{xt} + cg(t) - k_1 g(t) \\ &\quad \times \int_0^1 \psi_{tt} \psi_{0xx}(x) dx, \quad \forall t \geq t_0, \end{aligned} \tag{4.19}$$

where  $k_1$  is a positive constant.

By using Young’s inequality, the last term in (4.19) gives

$$-k_1 \int_0^1 \psi_{tt} \psi_{0xx}(x) dx \leq \frac{k_1}{2} \int_0^1 \psi_{tt}^2 dx + \frac{k_1}{2} \int_0^1 \psi_{0xx}^2 dx. \tag{4.20}$$

By exploiting (4.12) and (4.14), it follows that

$$\int_0^1 \psi_{tt}^2 dx \leq \frac{2}{\rho_2 \beta} \mathcal{E}(t) \leq c,$$

thus, (4.20) yields

$$-k_1 \int_0^1 \psi_{tt} \psi_{0xx}(x) dx \leq c. \tag{4.21}$$

By substituting (4.21) into (4.19), we get

$$\mathcal{L}'(t) \leq -k_0 E(t) + c g \circ \psi_x + c g \circ \psi_{xt} + c g'(t), \quad \forall t \geq t_0. \tag{4.22}$$

By multiplying (4.22) by  $\xi(t)$  and using  $(A_1)$  and  $(A_2)$ , we obtain

$$\xi(t) \mathcal{L}'(t) \leq -k_0 \xi(t) E(t) - c g' \circ \psi_x - c g' \circ \psi_{xt} - c g'(t), \quad \forall t \geq t_0,$$

that is

$$\xi(t) E(t) \leq -k_2 \xi(t) \mathcal{L}'(t) - c g' \circ \psi_x - c g' \circ \psi_{xt} - c g'(t), \quad \forall t \geq t_0, \tag{4.23}$$

where  $k_2 = \frac{1}{k_0}$ . Integrating (4.23) over  $[t_0, t]$ , we get

$$\begin{aligned} \int_{t_0}^t \xi(s) E(s) ds &\leq k_2 \left[ \xi(t_0) \mathcal{L}(t_0) - \xi(t) \mathcal{L}(t) + \int_{t_0}^t \xi'(s) \mathcal{L}(s) ds \right] - c \int_{t_0}^t g' \circ \psi_x(s) ds \\ &\quad - c \int_{t_0}^t g' \circ \psi_{xt}(s) ds + c, \quad \forall t \geq t_0. \end{aligned} \tag{4.24}$$

Recalling (3.1) and (4.15), we have

$$-g' \circ \psi_x \leq -\frac{2}{\beta} E'(t),$$

and

$$-g' \circ \psi_{xt} \leq -\frac{2}{\beta} \mathcal{E}'(t) - 2g(t) \int_0^1 \psi_{tt} \psi_{0xx} dx.$$

In addition, since  $E$  and  $\xi$  are both positive and nonincreasing, so (4.24) gives

$$\int_{t_0}^t \xi(s) E(s) ds \leq c + \int_{t_0}^t g(s) \left[ -2c \int_0^1 \psi_{tt}(x, s) \psi_{0xx} dx \right] ds, \quad \forall t \geq t_0.$$

By using (4.21) and  $(A_1)$ , we obtain

$$\int_{t_0}^t \xi(s) E(s) ds \leq c + c \int_{t_0}^t g(s) ds \leq c + c(\alpha - l) \leq c. \quad \forall t \geq t_0. \tag{4.25}$$

Consequently, we obtain

$$\begin{aligned} E(t) \int_0^t \zeta(s) ds &\leq \int_0^t \zeta(s) E(s) ds = \int_0^{t_0} \zeta(s) E(s) ds + \int_{t_0}^t \zeta(s) E(s) ds \\ &\leq t_0 \zeta(0) E(0) + c \leq c_2 \quad \forall t \geq t_0. \end{aligned}$$

Therefore,

$$E(t) \leq \frac{c_2}{\int_0^t \zeta(s) ds} \quad \forall t \geq t_0,$$

which is the conclusion of Theorem 4.5.

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