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General f-harmonic morphisms

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Abstract. In this paper, we prove that a map between Riemannian manifolds is an f-harmonic morphism in a general sense if and only if it is horizontally weakly conformal, satisfying some conditions, and we investigate the properties of f-harmonic morphism in a general sense.

Keywords: f-harmonic maps; f-harmonic morphism

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1. Introduction

f-harmonic morphisms are mappings between Riemannian manifolds which preserve Laplace's equation [4,15,14]. They can be characterized as f-harmonic maps which checks the property of horizontal weak conformality (called semiconformality).

In mathematical physics, f-harmonic maps relate to the equations of the motion of a continuous system of spins [6] and the gradient Ricci-soliton structure [16,1]. Recently the notion of f-harmonic maps (resp bi-f-harmonic maps) was developed by N. Course [7], M. Djaa and S. Ouakkas [15,8,4], and studied by many authors, including Y.J. Chiang [5], M. Rimoldi [16], Y.L. Ou [14], S. Feng [10], W.J. Lu [13] and others.

The goal of this work is the characterization of f-harmonic morphism (in a general sense) between Riemannian manifolds (Theorem 3.1), which generalizes the Fuglede–Ishihara characterization for harmonic morphisms [11,12], and we investigate the properties of

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f-harmonic morphism in a general sense. Also we construct some examples of f-harmonic morphism (Example 3.2).

2. f-HARMONIC MAPS

Consider a smooth map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds and $f : (x,y,r) \in M \times N \times \mathbb{R} \to f(x,y,r) \in (0,\infty)$ a smooth positive function. For any compact domain D of M the f-energy functional of φ is defined by

$$E_f(\varphi; D) = \int_D f\big(x, \varphi(x), e(\varphi)(x)\big)\, v_g, \tag{2.1}$$

where v_g is the volume element and

$$e(\varphi) = \frac{1}{2} \sum_i h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{2.2}$$

is the energy density of φ, here $\{e_i\}$ is an orthonormal frame on (M,g).

Definition 2.1 (*[4]*). A map is called f-harmonic if it is a critical point of the f-energy functional over any compact subset D of M.

2.1. The first variation of f-energy functional

Theorem 2.1 ([4]). *Let $\varphi : (M,g) \to (N,h)$ be a smooth map and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. If $v = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ denote the variation vector field of φ, then*

$$\frac{d}{dt} E_f(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_f(\varphi), v)\, v_g, \tag{2.3}$$

where

$$\tau_f(\varphi) = \left(\frac{\partial f}{\partial r}\right)_\varphi \tau(\varphi) + d\varphi\left(\operatorname{grad}^M \left(\frac{\partial f}{\partial r}\right)_\varphi\right) - (\operatorname{grad}^N f) \circ \varphi,$$

here $\left(\frac{\partial f}{\partial r}\right)_\varphi (x) = \frac{\partial f}{\partial r}\big(x, \varphi(x), e(\varphi)(x)\big)$ *and* $\tau(\varphi) = \operatorname{trace} \nabla d\varphi$.

Definition 2.2. $\tau_f(\varphi)$ is called an f-tension field of φ.

Theorem 2.2 ([4]). *Let $\varphi : (M,g) \to (N,h)$ be a smooth map between Riemannian manifolds and $f : M \times N \times \mathbb{R} \to (0,\infty)$ be a smooth function. Then φ is an f-harmonic map if and only if*

$$\tau_f(\varphi) = \left(\frac{\partial f}{\partial r}\right)_\varphi \tau(\varphi) + d\varphi\left(\operatorname{grad}^M \left(\frac{\partial f}{\partial r}\right)_\varphi\right) - (\operatorname{grad}^N f) \circ \varphi = 0. \tag{2.4}$$

Remarks 2.1. 1. If $f(x,y,r) = f_1(x)f_2(y)\,r$, then any smooth map $\varphi : (M^m, g) \to (N^n, h)$ $(m \neq 2)$ is an f-harmonic map if and only if $\varphi : (M^m, f_1^{\frac{2}{m-2}} g) \to (N^n, f_2 h)$ is a harmonic map.

2. If $f(x,y,r) = f(x)\,r$, then any smooth map $\varphi : (M^m, g) \to (N^n, h)$ $(m \neq 2)$ is an f-harmonic map if and only if $\varphi : (M^m, f^{\frac{2}{m-2}} g) \to (N^n, h)$ is a harmonic map (see [9]).

Examples 2.1 (*[14]*)**.** Inhomogeneous Heisenberg spin system

1. $\varphi : (\mathbb{R}^3, ds_0) \to (N^n, h)$ is an f-harmonic map if and only if

$$\varphi : (\mathbb{R}^3, f_1^2 ds_0) \to (N^n, h)$$

is a harmonic map with $f(x,y,r) = f_1(x)r$.

2. $\varphi : S^3 \setminus \{N\} \equiv (\mathbb{R}^3, \frac{4ds_0}{(1+|x|^2)^2}) \to (N^n, h)$ is a harmonic map if and only if

$$\varphi : (\mathbb{R}^3, ds_0) \to (N^n, h)$$

is an f-harmonic map with $f(x,y,r) = \frac{2}{(1+|x|^2)} r$.

3. When $(N^n, h) = S^2$, we have 1-1 correspondence between the set of harmonic maps $S^3 \to S^2$ and the set of stationary solutions of the inhomogeneous Heisenberg spin system on $\mathbb{R}^3$.

4.

$$\varphi : H^3 \equiv \left(\mathbb{D}^3, \frac{4ds_0}{(1+|x|^2)^2}\right) \to (N^n, h)$$

is a harmonic map if and only if

$$\varphi : (\mathbb{D}^3, ds_0) \to (N^n, h)$$

is an f-harmonic map with $f(x,y,r) = \frac{2}{(1+|x|^2)} r$.

3. f-HARMONIC MORPHISMS

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and $C_\varphi = \{x \in M \mid d_x\varphi = 0\}$ be the set of critical points of φ. Then φ is called horizontally weakly conformal or semi-conformal if for each $x \in M \backslash C_\varphi$ the restriction of $d_x\varphi$ to $\mathcal{H}_x$ is surjective and conformal, where the horizontal space $\mathcal{H}_x$ is the orthogonal complement of $\mathcal{V}_x = Ker\, d_x\varphi$. The horizontal conformality of φ implies that there exists a function $\lambda : M\backslash C_\varphi \to \mathbb{R}_+$ such that for all $x \in M\backslash C_\varphi$ and $X, Y \in \mathcal{H}_x$

$$h(d_x\varphi(X), d_x\varphi(Y)) = \lambda(x)^2 g(X,Y). \tag{3.1}$$

φ is horizontally weakly conformal at x with dilation $\lambda(x)$ if and only if in any local coordinates (y^α) on a neighborhood of $\varphi(x)$ we have

$$g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\beta) = \lambda^2 (h^{\alpha\beta} \circ \varphi) \quad (\alpha, \beta = 1, \ldots, n). \tag{3.2}$$

Definition 3.1. Let $f : M \times \mathbb{R} \times \mathbb{R} \to (0, \infty)$, $(x,t,r) \mapsto f(x,t,r)$ be a smooth function and U be an open subset of M. A C^2-function $u : U \to \mathbb{R}$ is called f-harmonic if

$$\Delta_f^M u \equiv \left(\frac{\partial f}{\partial r}\right)_u \Delta^M u + du\Big(\operatorname{grad}^M \left(\frac{\partial f}{\partial r}\right)_u\Big) - \left(\frac{\partial f}{\partial t}\right)_u = 0 \tag{3.3}$$

where

$$\left(\frac{\partial f}{\partial r}\right)_u : U \to (0,+\infty)$$
$$x \mapsto \left(\frac{\partial f}{\partial r}\right)_u(x) = \frac{\partial f}{\partial r}\big(x, u(x), e(u)(x)\big).$$

Definition 3.2. A map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is said to be an f-harmonic morphism if for every open subset V of N with $\varphi^{-1}(V) \neq \emptyset$ and every harmonic function $v : V \to \mathbb{R}$, the composition $v \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is f-harmonic.

Example 3.1 (*[14]*)**.** Let $\varphi, \psi, \phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as

$$\begin{aligned}
\varphi(x,y,z) &= (x,y),\\
\psi(x,y,z) &= (3x, xy),\\
\phi(x,y,z) &= (x, y+z).
\end{aligned}$$

Then both φ and ψ are f-harmonic with $f = r\,e^z$, φ is a horizontally conformal submersion whilst ψ is not. Also, ϕ is an f-harmonic map with $f = r\,e^{(y-z)}$, which is a submersion but not horizontally weakly conformal.

Theorem 3.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and $f : M \times \mathbb{R} \times \mathbb{R} \to (0,\infty)$ be a smooth function such that*

$$\begin{cases}
\dfrac{\partial f}{\partial r}(x,t,r) \neq 0, & \text{for all } (x,t,r) \in M \times \mathbb{R} \times \mathbb{R};\\
\dfrac{\partial f}{\partial t}(x,t,0) = 0, & \text{for all } (x,t) \in M \times \mathbb{R}.
\end{cases} \tag{3.4}$$

Then, the following are equivalent:
(1) *φ is an f-harmonic morphism;*
(2) *φ is a horizontally weakly conformal satisfying*

$$\left(\frac{\partial f}{\partial r}\right)_{\varphi^\alpha} \tau(\varphi)^\alpha + g\Big(\operatorname{grad}^M \left(\frac{\partial f}{\partial r}\right)_{\varphi^\alpha}, \operatorname{grad}^M \varphi^\alpha\Big) - \left(\frac{\partial f}{\partial t}\right)_{\varphi^\alpha} = 0, \tag{3.5}$$

for all $\alpha = 1, \dots, n$ and in any local coordinates (y^α) on N;
(3) *There exists a smooth positive function λ on M such that*

$$\Delta_f^M (v \circ \varphi) = \left(\frac{\partial f}{\partial r}\right)_{v\circ\varphi} \lambda^2\, (\Delta^N v) \circ \varphi,$$

for every smooth function v defined on an open subset V of N.

Proof. To prove Theorem 3.1 we need the following lemma.

Lemma 3.1 ([2])**.** *Let y_0 be a point in N^n and (y^γ) be local coordinates centered at y_0. Then for any constants $\{c_\gamma, c_{\alpha\beta}\}_{\alpha,\beta,\gamma=1}^n$ with $c_{\alpha\beta} = c_{\beta\alpha}$ and $\sum_{\alpha=1}^n c_{\alpha\alpha} = 0$, there exists a neighborhood V of y_0 in N and a harmonic function $v : V \to \mathbb{R}$ such that*

$$\frac{\partial v}{\partial y^\alpha}(y_0) = c_\alpha, \qquad \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta}(y_0) = c_{\alpha\beta}, \tag{3.6}$$

for all $\alpha, \beta, \gamma = 1, \dots, n$.

Suppose that $\varphi : (M^m, g) \to (N^n, h)$ is an f-harmonic morphism. For $x_0 \in M$ we consider a system of local coordinates (x^i) and (y^α) around x_0 and $y_0 = \varphi(x_0)$ respectively, where we assume that (y^α) are normal. By Lemma 3.1, for a sequence $(c_\gamma, c_{\alpha\beta})^n_{\alpha,\beta,\gamma=1}$ with $c_\gamma = 0$, $c_{\alpha\beta} = c_{\beta\alpha}$ and $\sum_\alpha c_{\alpha\alpha} = 0$, we can choose a local harmonic function v such that

$$\frac{\partial v}{\partial y^\alpha}(y_0) = 0, \qquad \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta}(y_0) = c_{\alpha\beta}, \tag{3.7}$$

for all $\alpha, \beta = 1, \ldots, n$. By assumption $v \circ \varphi$ is f-harmonic in a neighborhood of x_0, from Definition 3.1 we have

$$\begin{aligned} 0 &= \Delta^M_f(v \circ \varphi) \\ &= \left(\frac{\partial f}{\partial r}\right)_{v\circ\varphi} \Delta^M(v \circ \varphi) + d(v \circ \varphi)\Big(\mathrm{grad}^M \left(\frac{\partial f}{\partial r}\right)_{v\circ\varphi}\Big) - \left(\frac{\partial f}{\partial t}\right)_{v\circ\varphi}. \end{aligned} \tag{3.8}$$

In particular at x_0

$$d(v \circ \varphi)\Big(\mathrm{grad}^M \left(\frac{\partial f}{\partial r}\right)_{v\circ\varphi}\Big) = 0, \tag{3.9}$$

$$\left(\frac{\partial f}{\partial t}\right)_{v\circ\varphi} = 0, \tag{3.10}$$

since $\frac{\partial v}{\partial y^\alpha}(y_0) = 0$, $e(v \circ \varphi) = 0$ and $\frac{\partial f}{\partial t}(x, t, 0) = 0$ for all $(x, t) \in M \times \mathbb{R}$.

By (3.8)–(3.10) and $\frac{\partial f}{\partial r}(x, t, r) \neq 0$, we have

$$\begin{aligned} 0 &= \Delta^M(v \circ \varphi) \\ &= dv(\tau(\varphi)) + \mathrm{trace}_g \nabla dv(d\varphi, d\varphi) \\ &= \mathrm{trace}_g \nabla dv(d\varphi, d\varphi). \end{aligned} \tag{3.11}$$

Since at x_0

$$\nabla dv = \sum_{\alpha,\beta} \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta} dy^\alpha \otimes dy^\beta = \sum_{\alpha,\beta} c_{\alpha\beta} dy^\alpha \otimes dy^\beta, \tag{3.12}$$

from (3.7), (3.11) and (3.12), we obtain

$$\begin{aligned} 0 &= \sum_{\alpha,\beta} g(\mathrm{grad}^M \varphi^\alpha, \mathrm{grad}^M \varphi^\beta) c_{\alpha\beta} \\ &= \sum_{\alpha} g(\mathrm{grad}^M \varphi^\alpha, \mathrm{grad}^M \varphi^\alpha) c_{\alpha\alpha} + \sum_{\alpha\neq\beta} g(\mathrm{grad}^M \varphi^\alpha, \mathrm{grad}^M \varphi^\beta) c_{\alpha\beta} \end{aligned} \tag{3.13}$$

$$0 = \sum_{\alpha} g(\mathrm{grad}^M \varphi^1, \mathrm{grad}^M \varphi^1) c_{\alpha\alpha}, \tag{3.14}$$

by (3.13) and (3.14), we obtain

$$0 = \sum_{\alpha} \Big[g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\alpha) - g(\operatorname{grad}^M \varphi^1, \operatorname{grad}^M \varphi^1) \Big] c_{\alpha\alpha} + \sum_{\alpha \neq \beta} g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\beta) c_{\alpha\beta}. \tag{3.15}$$

Let $\alpha_0 \neq 1$ and let

$$c_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta = 1; \\ -1, & \text{if } \alpha = \beta = \alpha_0; \\ 0, & \text{if } \alpha = \beta \neq 1, \alpha_0; \\ 0, & \text{if } \alpha \neq \beta, \end{cases}$$

then by (3.15), we have

$$g(\operatorname{grad}^M \varphi^{\alpha_0}, \operatorname{grad}^M \varphi^{\alpha_0}) = g(\operatorname{grad}^M \varphi^1, \operatorname{grad}^M \varphi^1), \tag{3.16}$$

$$g(\operatorname{grad}^M \varphi^{\alpha}, \operatorname{grad}^M \varphi^{\alpha}) = g(\operatorname{grad}^M \varphi^1, \operatorname{grad}^M \varphi^1), \tag{3.17}$$

for all $\alpha = 1, \dots, n$. Let $\alpha_0 \neq \beta_0$ and

$$c_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \alpha_0 \text{ and } \beta = \beta_0; \\ 0, & \text{if } \alpha \neq \alpha_0 \text{ or } \beta \neq \beta_0; \\ 0, & \text{if } \alpha = \beta. \end{cases}$$

By (3.15), we have

$$g(\operatorname{grad}^M \varphi^{\alpha_0}, \operatorname{grad}^M \varphi^{\beta_0}) = 0, \tag{3.18}$$

and

$$g(\operatorname{grad}^M \varphi^{\alpha}, \operatorname{grad}^M \varphi^{\beta}) = 0, \tag{3.19}$$

for all $\alpha \neq \beta = 1, \dots, n$. From (3.17) and (3.19) we deduce that φ is horizontally weakly conformal map such that

$$g(\operatorname{grad}^M \varphi^{\alpha}, \operatorname{grad}^M \varphi^{\beta}) = \lambda^2 \delta_{\alpha\beta}, \tag{3.20}$$

for all $\alpha, \beta = 1, \dots, n$.

For every C^2-function $v : V \to \mathbb{R}$ defined on an open subset V of N, we have

$$\begin{aligned} \Delta_f^M (v \circ \varphi) &= \Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi} \Delta^M (v \circ \varphi) + dv\Big(d\varphi\Big(\operatorname{grad}^M \Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi}\Big)\Big) - \Big(\frac{\partial f}{\partial t}\Big)_{v\circ\varphi} \\ &= \Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi} dv(\tau(\varphi)) + \Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi} \operatorname{trace}_g \nabla dv(d\varphi, d\varphi) \\ &\quad + dv\Big(d\varphi\Big(\operatorname{grad}^M \Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi}\Big)\Big) - \Big(\frac{\partial f}{\partial t}\Big)_{v\circ\varphi}. \end{aligned} \tag{3.21}$$

Since, φ is a horizontally weakly conformal map, we obtain

$$\Delta_f^M(v\circ\varphi) = \Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi} dv(\tau(\varphi)) + \Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi} \lambda^2(\Delta^N v)\circ\varphi \\ + dv\Big(d\varphi\Big(\operatorname{grad}^M\Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi}\Big)\Big) - \Big(\frac{\partial f}{\partial t}\Big)_{v\circ\varphi}. \tag{3.22}$$

By special choice of harmonic function v we have

$$\Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi} dv(\tau(\varphi)) + dv\Big(d\varphi\Big(\operatorname{grad}^M\Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi}\Big)\Big) - \Big(\frac{\partial f}{\partial t}\Big)_{v\circ\varphi} = 0,$$

i.e., in any local coordinates (y^α) on N, we have

$$\Big(\frac{\partial f}{\partial r}\Big)_{\varphi^\alpha} \tau(\varphi)^\alpha + g\Big(\operatorname{grad}^M\Big(\frac{\partial f}{\partial r}\Big)_{\varphi^\alpha}, \operatorname{grad}^M \varphi^\alpha\Big) - \Big(\frac{\partial f}{\partial t}\Big)_{\varphi^\alpha} = 0,$$

for all $\alpha = 1, \dots, n$. Thus, we obtain the implication $(1) \Longrightarrow (2)$. Therefore, it follows from (3.22) that $(2) \Longrightarrow (3)$. Finally, $(3) \Longrightarrow (1)$ is clearly true. $\square$

Particular cases:

1. If $f(x,t,r) = r$ for all $(x,t,r) \in M\times\mathbb{R}\times\mathbb{R}$, the condition (3.5) is equivalent to $\tau(\varphi) = 0$, i.e. φ is harmonic. Then, a smooth map $\varphi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if $\varphi : M \to N$ is both harmonic and horizontally weakly conformal [2].
2. If $f(x,t,r) = f_1(x)r$ for all $(x,t,r) \in M\times\mathbb{R}\times\mathbb{R}$, where $f_1 \in C^\infty(M)$ be a smooth positive function, the condition (3.5) is equivalent to $f_1\,\tau(\varphi) + d\varphi(\operatorname{grad}^M f_1) = 0$, i.e. φ is f_1-harmonic. Then, a smooth map $\varphi : M \to N$ between Riemannian manifolds is an f_1-harmonic morphism if and only if $\varphi : M \to N$ is both f_1-harmonic and horizontally weakly conformal [14].
3. If $f(x,t,r) = f_1(x,t)r$ for all $(x,t,r) \in M\times\mathbb{R}\times\mathbb{R}$, where $f_1 \in C^\infty(M\times\mathbb{R})$ be a smooth positive function, then φ is an f-harmonic morphism if and only if φ is an f_1-harmonic morphism [3].
4. If $f(x,t,r) = F(r)$ for all $(x,t,r) \in M\times\mathbb{R}\times\mathbb{R}$, where $F \in C^\infty(\mathbb{R})$ be a smooth function such that $F' > 0$, then, the following are equivalent:
   (a) φ is an f-harmonic morphism;
   (b) φ is an F-harmonic morphism;
   (c) φ is horizontally weakly conformal satisfying
   $$F'\big(e(\varphi^\alpha)\big)\,\tau(\varphi)^\alpha + g(\operatorname{grad}^M F'\big(e(\varphi^\alpha)\big), \operatorname{grad}^M \varphi^\alpha) = 0,$$
   for all $\alpha = 1, \dots, n$ and in any local coordinates (y^α) on N;
   (d) There exists a smooth positive function λ on M such that
   $$\Delta_f^M(v\circ\varphi) = F'\big(e(v\circ\varphi)\big)\,\lambda^2\,(\Delta^N v)\circ\varphi,$$
   for every smooth function v defined on an open subset V of N.

Proposition 3.1. *Let $\varphi : M \to N$ be an f-harmonic morphism between Riemannian manifolds with dilation λ_1, $\psi : N \to P$ be a harmonic morphism between Riemannian manifolds with dilation λ_2 and f be a smooth positive function in $M\times\mathbb{R}\times\mathbb{R}$ satisfying (3.4). Then, the composition $\psi\circ\varphi : M \to P$ is an f-harmonic morphism with dilation $\lambda_1(\lambda_2\circ\varphi)$.*

Proof. This follows from

$$\varDelta_f^M(v\circ\varphi)=\Big(\frac{\partial f}{\partial r}\Big)_{v\circ\varphi}\lambda_1^2\,(\varDelta^N v)\circ\varphi,$$

for every smooth function v defined on an open subset V of N and

$$\varDelta^N(u\circ\psi)=\lambda_2^2\,(\varDelta^P u)\circ\psi,$$

for every smooth function u defined on an open subset U of P. So that

$$\begin{aligned}\varDelta_f^M(u\circ\psi\circ\varphi)&=\Big(\frac{\partial f}{\partial r}\Big)_{u\circ\psi\circ\varphi}\lambda_1^2\,(\varDelta^N(u\circ\psi))\circ\varphi\\&=\Big(\frac{\partial f}{\partial r}\Big)_{u\circ\psi\circ\varphi}\lambda_1^2\,(\lambda_2\circ\varphi)^2(\varDelta^P u)\circ\psi\circ\varphi.\quad\square\end{aligned}$$

Proposition 3.2. *Let (M,g) be a Riemannian manifold and $f:M\times\mathbb{R}\times\mathbb{R}\to(0,+\infty)$ be a smooth positive function satisfying (3.4). A smooth map*

$$\varphi:(M,g)\to(\mathbb{R}^n,\langle,\rangle_{\mathbb{R}^n}),\qquad x\mapsto(\varphi^1(x),\dots,\varphi^n(x))$$

is an f-harmonic morphism if and only if its components φ^α are f-harmonic functions whose gradients are orthogonal and of the same norm at each point.

Proof. Let us notice that the condition (3.5) of Theorem 3.1 becomes

$$\Big(\frac{\partial f}{\partial r}\Big)_{\varphi^\alpha}\varDelta^M\varphi^\alpha+g\Big(\operatorname{grad}^M\Big(\frac{\partial f}{\partial r}\Big)_{\varphi^\alpha},\operatorname{grad}^M\varphi^\alpha\Big)-\Big(\frac{\partial f}{\partial t}\Big)_{\varphi^\alpha}=0,$$

for all $\alpha=1,\dots,n$, i.e. the functions φ^α are f-harmonic. $\square$

Using Proposition 3.2, we can construct many non-trivial examples on $\mathbb{R}^n$.

Example 3.2. Let $M=\mathbb{R}_+^*\times\mathbb{R}\times\mathbb{R}$, then the map

$$\varphi:(M,\langle,\rangle_{\mathbb{R}^3})\to(\mathbb{R}^2,\langle,\rangle_{\mathbb{R}^2}),\qquad(x,y,z)\mapsto(\sqrt{x^2+y^2},z),$$

is an f-harmonic morphism with

$$f(x,y,z,t,r)=\frac{F\Big(\frac{y}{x}\Big)\,e^{-\frac12(x^2+y^2+z^2)+(t^2+1)\,r}}{x\,\sqrt{(x^2+y^2+1)(z^2+1)}},$$

where F is a smooth positive function. Indeed, we have

$$\begin{aligned}&\varphi^1(x,y,z)=\sqrt{x^2+y^2},\qquad\varphi^2(x,y,z)=z,\\&\varDelta^M\varphi^1=\frac{1}{\sqrt{x^2+y^2}},\qquad\varDelta^M\varphi^2=0,\\&e(\varphi^1)=\frac12,\qquad e(\varphi^2)=\frac12,\end{aligned}$$

$$\operatorname{grad}^M \varphi^1 = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right), \qquad \operatorname{grad}^M \varphi^2 = \big(0,0,1\big).$$

Let $f(x,y,z,t,r) = h(x,y,z)\, e^{(t^2+1)\,r}$ for all $(x,y,z,t,r) \in M \times \mathbb{R} \times \mathbb{R}$, where h is a smooth positive function in M, we get

$$\begin{cases} \dfrac{\partial f}{\partial r}(x,y,z,t,r) = h(x,y,z)\,(t^2+1)\, e^{(t^2+1)\,r} \neq 0, \\ \dfrac{\partial f}{\partial t}(x,y,z,t,0) = 2\, h(x,y,z)\, t\, r\, e^{(t^2+1)\,r}\Big|_{r=0} = 0, \end{cases}$$

for all $(x,y,z,t,r) \in M \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} \left(\frac{\partial f}{\partial r}\right)_{\varphi^1} &= h(x,y,z)\,(x^2+y^2+1)\, e^{\frac{x^2+y^2+1}{2}}, \\ \left(\frac{\partial f}{\partial r}\right)_{\varphi^2} &= h(x,y,z)\,(z^2+1)\, e^{\frac{z^2+1}{2}}, \\ \left(\frac{\partial f}{\partial t}\right)_{\varphi^1} &= h(x,y,z)\,\sqrt{x^2+y^2}\, e^{\frac{x^2+y^2+1}{2}}, \qquad \left(\frac{\partial f}{\partial r}\right)_{\varphi^2} = h(x,y,z)\, z\, e^{\frac{z^2+1}{2}}, \\ \operatorname{grad}^M\left(\frac{\partial f}{\partial r}\right)_{\varphi^1} &= e^{\frac{x^2+y^2+1}{2}}\left(\frac{\partial h}{\partial x}\,x^2 + \frac{\partial h}{\partial x}\,y^2 + \frac{\partial h}{\partial x} + 3\,h\,x + h\,x^3 + h\,x\,y^2\right)\frac{\partial}{\partial x} \\ &\quad + e^{\frac{x^2+y^2+1}{2}}\left(\frac{\partial h}{\partial y}\,x^2 + \frac{\partial h}{\partial y}\,y^2 + \frac{\partial h}{\partial y} + 3\,h\,y + h\,y^3 + h\,y\,x^2\right)\frac{\partial}{\partial y} \\ &\quad + e^{\frac{x^2+y^2+1}{2}}\left(\frac{\partial h}{\partial z}\,x^2 + \frac{\partial h}{\partial z}\,y^2 + \frac{\partial h}{\partial z}\right)\frac{\partial}{\partial z}, \\ \operatorname{grad}^M\left(\frac{\partial f}{\partial r}\right)_{\varphi^2} &= e^{\frac{z^2+1}{2}}\left(\frac{\partial h}{\partial x}\,z^2 + \frac{\partial h}{\partial x}\right)\frac{\partial}{\partial x} + e^{\frac{z^2+1}{2}}\left(\frac{\partial h}{\partial y}\,z^2 + \frac{\partial h}{\partial y}\right)\frac{\partial}{\partial y} \\ &\quad + e^{\frac{z^2+1}{2}}\left(\frac{\partial h}{\partial z}\,z^2 + \frac{\partial h}{\partial z} + 3\,h\,z + h\,z^3\right)\frac{\partial}{\partial z}. \end{aligned}$$

According to Proposition 3.2, the map φ is f-harmonic if and only if

$$\begin{cases} 3\,h\,x^2 + 3\,h\,y^2 + h + \dfrac{\partial h}{\partial x}\,x^3 + \dfrac{\partial h}{\partial x}\,x\,y^2 + \dfrac{\partial h}{\partial x}\,x + h\,x^4 \\ \quad + 2\,h\,x^2\,y^2 + \dfrac{\partial h}{\partial y}\,x^2\,y + \dfrac{\partial h}{\partial y}\,y^3 + \dfrac{\partial h}{\partial y}\,y + h\,y^4 = 0, \\ \dfrac{\partial h}{\partial z}\,z^2 + \dfrac{\partial h}{\partial z} + 2\,h\,z + h\,z^3 = 0. \end{cases} \tag{3.23}$$

Let $F \in C^\infty(\mathbb{R})$ be a smooth positive function, then the function of type

$$h(x,y,z) = \frac{F\left(\frac{y}{x}\right) e^{-\frac{1}{2}(x^2+y^2+z^2)}}{x\,\sqrt{(x^2+y^2+1)(z^2+1)}}$$

satisfies the system of differential equations (3.23).

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References

[1] P. Baird, R. Pantilie, On Ricci solitons and twistorial harmonic morphisms (2012) arXiv:1210.4688.
[2] P. Baird, J.C. Wood, Harmonic Morphisms between Riemannain Manifolds, Clarendon Press Oxford, 2003.
[3] A.M. Cherif, M. Djaa, On generaized f-harmonic morphisms, Comment. Math. Univ. Carolin. 55 (1) (2014) 17–27.
[4] A.M. Cherif, M. Djaa, Geometry of energy and bienergy variations between Riemannian manifolds, Kyungpook Math. J. 55 (2015) 715–730.
[5] Y.J. Chiang, f-biharmonic Maps between Riemannian Manifolds, Department of Mathematics, University of Mary Washington Fredericksburg, VA 22401, USA, 2012.
[6] J. Cieslinski, A. Sym, W. Wesselius, On the geometry of the inhomogeneous heisenberg ferromagnet model: non-integrable case, J. Phys. A: Math. Gen. 26 (1993) 1353–1364.
[7] N. Course, f-harmonic maps which map the boundary of the domain to one point in the target, New York J. Math. 13 (2007) 423–435.
[8] M. Djaa, A.M. Cherif, K. Zegga, S. Ouakkas, On the generalized of harmonic and bi-harmonic maps, Int. Electron. J. Geom. 5 (1) (2012) 1–11.
[9] J. Eells, L. Lemaire, A report on harmonic maps, Bull. Lond. Math. Soc. 10 (1978) 1–68.
[10] S. Feng, Y. Han, Liouville type theorems of f-harmonic maps with potential, Results Math. 66 (1–2) (2014) 43–64.
[11] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier (Grenoble) 28 (1978) 107–144.
[12] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto Univ. 19 (2) (1979) 215–229.
[13] W.J. Lu, On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds, Sci. China Math. 58 (2015) 1483–1498.
[14] Y.L. Ou, On f-harmonic morphisms between Riemannian manifolds, Chin. Ann. Math. Ser. B 35 (2014) 225–236.
[15] S. Ouakkas, R. Nasri, M. Djaa, On the f-harmonic and f-biharmonic maps, J. P. J. Geom. Top. 10 (1) (2010) 11–27.
[16] M. Rimoldi, G. Veronelli, f-Harmonic Maps and Applications to Gradient Ricci Solitons, Institut Elie Cartan Université de Lorraine, 2012, Journées $Nancéiennes$ de Géométrie 17 et 18 janvier.