ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 25 (2) (2019) 156–168

Fractional calculus of generalized p-k-Mittag-Leffler function using Marichev–Saigo–Maeda operators

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Received 25 April 2018; revised 17 May 2019; accepted 17 May 2019
Available online 24 May 2019

Abstract. In this paper, we establish fractional integral and derivative formulas involving the generalized p-k-Mittag-Leffler function by using Marichev–Saigo–Maeda type fractional integral and derivative operators. We also consider some special cases of derived results by considering specific values of the parameters of the generalized p-k-Mittag-Leffler function to give the application of our main results.

Keywords: Fractional calculus operators; Fox–Wright function; Fox H-function; Generalized p-k-Mittag-Leffler function

Mathematics Subject Classification: 26A33; 33C20; 33E12; 62E12

1. INTRODUCTION AND PRELIMINARIES

Fractional calculus is a valuable mathematical tool which has important applications in various branches of engineering and science, such as electromagnetic, viscosity, electrochemistry, biological population models, optics, and signal processing. A remarkably large number of works on the subject of fractional calculus have given an interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis. The special function, called Mittag-Leffler function $E_\alpha(z)$ has gained

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Peer review under responsibility of King Saud University.

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https://doi.org/10.1016/j.ajmsc.2019.05.003
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great popularity and importance due to its applications in the solutions of fractional order integral and differential equations, and comprehensive types of problems in several areas of mathematical analysis and mathematical physics. In recent years, a number of papers have been published by many authors (see, e.g., [3,7–12,19,20,22–25]), in which important properties, applications, and different extensions of fractional calculus operators involving a variety of special functions have been investigated. Motivated by the above discussion, we develop new fractional integral and derivative formulas involving the generalized p-k-Mittag-Leffler function by applying generalized fractional calculus operators. Also, we derive some special cases of our main results by substituting suitable values of the parameters of the generalized p-k-Mittag-Leffler function which provide unification and extension of known results given earlier by various authors.

For our present study, we consider the following definitions:

Definition 1. The Wright generalized hypergeometric function ${}_r\Psi_s[x]$, also called Fox–Wright function (see [5,27]) is defined as:

$$ {}_r\Psi_s[x] = {}_r\Psi_s\left[\begin{matrix}(\gamma_1,\acute{\gamma}_1),\ldots,(\gamma_r,\acute{\gamma}_s);\\ (l_1,\acute{l}_1),\ldots,(l_r,\acute{l}_s);\end{matrix}\; x\right] \tag{1} $$

$$ = \sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+\acute{\gamma}_1k),\ldots,\Gamma(\gamma_r+\acute{\gamma}_sk)}{\Gamma(l_1+\acute{l}_1k),\ldots,\Gamma(l_r+\acute{l}_sk)}\frac{x^k}{k!} \tag{2} $$

$$ = H^{1,r}_{r,s+1}\left[-x\left|\begin{matrix}(1-\gamma_1,\acute{\gamma}_1),\ldots,(1-\gamma_r,\acute{\gamma}_r)\\ (0,1),(1-l_1,\acute{l}_1),\ldots,(1-l_s,\acute{l}_s)\end{matrix}\right.\right], \tag{3} $$

where $H^{1,r}_{r,s+1}[x]$ denotes the Fox-H function [5], coefficients $\gamma_1',\ldots,\gamma_r',\, l_1',\ldots,l_s' \in \mathbb{R}^+$ and the series absolutely converges for all $x\in\mathbb{C}$ when $1+\sum_{j=1}^{s} l_j' - \sum_{m=1}^{r}\gamma_m' > 0$.

Gehlot [6] introduced p-k-Pochhammer symbol and p-k-gamma function in the following way

$$ {}_p(\omega)_{r,k} = \begin{cases} \dfrac{{}_p\Gamma_k(\omega+rk)}{{}_p\Gamma_k(\omega)}, & (p,k\in\mathbb{R}^+\setminus\{0\};\ \omega\in\mathbb{C}) \\ (\dfrac{\omega p}{k})(\dfrac{\omega p}{k}+p)....(\dfrac{\omega p}{k}+(r-1)p), & (r\in\mathbb{N};\ \omega\in\mathbb{C}) \end{cases} \tag{4} $$

and the relation with classical gamma function

$$ {}_p\Gamma_k(\omega) = \frac{p^{\frac{\omega}{k}}}{k}\Gamma\left(\frac{\omega}{k}\right), \qquad (\omega\in\mathbb{C}, \Re(\omega)>0;\ p,k\in\mathbb{R}^+\setminus\{0\}). \tag{5} $$

Further, let $\omega\in\mathbb{C}$; $p,k\in\mathbb{R}^+\setminus\{0\}$ and $\delta\in\mathbb{N}$, then the following identity holds

$$ {}_p(\omega)_{r\delta,k} = \left(\frac{p}{k}\right)^{r\delta}(\omega)_{r\delta,k} = (p)^{r\delta}\left(\frac{\omega}{k}\right)_{r\delta}. \tag{6} $$

Fore more detail of p-k-gamma function and p-k-Pochhammer symbol [6].

Recently, Kamarujjama et al. [8] investigated following a new definition of generalized k-MLf [2], called generalized p-k-Mittag-Leffler function (p-k-MLf).

Definition 2. Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $p, k \in \mathbb{R}^+ \setminus \{0\}$; $n, q \in \mathbb{N}$, then generalized p-k-MLf is defined as

$$ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}, \tag{7}$$

where ${}_p\Gamma_k$ denotes the p-k-gamma function defined by (5) and ${}_p(x)_{qn,k}$ is the Pochhammer symbol given by (6).

By using (7), we consider special cases and connections which are enumerated

(i) If set we $p = k$ in (7), we get generalized k-Mittag-Leffler function [2]

$$ {}_kE^{\gamma,q}_{k,\alpha,\beta}(z) = E^{\gamma,q}_{k,\alpha,\beta}(z). \tag{8}$$

(ii) If we set $p = k$ and $q = 1$ in (7), we get k-Mittag-Leffler function [4]

$$ {}_kE^{\gamma,1}_{k,\alpha,\beta}(z) = E^{\gamma}_{k,\alpha,\beta}(z). \tag{9}$$

(iii) If we take $p = k = 1$ in (7), we obtain GMLf given by Shukla and Prajapati [21]

$$ {}_1E^{\gamma,q}_{1,\alpha,\beta}(z) = E^{\gamma,q}_{\alpha,\beta}(z). \tag{10}$$

(iv) Setting $p = k = q = 1$ in (7), we get MLf has been given by Prabhakar [16]

$$ {}_1E^{\gamma,1}_{1,\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z). \tag{11}$$

(v) Setting $p = k = q = \gamma = 1$ in (7), we compute MLf has been given by Wiman [26]

$$ {}_1E^{1,1}_{1,\alpha,\beta}(z) = E_{\alpha,\beta}(z). \tag{12}$$

(vi) If we take $p = k = q = \gamma = \beta = 1$ in (7), we get MLf given by [15]

$$ {}_1E^{1,1}_{1,\alpha,1}(z) = E_{\alpha}(z). \tag{13}$$

(vii) Connection with Fox–Wright function ${}_r\Psi_s$[z]

$$ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{p^{\frac{\gamma-\beta}{k}}}{{}_p\Gamma_k(\gamma)}\, {}_1\Psi_1 \left[\begin{matrix} (\frac{\gamma}{k}, q) \\ (\frac{\beta}{k}, \frac{\alpha}{k}) \end{matrix} \middle| zp^{q-\frac{\alpha}{k}} \right]. \tag{14}$$

(viii) Connection with Fox H-function

$$ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{p^{\frac{\gamma-\beta}{k}}}{{}_p\Gamma_k(\gamma)} H^{1,1}_{1,2} \left[-zp^{q-\frac{\alpha}{k}} \middle| \begin{matrix} (1-\frac{\gamma}{k}, q) \\ (0,1),\ (1-\frac{\beta}{k}, \frac{\alpha}{k}) \end{matrix} \right]. \tag{15}$$

Here, we recall the generalized hypergeometric fractional integrals and derivatives, introduced by Marichev [13] and later extended by Saigo and Maeda [18]. These operators known as the Marichev–Saigo–Maeda operators. The generalized FC operators involving the third Appell function (or the Horn $F_3(\cdot)$ function in other words) in the kernel are defined in the following way.

Definition 3. Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma \in \mathbb{C}$, then fractional integral operators of a function f(x) are defined, for $\Re(\gamma) > 0$ as follows (see [18]):

$$\left(I_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma} f\right)(u) = \frac{u^{-\mu}}{\Gamma(\gamma)} \int_0^u (u-t)^{\gamma-1} t^{-\acute{\mu}} F_3 \times \left(\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}; \gamma; 1-t/u, 1-u/t\right) f(t)dt \tag{16}$$

and

$$\left(I_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma} f\right)(u) = \frac{u^{-\acute{\mu}}}{\Gamma(\gamma)} \int_u^\infty (u-t)^{\gamma-1} t^{-\mu} F_3 \times \left(\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}; \gamma; 1-u/t, 1-t/u\right) f(t)dt, \tag{17}$$

where F_3 is one of the Appell series defined by (see [24])

$$F_3\left(\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}; \gamma; u, v\right) = \sum_{m,n=0}^{\infty} \frac{(\mu)_m (\acute{\mu})_n (\varepsilon)_m (\acute{\varepsilon})_n}{(\gamma)_{m+n}} \frac{u^m}{m!} \frac{v^n}{n!}, \qquad (max\,\{|u|, |v|\} < 1).$$

For $\mu = \mu + \varepsilon$, $\acute{\mu} = \acute{\varepsilon} = 0$, $\varepsilon = -\tau$ and $\gamma = \mu$, (16) and (17) reduce into the following fractional integral operators [17] as follows:

$$\left(I_{0+}^{\mu+\varepsilon,0,-\tau,0,\mu} f\right)(u) = \left(I_{0+}^{\mu,\varepsilon,\tau} f\right)(u), \tag{18}$$

$$\left(I_{-}^{\mu+\varepsilon,0,-\tau,0,\mu} f\right)(x) = \left(I_{-}^{\mu,\varepsilon,\tau} f\right)(u), \tag{19}$$

where $\left(I_{0+}^{\mu,\varepsilon,\tau} f\right)(u)$ and $\left(I_{-}^{\mu,\varepsilon,\tau} f\right)(u)$ is the left and right sided fractional integral operators, and also known as Saigo hypergeometric fractional integral operators [17].

Definition 4. Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$ and $u \in \mathbb{R}^+$, then generalized fractional differentiation operators involving the Appell function F_3 in the kernel are defined as follows (see [18])

$$\left(D_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma} f\right)(u) = \left(I_{0+}^{-\acute{\mu},-\mu,-\acute{\varepsilon},-\varepsilon,\gamma} f\right)(u) \tag{20}$$

$$= \left(\frac{d}{du}\right)^n \left(I_{0+}^{-\acute{\mu},-\mu,-\acute{\varepsilon}+n,-\varepsilon,\gamma+n} f\right)(u) \quad (\Re(\gamma) > 0; n = [\Re(\gamma)] + 1)$$

$$= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{du}\right)^n (u^{-\acute{\mu}}) \int_0^u (u-t)^{n-\gamma-1} t^{-\mu} \times F_3\left(-\acute{\mu}, -\mu, n-\varepsilon, -\varepsilon, n-\gamma; 1-t/u, 1-u/t\right) f(t)dt \tag{21}$$

and

$$\left(D_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma} f\right)(u) = \left(I_{-}^{-m\acute{u},-\mu,-\acute{\varepsilon},-\varepsilon,\gamma} f\right)(u) \tag{22}$$

$$= \left(-\frac{d}{du}\right)^n \left(I_{-}^{-\acute{\mu},-\mu,-\acute{\varepsilon},-\varepsilon+n,\gamma+n} f\right)(u) \quad (\Re(\gamma) > 0; n = [\Re(\gamma)] + 1)$$

$$= \frac{1}{\Gamma(n-\gamma)}\left(-\frac{d}{du}\right)^{n}(u^{-\acute{\lambda}})\int_{u}^{\infty}(u-t)^{n-\gamma-1}t^{-\acute{\mu}}$$
$$\times F_3\left(-\acute{\mu},-\mu,n-\varepsilon,-\varepsilon,n-\gamma;1-u/t,1-t/u\right)f(t)dt. \tag{23}$$

For $\mu=\mu+\varepsilon$, $\acute{\mu}=\acute{\varepsilon}=0$, $\varepsilon=-\tau$ and $\gamma=\mu$, (21) and (23) reduces into the following fractional derivative operators [17] as follows:

$$\left(D_{0+}^{\mu+\varepsilon,0,-\tau,0,\mu}f\right)(u)=\left(D_{0-}^{\mu,\varepsilon,\tau}f\right)(u) \tag{24}$$

$$\left(D_{-}^{\mu+\varepsilon,0,-\tau,0,\mu}f\right)(u)=\left(D_{-}^{\mu,\varepsilon,\tau}f\right)(u). \tag{25}$$

Lemma 1. *Let $\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma \in \mathbb{C}$ with $\Re(\gamma)>0$ and $\Re(\rho)>max\left\{0,\Re(\mu+\acute{\mu}-\gamma),\Re(\acute{\mu}-\acute{\varepsilon})\right\}$, then (see [18])*

$$\left(I_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}t^{\rho-1}\right)(u)=\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\mu-\acute{\mu}-\varepsilon)\Gamma(\rho+\acute{\varepsilon}-\acute{\mu})}{\Gamma(\rho+\gamma-\mu-\acute{\mu})\Gamma(\rho+\gamma-\acute{\mu}-\varepsilon)\Gamma(\rho+\acute{\varepsilon})}x^{\rho-\mu-\acute{\mu}+\gamma-1}. \tag{26}$$

Lemma 2. *Let $\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma \in \mathbb{C}$ with $\Re(\gamma)>0$ and $\Re(\rho)<1+min\left\{\Re(-\varepsilon),\Re(\mu+\acute{\mu}-\gamma),\ \Re(\mu+\acute{\varepsilon}-\gamma)\right\}$, then (see [18])*

$$\left(I_{0-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}t^{\rho-1}\right)(u)=\frac{\Gamma(1+\mu+\acute{\mu}-\gamma-\varepsilon-\rho)\Gamma(1+\mu+\acute{\varepsilon}-\gamma-\rho)\Gamma(1-\varepsilon-\rho)}{\Gamma(1-\rho)\Gamma(1+\mu+\acute{\mu}+\acute{\varepsilon}-\gamma-\rho)\Gamma(1+\mu-\varepsilon-\rho)}x^{\rho-\mu-\acute{\mu}+\gamma-1}. \tag{27}$$

2. Fractional integral formulas

In this section, we evaluate some fractional integral formulas of generalized p-k-Mittag-Leffler function by applying left and right sided generalized fractional integral operators.

Theorem 1. *Let $\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma,\alpha_i,\beta_i,\delta_i,\rho\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\rho)>max\left\{0,\Re(\mu+\acute{\mu}+\varepsilon-\gamma),\Re(\acute{\mu}-\acute{\varepsilon})\right\}$ and $p_i,k_i\in\mathbb{R}^+\setminus\{0\}$; $q_i\in\mathbb{N}$ (where $i=1,\ldots,r$), then left-sided fractional integral formula holds*

$$\left(I_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1}\prod_{i=1}^{r}\ {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(t)\right)(u)=u^{\rho-\mu-\acute{\mu}-\gamma-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times\ {}_{r+3}\Psi_{r+3}\left[\begin{matrix}(\frac{\delta_i}{k_i},q_i)_1^r,\ (\rho,r),\ (\rho+\gamma-\mu-\acute{\mu}-\varepsilon,r),(\rho+\acute{\varepsilon}-\acute{\mu},\ r)\\ (\frac{\beta_i}{k_i},\frac{\alpha_i}{k_i})_1^r,\ (\rho+\acute{\varepsilon},\ r),\ (\rho+\gamma-\mu-\acute{\mu},\ r),(\rho+\gamma+\acute{\varepsilon}-\acute{\mu},\ r)\end{matrix}\ \middle|\ k_i^{(q_i-\frac{\alpha_i}{k_i})_1^r}x^r\right]. \tag{28}$$

Proof. Denoting the left hand side of (28) by L, using (7) and arranging the order of integration and summation (which is valid under the conditions given in Theorem 1), we get

$$L=\prod_{i=1}^{r}\left\{\sum_{n=0}^{\infty}\frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i+\beta_i)}\frac{1}{n!}\right\}\left(I_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho+nr-1}\right)(x). \tag{29}$$

Applying the result (26) in (29), we have

$$L=\prod_{i=1}^{r}\left\{\sum_{n=0}^{\infty}\frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i+\beta_i)}\frac{1}{n!}\right\}\frac{\Gamma(\rho+nr)\Gamma(\rho+nr+\gamma-\mu-\acute{\mu}-\varepsilon)}{\Gamma(\rho+nr+\acute{\varepsilon})\Gamma(\rho+\gamma-\mu-\acute{\mu}+nr)}$$
$$\times\frac{\Gamma(\rho+\acute{\varepsilon}-\acute{\mu}+nr)}{\Gamma(\rho+nr+\gamma-\acute{\mu}-\varepsilon)}\ x^{\rho-\mu-\acute{\mu}+\gamma-1}. \tag{30}$$

After using (5) and (6), (30) reduces to following expression

$$L=x^{\rho-\mu-\acute{\mu}+\gamma-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\delta_i}{k_i}+q_in)}{\Gamma(\frac{\beta_i}{k_i}+\frac{\alpha_i}{k_i}n)}\frac{1}{n!}\frac{\Gamma(\rho+nr)}{\Gamma(\rho+nr+\acute{\varepsilon})}\right.$$
$$\left.\times\frac{\Gamma(\rho+nr+\gamma-\mu-\acute{\mu}-\varepsilon)\Gamma(\rho+\acute{\varepsilon}-\acute{\mu}+nr)}{\Gamma(\rho+\gamma-\mu-\acute{\mu}+nr)\Gamma(\rho+nr+\gamma-\acute{\mu}-\varepsilon)}\right\}.$$

Finally, solving the above expression with the help of (1), we achieve the required result (28). □

Remark 2.1. Setting $\mu=\mu+\varepsilon$, $\acute{\mu}=\acute{\varepsilon}=0$, $\varepsilon=-\tau$ and $\gamma=\mu$ in (28), we deduce the following consequence of Theorem 1.

Corollary 2.1. *$\mu,\varepsilon,\alpha_i,\beta_i,\delta_i,\rho\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\tau)>0$, $\Re(\rho)>max\,\{0,\Re(\varepsilon-\tau)\}$ and $p_i,k_i\in\mathbb{R}^+\setminus\{0\}$; $q_i\in\mathbb{N}$ (where $i=1,\ldots,r$), then left-sided fractional integral formula holds*

$$\left(I_{0+}^{\mu,\varepsilon,\tau}\ t^{\rho-1}\prod_{i=1}^{r}{}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(t)\right)(x)=x^{\rho-\varepsilon-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times\ {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\frac{\delta_i}{k_i},q_i)_1^r,\ (\rho,\ r),\ (\rho+\tau-\varepsilon,\ r)\\(\frac{\beta_i}{k_i},\frac{\alpha_i}{k_i})_1^r,\ (\rho-\varepsilon,\ r),\ (\rho+\mu+\tau,\ r)\end{matrix}\middle|k_i^{(q_i-\frac{\alpha_i}{k_i})_1^r}x^r\right]. \tag{31}$$

Remark 2.2. For $p_i=k_i$ (where $i=1,\ldots,r$), (31) coincides with the known result of Chand et al. [2].

Theorem 2. *Let $\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma,\alpha_i,\beta_i,\delta_i,\rho\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\rho)<1+min\left\{\Re(-\varepsilon),\Re(\mu+\acute{\mu}-\gamma),\Re(\mu-\acute{\varepsilon}-\gamma)\right\}$; and $p_i,k_i\in\mathbb{R}^+\setminus\{0\}$, $q_i\in\mathbb{N}$*

(where $i = 1, \dots, r$), then right-sided fractional integral formula holds

$$\left(I_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1}\prod_{i=1}^{r}{}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(1/t)\right)(x) = x^{\rho-\mu-\acute{\mu}+\gamma-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times\ {}_{r+3}\Psi_{r+3}\left[\begin{matrix}(\frac{\delta_i}{k_i}, q_i)_1^r,\ (1-\rho-\varepsilon, r),\ (1-\rho-\gamma+\mu+\acute{\mu}, r),\\ (\frac{\beta_i}{k_i}, \frac{\alpha_i}{k_i})_1^r,\ (1-\rho, r),\ (1-\rho-\gamma+\mu+\acute{\mu}+\acute{\varepsilon},\ r),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\rho+\acute{\varepsilon}+\mu-\gamma, r)\\ (1-\rho+\mu-\varepsilon, r)\end{matrix}\ \middle|\ \frac{k_i^{(q_i-\frac{\alpha_i}{k_i})_1^r}}{x^r}\right]. \tag{32}$$

Proof. Denoting the left hand side of (32) by £, using (5) and inverting the order of integration and summation (which is valid under the conditions given in Theorem 2), we get

$$£ = \prod_{i=1}^{r}\left\{\sum_{n=0}^{\infty}\frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i+\beta_i)}\frac{1}{n!}\right\}\left(I_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-nr-1}\right)(x). \tag{33}$$

Applying the result (27) in (33), we have

$$£ = \prod_{i=1}^{r}\left\{\sum_{n=0}^{\infty}\frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i+\beta_i)}\frac{1}{n!}\right\}\frac{\Gamma(1-\rho-\varepsilon+nr)\Gamma(1-\rho+nr-\gamma+\mu+\acute{\mu})}{\Gamma(1-\rho+nr)\Gamma(1-\rho-\gamma+\mu+\acute{\mu}+nr)}$$
$$\times\frac{\Gamma(1-\rho+\acute{\varepsilon}+\mu-\gamma+nr)}{\Gamma(1-\rho+\mu-\varepsilon+nr)}\ x^{\rho-\mu-\acute{\mu}+\gamma-1}. \tag{34}$$

Using (5) and (6) in (34), gives

$$£ = x^{\rho-\mu-\acute{\mu}+\gamma-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\delta_i}{k_i}+q_in)}{\Gamma(\frac{\beta_i}{k_i}+\frac{\alpha_i}{k_i}n)}\frac{1}{n!}\frac{\Gamma(1-\rho-\varepsilon+nr)}{\Gamma(1-\rho+nr)}\right.$$
$$\left.\times\frac{\Gamma(1-\rho+nr-\gamma+\mu+\acute{\mu})\Gamma(1-\rho+\acute{\varepsilon}+\mu-\gamma+nr)}{\Gamma(1-\rho-\gamma+\mu+\acute{\mu}+nr)\Gamma(1-\rho+\mu-\varepsilon+nr)}\right\}.$$

Finally, solving the above expression with the help of (1), we achieve the desired result (32). □

Remark 2.3. Taking $\lambda = \lambda + \xi$, $\acute{\lambda} = \acute{\xi} = 0$, $\xi = -\tau$ and $\gamma = \lambda$ in (32), we deduce the following consequence of Theorem 2.

Corollary 2.2. *Let $\mu, \varepsilon, \tau, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\rho) < 1 + min\{\Re(\xi-\tau)\}$ and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \dots, r$), then following* *fractional integral formula holds*

$$\left(I_{-}^{\mu,\varepsilon,\tau}\ t^{\rho-1}\prod_{i=1}^{r}\ {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(1/t)\right)(x)=x^{\rho-\varepsilon-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times\ {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\frac{\delta_i}{k_i},q_i)_1^r,\ (1-\rho+\tau,r),\ (1-\rho+\varepsilon,r)\\ (\frac{\beta_i}{k_i},\frac{\alpha_i}{k_i})_1^r,\ (1-\rho,r),\ (1-\rho+\mu+\varepsilon+\tau,r)\end{matrix}\left|\frac{k_i^{(q_i-\frac{\alpha_i}{k_i})_1^r}}{x^r}\right.\right]. \tag{35}$$

Remark 2.4. For $p_i = k_i$ (where $i = 1, \dots, r$), (35) coincides with the known result of Chand et al. [2].

3. Fractional derivative formulas

Here, we investigate some fractional derivative formulas of generalized p-k-Mittag-Leffler function by applying left and right sided generalized fractional derivative operators.

Theorem 3. *Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) > max\left\{0, \Re(\gamma-\mu-\acute{\mu}-\acute{\varepsilon}), \Re(\mu-\varepsilon)\right\}$ and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \dots, r$), then left-sided fractional derivative formula holds*

$$\left(D_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1}\prod_{i=1}^{r}\ {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(t)\right)(x)=x^{\rho+\mu+\acute{\mu}-\gamma-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times\ {}_{r+3}\Psi_{r+3}\left[\begin{matrix}(\frac{\delta_i}{k_i},q_i)_r^r,\ (\rho,r),\ (\rho-\gamma+\mu+\acute{\mu}+\acute{\varepsilon},r),(\rho-\varepsilon+\mu,r)\\ (\frac{\beta_i}{k_i},\frac{\alpha_i}{k_i})_1^r,\ (\rho-\varepsilon,r),\ (\rho-\gamma+\mu+\acute{\mu},r),(\rho-\gamma+\acute{\varepsilon}+\acute{\mu},r)\end{matrix}\left|k_i^{(q_i-\frac{\alpha_i}{k_i})}x^r\right.\right]. \tag{36}$$

Proof. Denoting the left hand side of (36) by $\mathcal{L}$, using (7) and arranging the order of integration and summation (which is valid under the conditions given in Theorem 3), we get

$$\mathcal{L}=\prod_{i=1}^{r}\left\{\sum_{n=0}^{\infty}\frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i+\beta_i)}\frac{1}{n!}\right\}\left(D_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho+nr-1}\right)(x). \tag{37}$$

Applying the result (20) and (26) in (37), we have

$$\mathcal{L}=\prod_{i=1}^{r}\left\{\sum_{n=0}^{\infty}\frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i+\beta_i)}\frac{1}{n!}\right\}\frac{\Gamma(\rho+nr)\Gamma(\rho+nr-\gamma+\mu+\acute{\mu}-\acute{\varepsilon})}{\Gamma(\rho+nr-\varepsilon)\Gamma(\rho-\gamma+\mu+\acute{\mu}+nr)}$$
$$\times\frac{\Gamma(\rho-\varepsilon+\mu+nr)}{\Gamma(\rho+nr-\gamma+\mu-\acute{\varepsilon})}\ x^{\rho+nr+\mu+\acute{\mu}-\gamma-1}. \tag{38}$$

After using (5) and (6) in (38), we obtain following expression

$$\mathcal{L}=x^{\rho+\mu+\acute{\mu}-\gamma-1}\prod_{i=1}^{r}\frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\delta_i}{k_i}+q_in)}{\Gamma(\frac{\beta_i}{k_i}+\frac{\alpha_i}{k_i}n)}\frac{1}{n!}\frac{\Gamma(\rho+nr)}{\Gamma(\rho+nr-\varepsilon)}\right.$$
$$\left.\times\frac{\Gamma(\rho+nr-\gamma+\mu+\acute{\mu}-\acute{\varepsilon})\Gamma(\rho-\varepsilon+\mu+nr)}{\Gamma(\rho-\gamma+\mu+\acute{\mu}+nr)\Gamma(\rho+nr-\gamma+\mu-\acute{\varepsilon})}\right\}.$$

Finally, solving the above expression with the help of (1), we achieve the required result (36). □

Remark 3.1. Setting $\mu = \mu + \varepsilon$, $\acute{\mu} = \acute{\varepsilon} = 0$, $\varepsilon = -\tau$ and $\gamma = \mu$ in the result (36) of Theorem 3, we deduce the following consequence of Theorem 3.

Corollary 3.1. *Let $\mu, \varepsilon, \tau, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\rho) > max\,\{0, \Re(\varepsilon - \tau)\}$; and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \ldots, r$), then fractional derivative formula holds*

$$\left(D_{0+}^{\mu,\varepsilon,\tau}\; t^{\rho-1} \prod_{i=1}^{r} {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(t)\right)(x) = x^{\rho+\varepsilon-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\; p_i^{-\beta_i/k_i} \times {}_{r+2}\Psi_{r+2}\left[\begin{matrix} (\frac{\delta_i}{k_i}, q_i)_1^r,\; (\rho, r),\; (\rho+\mu+\tau+\varepsilon, r) \\ (\frac{\beta_i}{k_i}, \frac{\alpha_i}{k_i})_1^r,\; (\rho+\tau, r),\; (\rho+\varepsilon, r) \end{matrix} \middle| k_i^{(q_i - \frac{\alpha_i}{k_i})_1^r} x^r \right]. \tag{39}$$

Remark 3.2. For $r = 1$ and $p_i = p = 1$, $k_i = k = 1$, $\alpha_i = \alpha$, $\beta_i = \beta$, $q_i = q = 1$, $\delta_i = \delta$, (39) coincides with known result of Ahmad [1].

Theorem 4. *Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) < 1 + min\left\{\Re(-\acute{\varepsilon}), \Re(\mu + \acute{\mu} - \gamma), \Re(\mu + \varepsilon - \gamma)\right\}$; and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \ldots, r$), then right-sided fractional derivative formula holds*

$$\left(D_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\; t^{\rho-1} \prod_{i=1}^{r} {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(1/t)\right)(x) = x^{\rho+\mu+\acute{\mu}-\gamma-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\; p_i^{-\beta_i/k_i} \times {}_{r+3}\Psi_{r+3}\left[\begin{matrix} (\frac{\delta_i}{k_i}, q_i)_1^r,\; (1-\rho, r),\; (1-\rho+\gamma-\mu-\acute{\mu}-\varepsilon, r), \\ (\frac{\beta_i}{k_i}, \frac{\alpha_i}{k_i})_1^r,\; (1-\rho-\mu-\acute{\mu}+\gamma, r),\; (1-\rho+\gamma-\acute{\mu}-\varepsilon, r), \end{matrix} \begin{matrix} (1-\rho+\acute{\varepsilon}-\acute{\mu}, r) \\ (1-\rho+\acute{\varepsilon}, r) \end{matrix} \middle| \frac{k_i^{(q_i - \frac{\alpha_i}{k_i})_1^r}}{x^r} \right]. \tag{40}$$

Proof. Denoting the left hand side of (40) by $\mathcal{I}$, using (7) and arranging the order of integration and summation (which is valid under the conditions given in Theorem 4), we get

$$\mathcal{I} = \prod_{i=1}^{r} \left\{ \sum_{n=0}^{\infty} \frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{1}{n!} \right\} \left(D_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\; t^{\rho-nr-1}\right)(x). \tag{41}$$

Applying the results (22) and (27) in (41), we have

$$\mathcal{I} = \prod_{i=1}^{r} \left\{ \sum_{n=0}^{\infty} \frac{{}_{p_i}(\delta)_{nq_i,k_i}}{{}_{p_i}\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{1}{n!} \right\} \times \frac{\Gamma(1-\rho+nr)\Gamma(1-\rho+nr+\gamma-\mu-\acute{\mu}-\varepsilon)}{\Gamma(1-\rho+nr-\mu-\acute{\mu}+\gamma)\Gamma(1-\rho+\gamma-\acute{\mu}-\varepsilon+nr)} \times \frac{\Gamma(1-\rho+\acute{\varepsilon}-\acute{\mu}+nr)}{\Gamma(1-\rho+\acute{\varepsilon}+nr)}\; x^{\rho-\mu-\acute{\mu}+\gamma-1} \tag{42}$$

After using (5) and (6) in (42), we obtain the following expression

$$\mathcal{I} = x^{\rho-\lambda-\acute{\lambda}+\gamma-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})} \; p_i^{-\beta_i/k_i} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\delta_i}{k_i}+q_i n)}{\Gamma(\frac{\beta_i}{k_i}+\frac{\alpha_i}{k_i}n)} \frac{1}{n!} \frac{\Gamma(1-\rho+nr)}{\Gamma(1-\rho+\acute{\varepsilon}+nr)} \right.$$
$$\left. \times \frac{\Gamma(1-\rho+nr-\acute{\mu}-\acute{\varepsilon})\Gamma(1-\rho-\varepsilon-\mu+\gamma-\acute{\mu}+nr)}{\Gamma(1-\rho+\gamma-\mu-\acute{\mu}+nr)\Gamma(1-\rho-\acute{\mu}-\varepsilon+\gamma+nr)} \right\}.$$

Finally, solving the above expression with the help of (1), we achieve the desired result (40). □

Remark 3.3. For $r = 1$ and $p_1 = p = 1$, $k_1 = k = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $q_1 = q$, $\delta_1 = \delta$, (36) and (40) coincide with known results of Mishra et al. [14].

Remark 3.4. Setting $\mu = \mu + \varepsilon$, $\acute{\mu} = \acute{\varepsilon} = 0$, $\varepsilon = -\tau$ and $\gamma = \mu$ in (40), we deduce the following consequence of Theorem 4.

Corollary 3.2. *Let $\mu, \varepsilon, \tau, \gamma, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\tau) > 0$, $\Re(\rho) < 1 + min\{\Re(\mu-\tau), \Re(\mu+\varepsilon-\tau)\}$; and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \ldots, r$), then right-sided fractional derivative formula holds*

$$\left(D_{-}^{\mu,\varepsilon\tau} \; t^{\rho-1} \prod_{i=1}^{r} {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(1/t)\right)(x) = x^{\rho+\varepsilon-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})} \; p_i^{-\beta_i/k_i}$$
$$\times \; {}_{r+2}\Psi_{r+2}\left[\begin{matrix} (\frac{\delta_i}{k_i}, q_i), \; (1-\rho, r), \; (1-\rho-\varepsilon+\tau, r) \\ (\frac{\beta_i}{k_i}, \frac{\alpha_i}{k_i}), \; (1-\rho-\varepsilon, r), \; (1-\rho+\tau+\mu, r) \end{matrix} \middle| \frac{k_i^{(q_i-\frac{\alpha_i}{k_i})_1^r}}{x^r}\right]. \tag{43}$$

Remark 3.5. For r=1 and $p_1 = p$, $k_1 = k$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $q_1 = q = 1$, $\delta_1 = \delta = 1$, (43) coincides with known result of Ahmed [1].

4. Special cases

(1) For $\alpha > 0$ and $q \in \mathbb{N}$, generalized p-k-MLf function reduces to Fox H-function, therefore in view of (3) and (15), we get the following new results from Theorems 1–4

Corollary 4.1. *Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) > max\left\{0, \Re(\mu+\acute{\mu}+\varepsilon-\gamma), \Re(\acute{\mu}-\acute{\varepsilon})\right\}$; and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \ldots, r$), then left-sided fractional integral formula holds*

$$\left(I_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma} \; t^{\rho-1} \prod_{i=1}^{r} {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(t)\right)(x) = x^{\rho-\mu-\acute{\mu}-\gamma-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})} \; p_i^{-\beta_i/k_i}$$
$$\times H_{r+3,r+4}^{1,r+3}\left[-x^r(k_i^{q_i-\frac{\alpha_i}{k_i}})_1^r \middle| \begin{matrix} (1-\frac{\delta_i}{k_i}, q_i)_1^r, \; (1-\rho, r), \; (1-\rho-\gamma+\mu+\acute{\mu}+\varepsilon, r), \\ (0,1), (1-\frac{\beta_i}{k_i}, \frac{\alpha_i}{k_i})_1^r, \; (1-\rho-\acute{\varepsilon}, \; r), \end{matrix}\right.$$
$$\left. \begin{matrix} (1-\rho-\acute{\varepsilon}+\acute{\mu}, \; r) \\ (1-\rho-\gamma+\mu+\acute{\mu}, \; r), (1-\rho-\gamma-\acute{\varepsilon}+\acute{\mu}, \; r) \end{matrix}\right]. \tag{44}$$

Corollary 4.2. *Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) < 1 + min\left\{\Re(-\varepsilon), \Re(\mu + \acute{\mu} - \gamma), \Re(\mu - \acute{\varepsilon} - \gamma)\right\}$; and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \ldots, r$), then right-sided fractional integral formula holds*

$$\left(I_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1} \prod_{i=1}^{r} {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(1/t)\right)(x) = x^{\rho-\lambda-\acute{\mu}+\gamma-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times H_{r+3,r+4}^{1,r+3}\left[-x^r (k_i^{q_i-\frac{\alpha_i}{k_i}})_1^r \middle| \begin{matrix} (1-\frac{\delta_i}{k_i}, q_i)_1^r,\ (\rho+\varepsilon, r),\ (\rho+\gamma-\mu-\acute{\mu}, r), \\ (0,1),\ (1-\frac{\beta_1}{k_1}, \frac{\alpha_1}{k_1})_1^r,\ (\rho, r), \end{matrix}\right. \tag{45}$$

$$\left. \begin{matrix} (\rho-\acute{\varepsilon}-\mu+\gamma, r) \\ (\rho+\gamma-\mu-\acute{\mu}-\acute{\varepsilon},\ r),\ (\rho-\mu+\varepsilon, r) \end{matrix} \right]. \tag{46}$$

Corollary 4.3. *Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) > max\left\{0, \Re(\gamma - \mu - \acute{\mu} - \acute{\varepsilon}), \Re(\mu - \varepsilon)\right\}$; and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \ldots, r$), then left-sided fractional derivative formula holds*

$$\left(D_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1} \prod_{i=1}^{r} {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(t)\right)(x) = x^{\rho+\mu+\acute{\mu}-\gamma-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times H_{r+3,r+4}^{1,r+3}\left[-x^r (k_i^{q_i-\frac{\alpha_i}{k_i}})_1^r \middle| \begin{matrix} (1-\frac{\delta_i}{k_i}, q_i)_i^r,\ (1-\rho, r),\ (1-\rho+\gamma-\mu-\acute{\mu}-\acute{\varepsilon}, r), \\ (0,1),\ (1-\frac{\beta_i}{k_i}, \frac{\alpha_i}{k_i})_i^r,\ (1-\rho+\varepsilon, r),\ (1-\rho+\gamma-\mu-\acute{\mu}, r), \end{matrix}\right.$$
$$\left. \begin{matrix} (1-\rho+\varepsilon-\mu, r) \\ (1-\rho+\gamma-\acute{\varepsilon}-\acute{\mu}, r) \end{matrix} \right]. \tag{47}$$

Corollary 4.4. *Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma, \alpha_i, \beta_i, \delta_i, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) < 1 + min\left\{\Re(-\acute{\varepsilon}), \Re(\mu + \acute{\mu} - \gamma), \Re(\mu + \varepsilon - \gamma)\right\}$; and $p_i, k_i \in \mathbb{R}^+ \setminus \{0\}$; $q_i \in \mathbb{N}$ (where $i = 1, \ldots, r$), then right-sided fractional derivative formula holds*

$$\left(D_{-}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1} \prod_{i=1}^{r} {}_{p_i}E_{k_i,\alpha_i,\beta_i}^{\delta_i,q_i}(1/t)\right)(x) = x^{\rho+\mu+\acute{\mu}-\gamma-1} \prod_{i=1}^{r} \frac{k_i}{\Gamma(\frac{\delta_i}{k_i})}\ p_i^{-\beta_i/k_i}$$
$$\times H_{r+3,r+4}^{1,r+3}\left[-x^r (k_i^{q_i-\frac{\alpha_i}{k_i}})_1^r \middle| \begin{matrix} (1-\frac{\delta_i}{k_i}, q_i)_i^r,\ (\rho, r),\ (\rho-\gamma+\mu+\acute{\mu}+\varepsilon, r),\ (\rho-\acute{\varepsilon}+\acute{\mu}, r) \\ (0,1),\ (1-\frac{\beta_i}{k_i}, \frac{\alpha_i}{k_i})_i^r,\ (\rho+\mu+\acute{\mu}-\gamma, r),\ (\rho-\gamma+\acute{\mu}+\varepsilon, r), (\rho-\acute{\varepsilon}, r) \end{matrix}\right]. \tag{48}$$

(2) Taking $r = 1$, $p_1 = p$, $k_1 = k$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $q_1 = q$ and $\delta_1 = \delta$ in Theorems 1 and 3, we get the following results as follows

Corollary 4.5. *Let $\mu, \acute{\mu}, \varepsilon, \acute{\varepsilon}, \gamma, \alpha, \beta, \delta, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) > max\left\{0, \Re(\mu + \acute{\mu} + \varepsilon - \gamma), \Re(\acute{\mu} - \acute{\varepsilon})\right\}$ and $p, k \in \mathbb{R}^+ \setminus \{0\}$; $q \in \mathbb{N}$, then following* *fractional integral formula holds*

$$\left(I_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1}\ {}_pE_{k,\alpha,\beta}^{\delta,q}(t)\right)(x) = x^{\rho-\mu-\acute{\mu}-\gamma-1}\ \frac{k}{\Gamma(\frac{\delta}{k})}\ p^{-\beta/k}$$
$$\times\ {}_4\Psi_4\left[\begin{matrix}(\frac{\delta}{k},\ q),\ (\rho,1),\ (\rho+\gamma-\mu-\acute{\mu}-\varepsilon,1),(\rho+\acute{\varepsilon}-\acute{\mu},\ 1)\\ (\frac{\beta}{k},\frac{\alpha}{k}),\ (\rho+\acute{\varepsilon},\ 1),\ (\rho+\gamma-\mu-\acute{\mu},\ 1),(\rho+\gamma+\acute{\varepsilon}-\acute{\mu},\ 1)\end{matrix}\middle| k^{(q-\frac{\alpha}{k})}x\right]. \tag{49}$$

Corollary 4.6. *Let $\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma,\alpha,\beta,\delta,\rho\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\rho)>max\left\{0,\Re(\gamma-\mu-\acute{\mu}-\acute{\varepsilon}),\Re(\mu-\varepsilon)\right\}$; and $p,k\in\mathbb{R}^+\setminus\{0\}$; $q\in\mathbb{N}$, then following derivative formula holds*

$$\left(D_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1}\ {}_pE_{k,\alpha,\beta}^{\delta,q}(t)\right)(x) = x^{\rho+\mu+\acute{\mu}-\gamma-1}\ \frac{k}{\Gamma(\frac{\delta}{k})}\ p^{-\beta/k}$$
$$\times\ {}_4\Psi_4\left[\begin{matrix}(\frac{\delta}{k},\ q),\ (\rho,\ 1),\ (\rho-\gamma+\mu+\acute{\mu}+\acute{\varepsilon},1),(\rho-\varepsilon+\mu,r)\\ (\frac{\beta}{k},\frac{\alpha}{k}),\ (\rho-\varepsilon,\ 1),\ (\rho-\gamma+\mu+\acute{\mu},\ 1),(\rho-\gamma+\acute{\varepsilon}+\acute{\mu},r)\end{matrix}\middle| k^{(q-\frac{\alpha}{k})}x\right].$$

5. Concluding remarks

In the current study, we have investigated four image formulas of generalized fractional calculus (of Marichev–Saigo–Maeda) operators involving the generalized p-k-Mittag-Leffler function, which are expressed in terms of Fox–Wright function and Fox H-function. Moreover, the results derived in this paper also correspond to Saigo hypergeometric fractional calculus operators as special cases and it can be easily seen that, if we set $\varepsilon=-\mu$ and $\varepsilon=0$ in (18) and (24), they yield the Erdelyi–Kober, the Riemann–Liouville, and the Weyl fractional integral and derivative operators. Thereby, the corresponding results can also be obtained involving the above well known fractional operators.

The generalized p-k-MLF is related to remarkable function of other families of special functions like as Wright function, Bessel–Maitland function, Meijer-G function and Fox H-function, so we can deduce more interesting and useful results, from Theorems 1–4 as given in following example:

$${}_1E_{1,1,\beta}^{\delta,1}(z)=\frac{1}{\Gamma(\delta)}G_{1,2}^{1,1}\left[-z\middle|\begin{matrix}1-\delta\\0,\ 1-\beta\end{matrix}\right],\qquad(\Re(\delta)>0).$$

Example. Let $\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma,\beta,\delta,\rho\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\rho)>max\left\{0,\Re(\mu+\acute{\mu}+\varepsilon-\gamma),\Re(\acute{\mu}-\acute{\varepsilon})\right\}$, then following fractional integral formula holds

$$\left(I_{0+}^{\mu,\acute{\mu},\varepsilon,\acute{\varepsilon},\gamma}\ t^{\rho-1}\ G_{1,2}^{1,1}\left[-t\middle|\begin{matrix}1-\delta\\0,\ 1-\beta\end{matrix}\right]\right)(x)=x^{\rho-\mu-\acute{\mu}-\gamma-1}\ \frac{1}{\Gamma(\delta)}$$
$$\times\ G_{4,5}^{1,4}\left[-x\middle|\begin{matrix}1-\delta,\ 1-\rho,\ 1-\rho-\gamma+\mu+\acute{\mu}+\varepsilon,\ 1-\rho-\acute{\varepsilon}+\acute{\mu}\\0,\ 1-\beta,\ 1-\rho-\acute{\varepsilon},\ 1-\rho-\gamma+\mu+\acute{\mu},1-\rho-\gamma-\acute{\varepsilon}+\acute{\mu}\end{matrix}\right].$$

Therefore, the results presented here, being general in nature would help to deduce a large number of new and known results involving different special functions that could be useful in the problems of mathematical physics, science, and engineering, etc.

Declaration of competing interest

Authors do not have any conflict of interest.

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