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# Fitted finite difference method for third order singularly perturbed convection diffusion equations with integral boundary condition

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**Abstract.** A class of third order singularly perturbed convection diffusion type equations with integral boundary condition is considered. A numerical method based on a finite difference scheme on a Shishkin mesh is presented. The method suggested is of almost first order convergent. An error estimate is derived in the discrete norm. Numerical examples are presented, which validate the theoretical estimates.

Keywords: Singular perturbation problems; Finite difference scheme; Shishkin mesh; Integral boundary condition; Error estimate

Mathematics Subject Classification: 65L11; 65L12; 65L20

## 1. INTRODUCTION

We consider the following third order singularly perturbed convection diffusion equations with integral boundary condition:

$$-\varepsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x),\ x \in (0, 1) = \Omega, \tag{1.1}$$

$$u(0) = l_1,\ u'(0) = l_2,\ u'(1) = \varepsilon \int_0^1 g(x)u'(x)dx + l_3, \tag{1.2}$$

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where $0 < \varepsilon \ll 1$, $a(x) \geq \alpha > 0$, $b(x) \geq \beta \geq 0$, $\theta \leq c(x) \leq \theta_0 \leq 0$, $4\alpha + \beta + 16\theta > 0$, $l_1, l_2, l_3$ are real numbers, $g(x)$ is nonnegative with $\int_0^1 g(x)dx < 1$ and $a(x), b(x), c(x), f(x), g(x)$ are sufficiently smooth on $[0, 1] = \bar{\Omega}$.

Differential equation with a small parameter $\varepsilon$ multiplying the leading derivative term is called Singularly Perturbed Problem (SPP). Traditional numerical methods are not suitable for SPP because the solutions of such equations have rapid changes in small regions of the domain. It is necessary to expand appropriate numerical methods for these kinds of problems, such that the error estimates do not depend on the parameter $\varepsilon$. That is, methods in which the numerical solutions are convergent $\varepsilon$-uniformly [6,9,13]. One of the easiest and useful ways to derive such methods consists of using a class of piecewise uniform meshes (Shishkin and Bakhvalov mesh).

Boundary value problems with integral boundary conditions are an important class of problems which arise in the fields of electro-chemistry [7], thermo-elasticity [8], heat conduction [5] etc. The existence and uniqueness of the third order differential equations with integral boundary conditions and its applications are discussed in [1,2,10,11,14]. The existence of systems of second order differential equations with integral boundary condition and its applications are discussed in [3,6,15]. The above mentioned papers are concerned with regular case (without boundary layers). In [12] and [4] uniform convergence of the approximate solution on a uniform mesh is proved for second order differential equations with integral boundary condition. Motivated by the above works, in this paper a fitted finite difference method is discussed to solve a class of third order singularly perturbed convection diffusion equations with integral boundary condition (1.1)–(1.2).

This paper is arranged in the following manner. In Section 2 maximum principle, stability result and derivative estimate are derived for the continuous problem. Discretized problem is discussed in Section 3. Error estimate for the numerical method is established in Section 4. Numerical experiments are given in Section 5. The paper concluded with a discussion given in Section 6.

Throughout the paper, we assume that $\varepsilon \leq CN^{-1}$, $C$ denotes a positive constant. The norm used for studying the convergence of the numerical solution is supremum norm defined by $\|u\|_D := \sup_{x \in D} |u(x)|$.

## 2. Properties of the exact solution

The boundary value problem (1.1)–(1.2) can be transformed into the following equivalent problem:

$$L_1\bar{u}(x) = u_1'(x) - u_2(x) = 0, \ x \in \Omega \cup \{1\} \tag{2.1}$$

$$L_2\bar{u}(x) = -\varepsilon u_2''(x) + a(x)u_2'(x) + b(x)u_2(x) + c(x)u_1(x) = f(x), \ x \in \Omega, \tag{2.2}$$

where $\bar{u}(x) = (u_1(x), u_2(x))$ with the boundary conditions

$$u_1(0) = l_1, \ u_2(0) = l_2, \ Bu_2(1) = u_2(1) - \varepsilon \int_0^1 g(x)u_2(x)dx = l_3. \tag{2.3}$$

**Theorem 2.1** (*Maximum Principle*)**.** *Let* $\bar{u}(x) = (u_1(x), u_2(x))$ *be any function satisfying* $u_1(0) \geq 0, u_2(0) \geq 0, Bu_2(1) \geq 0, L_1\bar{u}(x) \geq 0, x \in \Omega \cup \{1\}$ *and* $L_2\bar{u}(x) \geq 0, \forall\, x \in \Omega$. *Then* $\bar{u}(x) \geq 0, \forall\, x \in \bar{\Omega}$.

**Proof.** Define $\bar{s}(x) = (s_1(x), s_2(x))$ as $s_1(x) = 1 + x$, $s_2(x) = \frac{1}{8} + \frac{x}{2}$. Note that $\bar{s}(x) > 0,\ x \in \bar{\Omega}$, $L_1\bar{s}(x) > 0$, $L_2\bar{s}(x) > 0$, $s_1(0) > 0$, $s_2(0) > 0$ and $Bs_2(1) > 0$. Further we define

$$\mu = \max\left\{\max_{x\in\bar{\Omega}}\left(\frac{-u_1(x)}{s_1(x)}\right), \max_{x\in\bar{\Omega}}\left(\frac{-u_2(x)}{s_2(x)}\right)\right\}.$$

Then there exists at least one $x_0 \in \Omega$, such that $\left(\frac{-u_1(x_0)}{s_1(x_0)}\right) = \mu$ or $\left(\frac{-u_2(x_0)}{s_2(x_0)}\right) = \mu$ or both. Also $(\bar{u} + \mu\bar{s})(x) \geq 0,\ x \in \bar{\Omega}$. Therefore either $(u_1 + \mu s_1)$ or $(u_2 + \mu s_2)$ attains minimum at $x = x_0$. Suppose the theorem is not true, then $\mu > 0$.

Case (i): Assume that $(u_1 + \mu s_1)(x_0) = 0$, for $x_0 = 0$. Therefore $(u_1 + \mu s_1)$ attains its minimum at $x = x_0$. Then,

$$0 = (u_1 + \mu s_1)(0) = u_1(0) + \mu s_1(0) > 0.$$

Case (ii): Assume that $(u_1 + \mu s_1)(x_0) = 0$, for $x_0 \in \Omega \cup \{1\}$. Therefore $(u_1 + \mu s_1)$ attains its minimum at $x = x_0$. Then,

$$0 < L_1(\bar{u} + \mu\bar{s})(x_0) = (u_1 + \mu s_1)'(x_0) - (u_2 + \mu s_2)(x_0) \leq 0.$$

Case (iii): Assume that $(u_2 + \mu s_2)(x_0) = 0$, for $x_0 = 0$. Therefore $(u_2 + \mu s_2)$ attains its minimum at $x = x_0$. Then,

$$0 < (u_2 + \mu s_2)(0) = u_2(0) + \mu s_2(0) = 0.$$

Case (iv): Assume that $(u_2 + \mu s_2)(x_0) = 0$, for $x_0 \in \Omega$. Therefore $(u_2 + \mu s_2)$ attains its minimum at $x = x_0$. Then,

$$\begin{aligned} 0 < L_2(\bar{u} + \mu\bar{s})(x_0) &= -\varepsilon(u_2 + \mu s_2)''(x_0) + a(x)(u_2 + \mu s_2)'(x_0) \\ &\quad + b(x)(u_2 + \mu s_2)(x_0) + c(x)(u_1 + \mu s_1)(x_0) \leq 0. \end{aligned}$$

Case (v): Assume that $(u_2 + \mu s_2)(x_0) = 0$, for $x_0 = 1$. Therefore $(u_2 + \mu s_2)$ attains its minimum at $x = x_0$. Then,

$$0 < B(u_2 + \mu s_2)(1) = (u_2 + \mu s_2)(1) - \varepsilon \int_0^1 g(x)(u_2 + \mu s_2)(x)dx \leq 0.$$

Observe that in all the cases we have a contradiction. Therefore $\mu > 0$ is not possible. Hence $\bar{u}(x) \geq 0, \forall\, x \in \bar{\Omega}$. □

**Corollary 2.2** (*Stability Result*)**.** *The solution $\bar{u}(x)$ of problem (2.1)–(2.3) satisfies the bound*

$$|u_i(x)| \leq C \max\{|u_1(0)|, |u_2(0)|, |Bu_2(1)|, \|L_1\bar{u}\|_{\Omega}, \|L_2\bar{u}\|_{\Omega}\},\ x \in \bar{\Omega},\ i = 1, 2.$$

**Proof.** Let $C > 0$ be a constant. Define $\psi_i^{\pm}(x) = CMs_i(x) \pm u_i(x), x \in \bar{\Omega}, i = 1,2$, where $M = \max\{|u_1(0)|, |u_2(0)|, |Bu_2(1)|, \|L_1\bar{u}\|_{\Omega}, \|L_2\bar{u}\|_{\Omega}\}$.

Note that $\psi_1^{\pm}(0) \geq 0$, $\psi_2^{\pm}(0) \geq 0$, $B\psi_2^{\pm}(1) \geq 0$ by proper choice of $C > 0$. It is easy to see that $L_1\bar{\psi}^{\pm}(x) \geq 0$, $L_2\bar{\psi}^{\pm}(x) \geq 0$. Then by maximum principle, we get the required result. □

Bounds for the derivatives of the solution $\bar{u}(x)$ are given in the following lemma.

**Lemma 2.3.** *Let $\bar{u}(x)$ be the solution of (2.1)–(2.3). Then we have the following bounds:*

$$\|u_1^{(k)}\|_{\bar{\Omega}} \leq C\varepsilon^{1-k}, \ k = 1, 2, 3,$$
$$\|u_2^{(k)}\|_{\bar{\Omega}} \leq C\varepsilon^{-k}, \ k = 1, 2, 3.$$

**Proof.** Using Corollary 2.2 and applying the arguments as given in [9] this lemma can be proved. □

The uniform error estimate can be derived using the sharper bounds on the derivatives of the solution $\bar{u}(x)$. To get sharper bounds we write the analytical solution in the form $\bar{u}(x) = \bar{v}(x) + \bar{w}(x)$, where $\bar{v}(x) = (v_1(x), v_2(x))$ and $\bar{w}(x) = (w_1(x), w_2(x))$. The regular component $\bar{v}(x)$ can be written as $\bar{v}(x) = \bar{v}_0(x) + \varepsilon\bar{v}_1(x) + \varepsilon^2\bar{v}_2(x)$, where $\bar{v}_0(x) = (v_{01}(x), v_{02}(x))$, $\bar{v}_1(x) = (v_{11}(x), v_{12}(x))$, $\bar{v}_2(x) = (v_{21}(x), v_{22}(x))$ respectively satisfy the following equations:

$$\begin{cases} v'_{01}(x) = v_{02}(x), \\ a(x)v'_{02}(x) + b(x)v_{02}(x) + c(x)v_{01}(x) = f(x), \\ v_{01}(0) = u_1(0), \ v_{02}(0) = u_2(0), \end{cases} \tag{2.4}$$

$$\begin{cases} v'_{11}(x) = v''_{12}(x), \\ a(x)v'_{12}(x) + b(x)v_{12}(x) + c(x)v_{11}(x) = v''_{02}(x), \\ v_{11}(0) = 0, \ v_{12}(0) = 0, \end{cases} \tag{2.5}$$

$$\begin{cases} L_1\bar{v}_2(x) = v_{21}(x) = v''_{22}(x), \\ L_2\bar{v}_2(x) = -\varepsilon v''_{22}(x) + a(x)v'_{22}(x) + b(x)v_{22}(x) + c(x)v_{21}(x) = v''_{12}(x), \\ v_{21}(0) = 0, \ v_{22}(0) = 0, \ Bv_{22}(1) = 0. \end{cases} \tag{2.6}$$

Thus the regular component $\bar{v}(x)$ is the solution of

$$\begin{cases} L_1\bar{v}(x) = v'_1(x) - v_2(x) = 0, \\ L_2\bar{v}(x) = -\varepsilon v''_2(x) + a(x)v'_2(x) + b(x)v_2(x) + c(x)v_1(x) = f(x), \\ v_1(0) = u_1(0), \ v_2(0) = u_2(0), \ Bv_2(1) = Bv_{02}(1) + \varepsilon Bv_{12}(1), \end{cases} \tag{2.7}$$

and layer component $\bar{w}(x)$ is the solution of

$$\begin{cases} L_1\bar{w}(x) = w'_1(x) - w_2(x) = 0, \\ L_2\bar{w}(x) = -\varepsilon w''_2(x) + a(x)w'_2(x) + b(x)w_2(x) + c(x)w_1(x) = 0, \\ w_1(0) = 0, \ \ w_2(0) = 0, \ \ Bw_2(1) = Bu_2(1) - Bv_2(1). \end{cases} \tag{2.8}$$

**Theorem 2.4.** *Let $\bar{u}(x)$ be the solution of the problem (2.1)–(2.3) and $\bar{v}_0(x)$ be its reduced problem solution defined in (2.4). Then*

$$|u_j(x) - v_{0j}(x)| \leq C(\varepsilon + e^{-\alpha(1-x)/\varepsilon}), \ \ x \in \bar{\Omega}, \ \ j = 1, 2.$$

**Proof.** Consider the barrier functions $\bar{\psi}^{\pm}(x) = (\psi_1^{\pm}(x), \psi_2^{\pm}(x))$, where

$$\psi_j^{\pm}(x) = C(\varepsilon s_j(x) + \varepsilon^{2-j}e^{-\alpha(1-x)/\varepsilon}) \pm (u_j(x) - v_{0j}(x)), \ \ x \in \bar{\Omega}, \ \ j = 1, 2.$$

It is easy to see that, $\psi_1^{\pm}(0) \geq 0$, $\psi_2^{\pm}(0) \geq 0$ for a suitable choice of $C > 0$.

Let $x \in \Omega$. Then

$$L_1\bar{\psi}^{\pm}(x) = C\left(\varepsilon(1 - s_2(x)) + (\alpha - 1)e^{-\alpha(1-x)/\varepsilon}\right) \pm L_1(\bar{u} - \bar{v}_0)(x) \geq 0,$$

and

$$L_2\bar{\psi}^{\pm}(x) = C[\frac{\alpha}{\varepsilon}(a(x)-\alpha)+b(x)+\varepsilon c(x)]e^{-\alpha(1-x)/\varepsilon}$$
$$+\varepsilon[a(x)s_2'(x)+b(x)s_2(x)+c(x)s_1(x)] \pm \varepsilon v_{02}''(x) \geq 0,$$

by a proper choice of $C > 0$.

Further

$$B\psi_2^{\pm}(1) = \psi_2^{\pm}(1) - \varepsilon\int_0^1 g(x)\psi_2^{\pm}(x)dx$$
$$\geq C(2\varepsilon+1) - 2C\varepsilon\int_0^1 g(x)dx - C\varepsilon\int_0^1 g(x)dx \pm B(u_2 - v_{02})(1) \geq 0$$

for a suitable choice of $C > 0$.

Then by Theorem 2.1, we have $\bar{\psi}_j^{\pm}(x) \geq 0,\ x \in \bar{\Omega},\ j = 1, 2.$ □

**Lemma 2.5.** *The regular component $\bar{v}(x)$ and the singular component $\bar{w}(x)$ of the solution $\bar{u}(x)$ of the problem (2.1)–(2.3) satisfy the following bounds:*

$$\|v_1^{(k)}\|_{\bar{\Omega}} \leq C(1+\varepsilon^{(3-k)}),\ k = 0, 1, 2, 3 \tag{2.9}$$
$$\|v_2^{(k)}\|_{\bar{\Omega}} \leq C(1+\varepsilon^{(2-k)}),\ k = 0, 1, 2, 3 \tag{2.10}$$
$$|w_1^{(k)}(x)| \leq C\varepsilon^{1-k}e^{-\alpha(1-x)/\varepsilon},\ x \in \bar{\Omega},\ k = 0, 1, 2, 3 \tag{2.11}$$
$$|w_2^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon},\ \ x \in \bar{\Omega},\ k = 0, 1, 2, 3. \tag{2.12}$$

**Proof.** Integrating (2.4), (2.5) and using the stability result one can prove the inequalities (2.9) and (2.10). To prove the inequalities (2.11) and (2.12) consider the barrier functions $\bar{\psi}^{\pm}(x) = (\psi_1^{\pm}(x), \psi_2^{\pm}(x))$, where

$$\psi_1^{\pm}(x) = C\varepsilon e^{-\alpha(1-x)/\varepsilon} \pm w_1(x),\ x \in \bar{\Omega},$$
$$\psi_2^{\pm}(x) = Ce^{-\alpha(1-x)/\varepsilon} \pm w_2(x),\ x \in \bar{\Omega}.$$

It is easy to see that $\psi_1^{\pm}(0) \geq 0$ and $\psi_2^{\pm}(0) \geq 0$, for a suitable choice of $C > 0$.

Further

$$L_1\bar{\psi}^{\pm}(x) = C[e^{-\alpha(1-x)/\varepsilon} - e^{-\alpha(1-x)/\varepsilon}] \pm L_1\bar{w} \geq 0$$
$$L_2\bar{\psi}^{\pm}(x) = C\left[\frac{\alpha}{\varepsilon}(a(x)-\alpha)+b(x)+\varepsilon c(x)\right]e^{-\alpha(1-x)/\varepsilon} \pm L_2\bar{w} \geq 0$$

$$B\psi_2^{\pm}(1) = \psi_2^{\pm}(1) - \varepsilon\int_0^1 g(x)\psi_1^{\pm}(x)dx \geq C(1-\varepsilon\int_0^1 g(x)dx) \pm Bw_2(1) \geq 0,$$

for a suitable choice of $C > 0$. Hence by the maximum principle, we have the desired result. From the differential equation (2.8), one can derive the rest of derivative estimates (2.11) and (2.12). □

Note: From the above theorem, it is easy to see that,

$$|u_1(x) - v_1(x)| \leq C(\varepsilon^2 + \varepsilon e^{-\alpha(1-x)/\varepsilon}),\ \ x \in \bar{\Omega}, \tag{2.13}$$
$$|u_2(x) - v_2(x)| \leq C(\varepsilon + e^{-\alpha(1-x)/\varepsilon}),\ \ x \in \bar{\Omega}, \tag{2.14}$$

## 3. Mesh and scheme

On $\bar{\Omega}$ a piecewise uniform Shishkin mesh of $N$ ($\geq 4$) mesh intervals is constructed. The domain $\bar{\Omega}$ is partitioned into two subintervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$, where $\sigma$ is the transition parameter defined by $\sigma = \min\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\alpha}\}$. On $[0, 1-\sigma]$ and $[1-\sigma, 1]$ a uniform mesh with $\frac{N}{2}$ mesh intervals is placed. The interior points of the mesh are denoted by

$$\Omega^N = \{x_i : 1 \leq i \leq \frac{N}{2}\} \cup \{x_i : \frac{N}{2} + 1 \leq i \leq N\}.$$

Clearly, $\bar{\Omega}^N = \{x_i\}_0^N$. Let $h_i = x_i - x_{i-1}$ be the mesh step and $\hbar_i = \frac{h_{i+1}+h_i}{2}$.

The discrete problem corresponding to (2.1)–(2.3) is:

Find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))$ such that

$$L_1^N \bar{U}(x_i) = D^- U_1(x_i) - U_2(x_i) = 0, \tag{3.1}$$

$$\begin{aligned} L_2^N \bar{U}(x_i) = &-\varepsilon \delta^2 U_2(x_i) + a(x_i) D^- U_2(x_i) + b(x_i) U_2(x_i) \\ &+ c(x_i) U_1(x_i) = f(x_i), \end{aligned} \tag{3.2}$$

$$\begin{cases} U_1(x_0) = l_1, \\ U_2(x_0) = l_2, \\ B^N U_2(x_N) = U_2(x_N) - \varepsilon \displaystyle\sum_{i=1}^{N} \frac{g(x_{i-1})U_2(x_{i-1}) + g(x_i)U_2(x_i)}{2} h_i = l_3, \forall x_i \in \bar{\Omega}^N. \end{cases} \tag{3.3}$$

where

$$\delta^2 U_2(x_i) = \frac{1}{\hbar_i}\left(\frac{U_2(x_{i+1}) - U_2(x_i)}{h_{i+1}} - \frac{U_2(x_i) - U_2(x_{i-1})}{h_i}\right),$$

$$D^- U_2(x_i) = \frac{U_2(x_i) - U_2(x_{i-1})}{h_i}.$$

## 4. Analysis of the method

**Theorem 4.1** (*Discrete Maximum Principle*). *Let $\bar{\Psi}(x_i) = (\Psi_1(x_i), \Psi_2(x_i))$ be the mesh function satisfying $\Psi_1(x_0) \geq 0$, $\Psi_2(x_0) \geq 0$, $B^N \Psi_2(x_N) \geq 0$, $L_1^N \bar{\Psi}(x_i) \geq 0$, and $L_2^N \bar{\Psi}(x_i) \geq 0$. Then $\bar{\Psi}(x_i) \geq 0$, $x_i \in \bar{\Omega}^N$.*

**Proof.** Define $\bar{S}(x_i) = (S_1(x_i), S_2(x_i))$, where $S_1(x_i) = 1 + x_i$ and $S_2(x_i) = \frac{1}{8} + \frac{x_i}{2}$. Note that $S_k(x_i) > 0$, $x_i \in \bar{\Omega}^N$, $k = 1, 2$, $L_1^N \bar{S}(x_i) > 0, \forall x_i \in \bar{\Omega}^N \cap \Omega \cup \{x_N\}$, $L_2^N \bar{S}(x_i) > 0, \forall x_i \in \bar{\Omega}^N$. Let

$$\gamma = \max\left\{\max_{x_i \in \bar{\Omega}^N}\left(\frac{-\Psi_1(x_i)}{S_1(x_i)}\right), \max_{x_i \in \bar{\Omega}^N}\left(\frac{-\Psi_2(x_i)}{S_2(x_i)}\right)\right\}.$$

Then there exists one $x_k \in \bar{\Omega}^N$ such that $\Psi_1(x_k) + \gamma S_1(x_k) = 0$ or $\Psi_2(x_k) + \gamma S_2(x_k) = 0$ or both. We have $\Psi_j(x_i) + \gamma S_j(x_i) \geq 0$, $x_i \in \bar{\Omega}^N$, $j = 1, 2$. Therefore either $(\Psi_1 + \gamma S_1)$ or $(\Psi_1 + \gamma S_1)$ attains minimum at $x_i = x_k$. Suppose the theorem is not true, then $\gamma > 0$.

Case (i): Assume that $(\Psi_1 + \gamma S_1)(x_k) = 0$, for $x_k = 0$. Therefore $(\Psi_1 + \gamma S_1)$ attains its minimum at $x_i = x_k$. Then,

$$0 = (\Psi_1 + \gamma S_1)(x_0) = \Psi_1(x_0) + \gamma S_1(x_0) > 0.$$

Case (ii): Assume that $(\Psi_1+\gamma S_1)(x_k)=0$, for $x_k\in\Omega^N\cup\{1\}$. Therefore $(\Psi_1+\gamma S_1)$ attains its minimum at $x_i=x_k$. Then,

$$0<L_1^N(\bar{\Psi}+\gamma\bar{S})(x_i)=D^-(\Psi_1+\gamma S_1)(x_i)-(\Psi_2+\gamma S_2)(x_i)\le 0.$$

Case (iii): Assume that $(\Psi_2+\gamma S_2)(x_k)=0$, for $x_k=0$. Therefore $(\Psi_2+\gamma S_2)$ attains its minimum at $x_i=x_k$. Then,

$$0<(\Psi_2+\gamma S_2)(x_0)=\Psi_2(x_0)+\gamma S_2(x_0)=0.$$

Case (iv): Assume that $(\Psi_2+\gamma S_2)(x_k)=0$, for $x_k\in\Omega^N$. Therefore $(\Psi_2+\gamma S_2)$ attains its minimum at $x_i=x_k$. Then,

$$\begin{aligned}0&<L_2^N(\bar{\Psi}+\mu\bar{S})(x_i)\\&=-\varepsilon\delta^2(\Psi_2+\mu S_2)(x_i)+a(x_i)D^-(\Psi_2+\mu S_2)(x_i)+b(x_i)(\Psi_2+\mu S_2)(x_i)\\&\quad+c(x_i)(\Psi_1+\mu S_1)(x_i)\le 0.\end{aligned}$$

Case (v): Assume that $(\Psi_2+\gamma S_2)(x_k)=0$, for $x_k=x_N$. Therefore $\Psi_2+\gamma S_2$ attains its minimum at $x_i=x_k$. Then

$$\begin{aligned}0&<B^N(\Psi_2+\gamma S_2)(x_N)\\&=(\Psi_2+\gamma S_2)(x_N)\\&\quad-\varepsilon\sum_{i=1}^{N}\frac{(\psi_2(x_{i-1})+\gamma S_2(x_{i-1}))g(x_{i-1})+(\Psi_2(x_i)+\gamma S_2(x_i))g(x_i)}{2}h_i\le 0.\end{aligned}$$

Observe that in all the cases we have a contradiction. Therefore $\gamma>0$ is not possible. Hence $\bar{\Psi}(x_i)\ge 0, \forall\, x_i\in\bar{\Omega}^N$. □

**Lemma 4.2** (*Discrete Stability Result*). *Let $\bar{U}(x_i)=(U_1(x_i),U_2(x_i))$ be any mesh function. Then*

$$\begin{aligned}|U_k(x_i)|\le C\max\Big\{&|U_1(x_0)|,|U_2(x_0)|,|BU_2(x_N),|\max_{x_j\in\Omega^N\cup\{x_N\}}|L_1^N\bar{U}(x_j)|\\&\max_{x_j\in\Omega^N}|L_2^N\bar{U}(x_j)|\Big\},\quad x_i\in\bar{\Omega}^N,\quad k=1,2.\end{aligned}$$

**Proof.** By choosing suitable barrier functions and using Theorem 4.1, one can establish the above inequality. □

Analogous to the continuous case, the discrete solution $\bar{U}(x_i)$ can be decomposed as

$$\bar{U}(x_i)=\bar{V}(x_i)+\bar{W}(x_i),$$

where $V(x_i)$ and $W(x_i)$ are respectively the solutions of the problems:

$$\begin{cases}L_1^N\bar{V}(x_i)=D^-V_i(x_i)-V_2(x_i)=0,\quad x_i\in\Omega^N\cup\{x_N\},\\ L_2^N\bar{V}(x_i)=-\varepsilon\delta^2V_2(x_i)+a(x_i)D^-V_2(x_i)+b(x_i)V_2(x_i)\\ \qquad\qquad +\ c(x_i)V_1(x_i),\quad x_i\in\Omega^N,\\ V_1(x_0)=v_1(0),\quad V_2(x_0)=v_2(0),\quad B^NV_2(x_N)=Bv_2(1)\end{cases}\tag{4.1}$$

and

$$\begin{cases} L_1^N \bar{W}(x_i) = D^- W_i(x_i) - W_2(x_i) = 0, \quad x_i \in \Omega^N \cup \{x_N\}, \\ L_2^N \bar{W}(x_i) = -\varepsilon\delta^2 W_2(x_i) + a(x_i)D^- W_2(x_i) + b(x_i)W_2(x_i) \\ \qquad\qquad + c(x_i)W_1(x_i), \; x_i \in \Omega^N, \\ W_1(x_0) = w_1(0), \quad W_2(x_0) = w_2(0), \quad B^N W_2(x_N) = Bw_2(1) \end{cases} \tag{4.2}$$

The following theorem gives an estimate for the difference of the solutions of (3.1)–(3.2) and (4.1).

**Theorem 4.3.** *Let $\bar{U}(x_i)$ be a numerical solution of (2.1)–(2.3) defined by (3.1)–(3.3) and $V(x_i)$ be a numerical solution of (2.7) defined by (4.1). Then*

$$|U_j(x_i) - V_j(x_i)| \le C \begin{cases} N^{-1}, & i = 0, 1, \dots, \dfrac{N}{2} \\ N^{-1} + |l_3 - B^N V_2(x_N)|, & i = \dfrac{N}{2} + 1, \dots, N. \end{cases} \quad j = 1, 2.$$

**Proof.** Consider mesh functions $\bar{\Psi}^{\pm}(x_i) = (\Psi_1^{\pm}(x_i), \Psi_2^{\pm}(x_i))$, where

$$\Psi_1^{\pm}(x_i) = CN^{-1}S_1(x_i) + Cx_i\varphi(x_i) \pm (U_1(x_i) - V_1(x_i)), \quad x_i \in \bar{\Omega}^N,$$

$$\Psi_2^{\pm}(x_i) = CN^{-1}S_2(x_i) + Cx_i\varphi(x_i) \pm (U_2(x_i) - V_2(x_i)), \quad x_i \in \bar{\Omega}^N,$$

$$\varphi(x_i) = \begin{cases} 0, & i = 0, 1, \dots, \dfrac{N}{2} \\ |l_3 - B^N V_2(x_N)|, & i = \dfrac{N}{2} + 1, \dots, N. \end{cases}$$

Now

$$\begin{aligned} L_1^N \bar{\Psi}^{\pm}(x_i) &= CN^{-1}[D^- S_1(x_i) - S_2(x_i)] + C[1 - x_i]\varphi(x_i) \pm 0 \ge 0, \\ L_2^N \bar{\Psi}^{\pm}(x_i) &= CN^{-1}[\frac{a(x_i)}{2} + b(x_i)S_2(x_i) + c(x_i)S_1(x_i)] \\ &\quad + CN^{-1}\varphi(x_i)[a(x_i) + x_i(b(x_i) + c(x_i))] \ge 0, x_i \in \Omega^N, \\ B\,\Psi_2^{\pm}(x_N) &= \Psi_2^{\pm}(x_N) - \varepsilon \sum_{i=1}^{i=N} \frac{g(x_{i-1})\Psi_2^{\pm}(x_{i-1}) + g(x_i)\;\Psi_2^{\pm}(x_i)}{2} h_i \ge 0. \end{aligned}$$

Then by Theorem 4.1 we get the result. □

We obtain separate error estimates for each component of the numerical solution.

**Lemma 4.4.** *Let $\bar{V}(x_i)$ be a numerical solution of (2.7) defined by (4.1). Then*

$$|(v_j(x_i) - V_j(x_i))| \le CN^{-1}, \; x_i \in \bar{\Omega}^N, \; j = 1, 2.$$

**Proof.** Now

$$L_1^N(\bar{v}(x_i) - \bar{V}(x_i)) = L_1^N \bar{v}(x_i) - L_1^N \bar{V}(x_i) = \left(D^- - \frac{d}{dx}\right) v_1(x_i),$$

$$L_2^N(\bar{v}(x_i) - \bar{V}(x_i)) = -\varepsilon\left(\delta^2 - \frac{d^2}{dx^2}\right) v_2(x_i) + a(x_i)\left(D^- - \frac{d}{dx}\right) v_2(x_i).$$

Therefore

$$L_j^N(\bar{v}(x_i) - \bar{V}(x_i)) \leq CN^{-1}, \; x_i \in \Omega^N, \; j = 1, 2.$$

Further

$$\begin{aligned} B^N(v_2 - V_2)(x_N) &= B^N v_2(x_N) - B^N V_2(x_N) \\ &= B^N v_2(x_N) - B v_2(1) \\ |B^N(v_2 - V_2)(x_N)| &\leq C\varepsilon(h_1^3 v''(\chi_1) + \cdots + h_N^3 v''(\chi_N)) \\ &\leq CN^{-2} \end{aligned}$$

where $x_{i-1} \leq \chi_i \leq x_i$, $1 \leq i \leq N$. Then by discrete stability result, we have $|(v_j(x_i) - V_j(x_i))| \leq CN^{-1}, \;\; x_i \in \bar{\Omega}^N, \;\; j = 1,2.$ □

**Lemma 4.5.** *Let $\bar{W}(x_i)$ be a numerical solution of (2.8) defined in (4.2). Then*

$$|(w_j - W_j)(x_i)| \leq CN^{-1}(\ln N)^2, \;\; x_i \in \bar{\Omega}^N, \;\; j = 1, 2.$$

**Proof.** Note that

$$|w_j(x_i) - W_j(x_i)| \leq |u_j(x_i) - U_j(x_i)| + |v_j(x_i) - V_j(x_i)|, \;\; j = 1, 2.$$

Then by (2.13), (2.14), we have

$$\begin{aligned} |u_j(x_i) - U_j(x_i)| \leq{} & |U_j(x_i) - V_j(x_i)| + |v_j(x_i) - V_j(x_i)| \\ & + |u_j(x_i) - v_j(x_i)|, \;\; j = 1, 2. \end{aligned}$$

Therefore

$$\begin{aligned} |w_j(x_i) - W_j(x_i)| &\leq |u_j(x_i) - U_j(x_i)| + |v_j(x_i) - V_j(x_i)| \\ &\leq Ce^{-\alpha(1-x_i)/\varepsilon} + CN^{-1} \\ &\leq Ce^{-\alpha\sigma/\varepsilon} + CN^{-1} \leq CN^{-1}, \quad 0 \leq i \leq \frac{N}{2} \end{aligned}$$

Now consider a mesh function $\bar{\Psi}^{\pm}(x_i) = (\Psi_1^{\pm}(x_i), \; \Psi_2^{\pm}(x_i)), \; x_i \in [1-\sigma, 1]$, where

$$\Psi_1^{\pm}(x_i) = CN^{-1}S_1(x_i) + 2CN^{-1}\frac{\sigma}{\varepsilon^2}(x_i - (1-\sigma)) \pm (w_1(x_i) - W_1(x_i)),$$
$$\Psi_2^{\pm}(x_i) = CN^{-1}S_2(x_i) + CN^{-1}\frac{\sigma}{\varepsilon^2}(x_i - (1-\sigma)) \pm (w_2(x_i) - W_2(x_i)).$$

It is easy to see that $\Psi_j^{\pm}(x_{N/2}) \geq 0, \;\; j = 1,2$ for a proper choice of $C > 0$.

$$L_1^N \bar{\Psi}^{\pm}(x_i) = CN^{-1}[1 - S_2(x_i)] + N^{-1}\frac{\sigma}{\varepsilon^2}(2 - x_i - \sigma) \pm (L_1^N - L_1)\bar{w}(x_i) \geq 0,$$

$$\begin{aligned} L_1^N \bar{\Psi}^{\pm}(x_i) = {}& CN^{-1}\left[\frac{a(x_i)}{2} + b(x_i)S_2(x_i) + c(x_i)S_1(x_i)\right] \\ & + CN^{-1}\frac{\sigma}{\varepsilon^2}[a(x_i) + [b(x_i) + 2c(x_i)](x_i + \sigma - 1)] \\ & \pm (L_2^N - L_2)(\bar{w}(x_i)) \geq 0, \end{aligned}$$

$$B\Psi_2^{\pm}(x_N) = \Psi_2^{\pm}(x_N) - \varepsilon \sum_{i=N/2}^{i=N} \frac{g(x_{i-1})\Psi_2^{\pm}(x_{i-1}) + g(x_i)\;\Psi_2^{\pm}(x_i)}{2} h_i \geq 0.$$

**Table 1**
Numerical results for Example 5.1.

| Number of mesh points N | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| $D_1^N$ | 2.073e−03 | 9.842e−04 | 4.782e−04 | 2.356e−04 | 1.169e−04 | 5.824e−05 | 2.906e−05 |
| $P_1^N$ | 1.075 | 1.041 | 1.021 | 1.010 | 1.005 | 1.002 | – |
| $D_2^N$ | 5.184e−03 | 3.269e−03 | 2.424e−03 | 1.652e−03 | 1.049e−03 | 6.326e−04 | 3.673e−04 |
| $P_2^N$ | 0.665 | 0.431 | 0.553 | 0.654 | 0.730 | 0.784 | – |

**Table 2**
Numerical results for Example 5.2.

| Number of mesh points N | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| $D_1^N$ | 2.768e−03 | 1.322e−03 | 6.488e−04 | 3.212e−04 | 1.596e−04 | 7.953e−05 | 3.969e−05 |
| $P_1^N$ | 1.066 | 1.027 | 1.014 | 1.009 | 1.005 | 1.002 | – |
| $D_2^N$ | 6.268e−03 | 3.826e−03 | 2.743e−03 | 1.815e−03 | 1.129e−03 | 6.716e−04 | 3.868e−04 |
| $P_2^N$ | 0.712 | 0.480 | 0.595 | 0.684 | 0.749 | 0.795 | – |

Then by discrete maximum principle, we have $\Psi_j^{\pm}(x_i) \geq 0, x_i \in [1-\sigma, 1], \ j = 1,2.$

Therefore $|w_j(x_i) - W_j(x_i)| \leq CN^{-1}(\ln N)^2, \ x_i \in [1-\sigma, 1], \ j = 1, 2.$ □

**Theorem 4.6.** *Let $\bar{U}(x_i)$ be the solution of (2.1)–(2.3) defined in (3.1)–(3.2). Then*

$$|u_j(x_i) - U_j(x_i)| \leq CN^{-1}(\ln N)^2, \ x_i \in \bar{\Omega}^N, \ j = 1, 2.$$

**Proof.** Combining Lemmas 4.4 and 4.5, completes the proof. □

## 5. Numerical results

The analytical solution of the test problems is not available. Therefore, we estimate the error using double mesh principle which is defined as $D_\varepsilon^N = \max_{x_i \in \bar{\Omega}^N} |U^N(x_i) - U^{2N}(x_i)|$ and $D^N = \max_\varepsilon D_\varepsilon^N$ where $U^N(x_i)$ and $U^{2N}(x_i)$ denote the numerical solution computed using $N$ and $2N$ mesh points. From these quantities the order of convergence is defined as $P^N = \log_2(\frac{D^N}{D^{2N}})$. In Tables 1 and 2, $D_1^N$ and $D_2^N$ denote the maximum pointwise errors of $U_1$ and $U_2$ respectively, $P_1^N$ and $P_2^N$ denote the order of convergence with respect to $U_1$ and $U_2$ respectively.

**Example 5.1.**

$$\begin{cases} -\varepsilon u'''(x) + (16+x)u''(x) + u'(x) - u(x) = x, \quad x \in \Omega \\ u(0) = 0, \ u'(0) = 0 \, u'(1) = \varepsilon \displaystyle\int_0^1 \frac{x}{2} u'(x)dx. \end{cases}$$

**Example 5.2.**

$$\begin{cases} -\varepsilon u'''(x) + (12+x^2)u''(x) - u(x) = x, \quad x \in \Omega \\ u(0) = 0, \ u'(0) = 0 \, u'(1) = \varepsilon \displaystyle\int_0^1 \frac{x}{2} u'(x)dx. \end{cases}$$

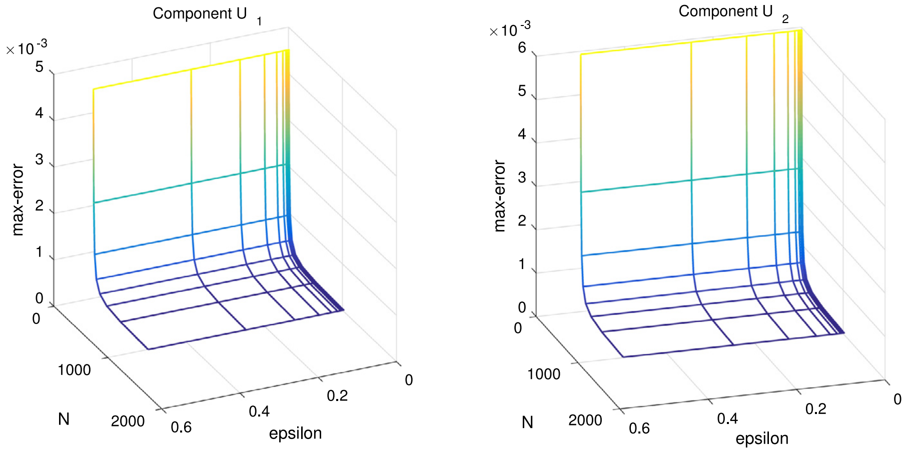


**Fig. 1.** Maximum pointwise errors of the numerical solution of Example 5.1.

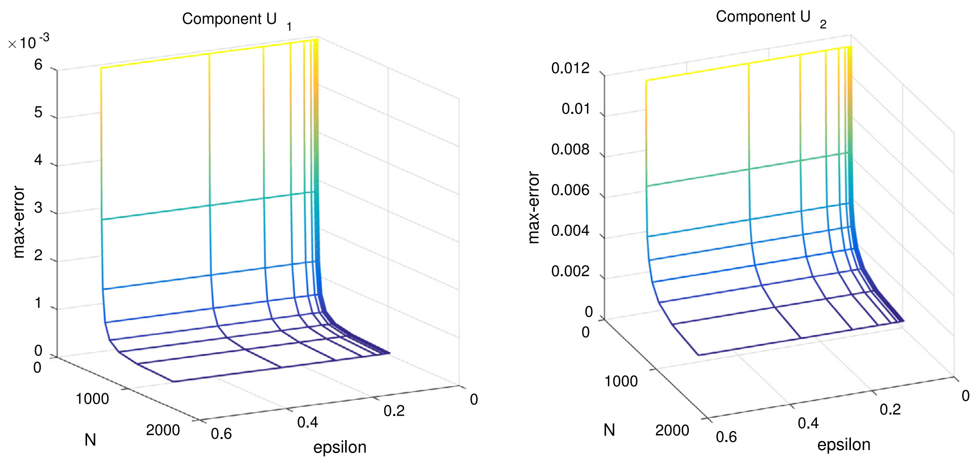


**Fig. 2.** Maximum pointwise errors of the numerical solution of Example 5.2.

## 6. Discussion

We have solved a class of third order singularly perturbed boundary value problems with integral boundary condition, using finite difference method on piecewise uniform mesh. Two examples are presented which authenticate our proposed numerical method. We have proved that the order of our numerical method is $O(N^{-1}\ln^2 N)$ (see Tables 1, 2). Maximum pointwise errors of Examples 5.1 and 5.2 are given in Figs. 1 and 2.

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