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Existence results for systems of first-order nabla dynamic inclusions on time scales

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Abstract. In this article, we study the existence of solutions to systems of first-order ∇-dynamic inclusions on time scales with terminal or periodic boundary conditions. We employ the method of solution-tube and Kakutani fixed point theorem.

Keywords: ∇-dynamic inclusions; Systems of first-order dynamic inclusions; Solution-tube; Existence theorems; Kakutani fixed-point theorem

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1. INTRODUCTION

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was born in 1988 with the Ph.D. thesis of Hilger [19]. The aim of this theory is to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. The reader interested

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on the subject of time scales is referred to [2,8,9,11,18,20]. The study of dynamic inclusions on time scales can be found, for example, in [3,7,14,15,17,24,25,27].

This paper considers the systems of first-order ∇-dynamic inclusions on time scales:

$$\begin{cases} x^{\nabla}(t) \in F(t, x(\rho(t))), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x \in (BC). \end{cases} \tag{1}$$

Here $\mathbb{T}$ is an arbitrary compact time scale, such that $a = \min \mathbb{T}$, $b = \max \mathbb{T}$, $\mathbb{T}_0 = \mathbb{T} \setminus \{a\}$, $F : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a multivalued map with compact and convex values, and (BC) denotes the terminal value or the periodic boundary value conditions:

$$x(b) = x_0, \tag{2}$$

$$x(a) = x(b). \tag{3}$$

In the particular case where $n = 1$, existence results for first order ∇-dynamic inclusion on time scales were obtained in [3] for the general boundary conditions:

$$x^{\nabla}(t) \in F(t, x(t)), \text{ a.e. } on\ \mathbb{T}_{\kappa}, \quad \text{and} \quad L(x(a), x(b)) = 0,$$

with $F : \mathbb{T}_{\kappa} \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ a multivalued map with compact and convex values and L is a continuous single-valued map. Their results were established with the method of lower and upper solutions. Existence results for systems of first order ∇-dynamic inclusions were obtained in [17] for the initial value problem. In [14] Frigon and Gilbert introduced the notion of solution-tube to systems of first order Δ-dynamic inclusions (with an initial or a periodic boundary value condition) which generalizes the notions of lower and upper solutions given in [3]. A notion of solution-tube was introduced for first order systems of differential inclusions by B. Mirandette [23]. In order to obtain the existence results for problem (1), we introduce the notion of solution-tube of (1) which generalizes the notions of lower and upper solutions. The results in this paper were motivated by results in [14,15].

This paper is organized as follows. We start with some notations, definitions and results which are used throughout this paper. The third section presents existence results for the problem (1).

2. Preliminaries

In this section, we establish notations, definitions, and results which we will use in this article.

2.1. *Lebesgue ∇-measure on time scales*

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \ \ and \ \ \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. We say that t is right-scattered (resp., left-scattered) if $\sigma(t) > t$ (resp., if $\rho(t) < t$); that t is isolated if it is right-scattered and left-scattered. Also, if $t < \sup \mathbb{T}$ and $t = \sigma(t)$, we say that t is right-dense. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, we say that t is left dense. Points that are right dense and left dense are called dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum, then

$\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{\sup \mathbb{T}\}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0, \infty)$ is defined by $\nu(t) := t - \rho(t)$. If $\mathbb{T}$ has a right-scattered minimum, then $\mathbb{T}_{\kappa} = \mathbb{T} \setminus \{\inf \mathbb{T}\}$, otherwise, $\mathbb{T}_{\kappa} = \mathbb{T}$. If $\mathbb{T}$ is bounded, then $\mathbb{T}_0 \subseteq \mathbb{T}_{\kappa}$ where $\mathbb{T}_0 = \mathbb{T} \setminus \{\min \mathbb{T}\}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$.

Definition 2.1. The function $f : \mathbb{T} \to \mathbb{R}^n$ is called ld-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and has a right-sided limit that exists at right-dense points in $\mathbb{T}$, write $f \in C_{ld}(\mathbb{T}, \mathbb{R}^n)$.

Definition 2.2 (*[26]*)**.** For $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}_{\kappa}$, the ∇-derivative of f at t, denoted by $f^{\nabla}(t)$, is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left\| f^{\rho}(t) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right\| \leq \varepsilon \left| \rho(t) - s \right|, \quad for\ all\ s \in U.$$

We say that f is ∇-differentiable if $f^{\nabla}(t)$ exists for every $t \in \mathbb{T}_{\kappa}$. The function $f^{\nabla} : \mathbb{T} \to \mathbb{R}^n$ is then called the ∇-derivative of f on $\mathbb{T}_{\kappa}$.

The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ which are ∇-differentiable and whose ∇-derivative is ld-continuous is denoted by $C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$.

Example 2.3 (*[6]*)**.** Assume $x : \mathbb{T} \to \mathbb{R}^n$ is ∇-differentiable at $t \in \mathbb{T}$. We know that $\|.\| : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ is differentiable. If $t = \rho(t)$, we have

$$\|x(t)\|^{\nabla} = \frac{\langle x(t), x^{\nabla}(t) \rangle}{\|x(t)\|}.$$

Definition 2.4 (*[9]*)**.** The function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive if

$$1 - \nu(t)p(t) \neq 0, \ for\ all t \in \mathbb{T}_{\kappa}.$$

The set of all ν-regressive and ld-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}_{\nu} = \mathcal{R}_{\nu}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^{+}_{\nu} = \{p \in \mathcal{R}_{\nu} : 1 - \nu(t)p(t) > 0\}\ for\ all\ t \in \mathbb{T}$.

Definition 2.5 (*[9]*)**.** If $p \in \mathcal{R}_{\nu}$, then we define the (nabla) exponential function $\hat{e}_p$ by:

$$\hat{e}_p(t, s) = \exp\Big(\int_s^t \hat{\xi}_{\nu(\tau)}(p(\tau)) \nabla \tau \Big),$$

for $t, s \in \mathbb{T}$, where the ν-cylinder transformation is as in:

$$\hat{\xi}_h(z) = \begin{cases} -\dfrac{1}{h} \log(1 - zh); & if\ h > 0; \\ z; & if\ \ h = 0, \end{cases}$$

where log is the principal logarithm function.

Theorem 2.6 (*[9]*)**.** *For $p \in \mathcal{R}_{\nu}$, the (nabla) exponential function $\hat{e}_p(., t_0) : \mathbb{T} \to \mathbb{R}$ is defined as the unique solution to the initial value problem*

$$x^{\nabla}(t) = p x(t), \qquad x(t_0) = 1.$$

We recall some notions and results related to the theory of ∇-measure and ∇-Lebesgue integration for an arbitrary bounded time scale $\mathbb{T}$ where $a = \min \mathbb{T} < \max \mathbb{T} = b$ introduced in [4,9,18].

Definition 2.7. Let $\mathbb{F}$ denote the family of all right closed and left open intervals of $\mathbb{T}$ of the form

$$(r, s] = \{t \in \mathbb{T} : r \leq t < s\},$$

with $r, s \in \mathbb{T}$ and $r \leq s$. The interval $(r, r]$ is understood as the empty set. We define an additive measure $m_1 : \mathbb{F} \to [0, \infty)$ by $m_1((r, s]) = s - r$. Using m_1, the outer measure $m_1^* : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$, defined for each $E \subset \mathbb{T}$ as:

$$m_1^*(E) = \begin{cases} \inf \left\{ \sum_{k=1}^{k=m} (s_k - r_k) : E \subset \bigcup_{k=1}^{k=m} (r_k, s_k] \text{ with } (r_k, s_k] \in \mathbb{F} \right\} & \text{if } a \notin E, \\ +\infty & \text{if } a \in E. \end{cases}$$

Definition 2.8. A set $A \subset \mathbb{T}$ is said to be ∇-measurable if, for every set $E \subset \mathbb{T}$

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)).$$

The Lebesgue ∇-measure on $\mathcal{M}(m_1^*) = \{A \subset \mathbb{T} : A \text{ is } \nabla\text{-measurable}\}$, denoted by μ_∇, is the restriction of m_1^* to $\mathcal{M}(m_1^*)$. So, $(\mathbb{T}, \mathcal{M}(m_1^*), \mu_\nabla)$ is a complete measurable space.

The following lemma can be proved analogously to Lemma 3.1 in [12].

Lemma 2.9. *The set of all left-scattered points of $\mathbb{T}$ is at most countable, that is, there are $J \subseteq \mathbb{N}$ and $\{t_j\}_{j \in J} \subset \mathbb{T}$ such that $\mathcal{L}_\mathbb{T} := \{t \in \mathbb{T}, \rho(t) < t\} = \{t_j\}_{j \in J}$.*

The notions of ∇-measurable and ∇-integrable functions $f : \mathbb{T} \to \mathbb{R}^n$ can be defined similarly to the theory of Lebesgue integral.

Definition 2.10. We say that $f : \mathbb{T} \longrightarrow \overline{\mathbb{R}} = [-\infty, +\infty]$ is ∇-measurable if for every $\alpha \in \mathbb{R}$, the set $f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$ is ∇-measurable.

In order to compare the Lebesgue ∇-integral on on $\mathbb{T}$ and Lebesgue integrable on $[a, b]$, given a function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$, we need an auxiliary function which extends $\widetilde{f}$ to the interval $[a, b]$ defined as

$$\widetilde{f}(t) := \begin{cases} f(t), & \text{if } t \in \mathbb{T}, \\ f(t_j), & \text{if } t \in (\rho(t_j), t_j), \text{ for all } j \in J. \end{cases} \tag{4}$$

Let $E \subset \mathbb{T}$, we define $J_E := \{j \in J : t_j \in E \cap \mathcal{L}_\mathbb{T}\}$ and

$$\widetilde{E} := E \cup \bigcup_{j \in J_E} (\rho(t_j), t_j). \tag{5}$$

The following theorem can be proved analogously to Theorem 5.1 in [12].

Theorem 2.11. *Let $E \subset \mathbb{T}$ be a ∇-measurable such that $a \notin E$, let $\widetilde{E}$ be the set defined in (5), let $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ be a ∇-measurable function and $\widetilde{f} : [a, b] \longrightarrow \mathbb{R}^n$ be the extension of f to $[a, b]$. Then, f is Lebesgue ∇-integrable on E if and only if $\widetilde{f}$ is Lebesgue integrable on $\widetilde{E}$. In this case we have*

$$\int_E f(t) \nabla t = \int_{\widetilde{E}} \widetilde{f}(t)\, dt. \tag{6}$$

2.2. *First order Sobolev's spaces on time scales*

In this section, we develop the Sobolev's spaces on bounded time scale $\mathbb{T}$ where $a = \min \mathbb{T} < \max \mathbb{T} = b$, $\mathbb{T}_0 = \mathbb{T} \setminus \{a\}$ and their important properties.

Definition 2.12. Let $E \subset \mathbb{T}$ be a ∇-measurable set and $f : \mathbb{T} \to \mathbb{R}^n$ be a ∇-measurable function. We say that $f \in L^1_\nabla(E, \mathbb{R}^n)$ provided

$$\int_E \|f(s)\| \nabla s < +\infty.$$

Proposition 2.13. *Assume $f \in L^1_\nabla(E, \mathbb{R}^n)$. Then,*

$$\left\| \int_E f(s) \nabla s \right\| \leq \int_E \|f(s)\| \, \nabla s.$$

The following theorem can be proved analogously to Theorem 2.5 in [1].

Theorem 2.14. *The set $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ is a Banach space equipped with the norm*

$$\|f\|_{L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)} := \int_{\mathbb{T}_0} \|f(t)\| \nabla t.$$

Here is an analog of the Lebesgue dominated convergence theorem.

Theorem 2.15. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$. If there exists a function $f : \mathbb{T}_0 \to \mathbb{R}^n$ such that $f_k(t) \to f(t)$ ∇-a.e. $t \in \mathbb{T}_0$ and if there exists a function $g \in L^1_\nabla(\mathbb{T}_0)$ such that $\|f_k(t)\| \leq g(t)$ ∇-a.e. $t \in \mathbb{T}_0$ and for every $k \in \mathbb{N}$. Then $f_k \to f$ in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$.*

Using Theorem 2.11, we obtain the following result.

Theorem 2.16. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$. If $\{\widetilde{f_k}\}$ converges weakly to γ in $L^1([a,b], \mathbb{R}^n)$, then γ is the extension $\widetilde{f}$ of a function f defined on $\mathbb{T}_0$ in the sense of definition (4). Moreover, for every ∇-measurable set $E \subset \mathbb{T}_0$ and every continuous function $g : \mathbb{T} \to \mathbb{R}$, we have*

$$\lim_{k\to\infty} \int_E g(s) f_k(s) \nabla s = \int_E g(s) f(s) \nabla s.$$

Proof. Since $\{\widetilde{f_k}\}$ converges weakly to γ in $L^1([a,b], \mathbb{R}^n)$, we have for every continuous function $g : \mathbb{T} \to \mathbb{R}$,

$\int_A \widetilde{g}(s) \widetilde{f_k}(s) ds \to \int_A \widetilde{g}(s) \gamma(s) ds$ for every measurable set $A \subset [a,b]$.

Thus, for $t_i \in R_{\mathbb{T}}$,

$$\int_{(\rho(t_i), t_i)} \widetilde{g}(s) \widetilde{f_k}(s) ds = \int_{(\rho(t_i), t_i)} g(t_i) f_k(t_i) ds = g(t_i) f_k(t_i) \nu(t_i)$$
$$\to \int_{(\rho(t_i), t_i)} \widetilde{g}(s) \gamma(s) ds.$$

So, $\{f_k(t_i)\}_{k\in\mathbb{N}}$ converges to some $f(t_i) \in \mathbb{R}^n$. Thus, $\{\widetilde{f_k}\}$ converges strongly to the constant function $f(t_i)$ in $L^1_\nabla((\rho(t_i), t_i), \mathbb{R}^n)$, and we can assume that $\gamma = f(t_i)$ on $(\rho(t_i), t_i]$.

The first part of the proposition is proved if we define $f = \gamma|_{\mathbb{T}}$. Finally, by Theorem 2.11,

$$\int_E g(s) f_k(s) \nabla s = \int_{\widetilde{E}} \widetilde{g}(s) \widetilde{f}_k(s) ds$$
$$\to \int_{\widetilde{E}} \widetilde{g}(s) \gamma(s) ds = \int_{\widetilde{E}} \widetilde{g}(s) f(s) ds = \int_E g(s) f(s) \nabla s. \quad \square$$

Now we introduce the concept of absolutely continuous function on $\mathbb{T}$.

Definition 2.17. A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be absolutely continuous on $\mathbb{T}$ if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{(a_k, b_k], k = 1, \ldots, m\}$, with $a_k, b_k \in \mathbb{T}$, is a finite pairwise disjoint family of subintervals of $\mathbb{T}$ satisfying

$$\sum_{k=1}^{k=m} (b_k - a_k) < \eta \ \ then \ \ \sum_{k=1}^{k=m} \| f(b_k) - f(a_k) \| < \varepsilon.$$

The following theorem can be proved analogously to Theorem 4.1 in [11].

Theorem 2.18. *A function $f : \mathbb{T} \to \mathbb{R}^n$ is absolutely continuous on $\mathbb{T}$ if and only if f is ∇-differentiable ∇-almost everywhere on $\mathbb{T}_0$, $f^{\nabla} \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$ and*

$$\int_{(t,b]\cap\mathbb{T}} f^{\nabla}(s) \nabla s = f(b) - f(t), \quad \text{for every } t \in \mathbb{T}.$$

The following two propositions can be proved analogously to Proposition 2.19 and Proposition 2.20 in [16].

Proposition 2.19. *Let $f \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$, then $F : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$F(t) = \int_{(t,b]\cap\mathbb{T}} f(s) \nabla s \ satisfies \ F^{\nabla}(t) = f(t), \ \nabla\text{-a.e. on } \mathbb{T}_0.$$

Proposition 2.20. *Let $u : \mathbb{T} \to \mathbb{R}$ be an absolutely continuous function, then the ∇-measure of the set $\{t \in \mathbb{T}_0 \setminus \mathcal{L}_{\mathbb{T}_0} : u(t) = 0 \text{ and } u^{\nabla}(t) \neq 0\}$ is zero.*

We now define a notion of Sobolev's space.

Definition 2.21. We say that a function $f : \mathbb{T} \to \mathbb{R}^n$ belongs to $W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ if and only if $f \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$ and there exists $g : \mathbb{T}_\kappa \to \mathbb{R}^n$ such that $g \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$ and

$$\int_{\mathbb{T}_0} f(s) \phi^{\nabla}(s) \nabla s = -\int_{\mathbb{T}_0} g(s) \phi^{\rho}(s) \nabla s, \quad \text{for all } \phi \in C^1_{0,ld}(\mathbb{T}), \tag{7}$$

with

$$C^1_{0,ld}(\mathbb{T}) := \left\{ \phi \in C^1_{ld}(\mathbb{T}) : \phi(a) = \phi(b) = 0 \right\}.$$

Theorem 2.22. *The set $W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ is a Banach space together with the norm defined for every $f \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ as*

$$\|f\|_{W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n)} = \|f\|_{L^1_{\nabla}(\mathbb{T},\mathbb{R}^n)} + \left\| f^{\nabla} \right\|_{L^1_{\nabla}(\mathbb{T}_\kappa,\mathbb{R}^n)}.$$

The proof is analogous to that of Theorem 3.5 in [1].

2.3. Multivalued maps

Now, we recall some definitions and classical results for multivalued maps. The reader is referred to [5,10,21,22] for more details on multivalued maps. Let X, Y be metric spaces and $G : X \to Y$ a multivalued map. The map G is upper semi-continuous ($u.s.c.$) if $\{x \in X : G(x) \cap C \neq \emptyset\}$ is closed for every closed set $C \subset Y$ and it is compact if $G(X) = \cup_{x \in X} G(x)$ is relatively compact. Let Ω be a measurable space, we say that a multivalued map $G : \Omega \to X$ is measurable (resp. weakly measurable) if $\{t \in \Omega : G(t) \cap C \neq \emptyset\}$ is measurable for every closed (resp. open) set $C \subset X$.

Proposition 2.23. *Let $G : \Omega \to X$ be a multivalued map.*

(a) *If G is measurable then it is weakly measurable.*
(b) *If G is weakly measurable and has compact values, then it is measurable.*
(c) *The map G is weakly measurable if and only if the multivalued map $\overline{G} : \Omega \to X$ defined by $\overline{G}(t) = \overline{G(t)}$ is weakly measurable.*

Proposition 2.24. *For $n \in \mathbb{N}$, let $G_n : \Omega \to X$ be measurable multivalued maps.*

(a) *The map $G : \Omega \to X$ defined by $G(t) = \cup_{n \in \mathbb{N}} G_n(t)$ is measurable.*
(b) *If X is separable, G_n has closed values, and for each t, at least one $G_{n_t}(t)$ is compact, then $G : \Omega \to X$ defined by $G(t) = \cap_{n \in \mathbb{N}} G_n(t)$ is measurable.*

Theorem 2.25 (*Kuratowski, Ryll, Nardzewski*)**.** *Let X be a separable Banach space and let $G : \Omega \to X$ be a measurable multivalued map. Then G has a measurable selection, i.e. there exists a single-valued measurable map $g : \Omega \to X$ such that $g(t) \in G(t)$ for almost every $t \in \Omega$.*

Theorem 2.26 (*Kakutani Fixed Point Theorem*)**.** *Let C be a nonempty compact convex subset of $\mathbb{R}^n$. Let $T : C \to \mathcal{P}(C)$ satisfy*

(i) *for each $x \in C$, $T(x)$ is nonempty closed and convex,*
(ii) *T is upper semi-continuous.*

Then T has a fixed point. ($i.e.\ there\ exists\ x \in C\ such\ that\ x \in T(x)$).

Next, we define a notion of ∇-Carathéodory multivalued map on a compact time scale.

Definition 2.27. A multivalued map $F : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is called a ∇-Carathéodory if the three following conditions hold.

(i) for every $x \in \mathbb{R}^n$, the function $t \mapsto F(t, x)$ is ∇-measurable;
(ii) the function $x \mapsto F(t, x)$ is $u.s.c.$ for ∇-almost every $t \in \mathbb{T}_0$;
(iii) for every $q > 0$, there exists a function $h_q \in L^1_\nabla(\mathbb{T}_0, [0, \infty))$ such that

$$sup \{\|y\| : y \in F(t, x), \|x\| \leq q\} \leq h_q(t), \quad \nabla\text{-}a.e.\ t \in \mathbb{T}_0.$$

3. Main results

In this section, we are concerned with the existence of solutions for the problem (1). A solution of the problem (1) will be a function $x \in W_{\nabla}^{1,1}(\mathbb{T}, \mathbb{R}^n)$ for which (1) is satisfied, we introduce the notion of solution-tube of this problem.

Definition 3.1. Let $(v, M) \in W_{\nabla}^{1,1}(\mathbb{T}, \mathbb{R}^n) \times W_{\nabla}^{1,1}(\mathbb{T}, [0, \infty))$. We say that (v, M) is a solution tube of (1) if

(1) ∇-a.e. $t \in \mathbb{T}_0$ and for every $x \in \mathbb{R}^n$ such that $\|x - v(\rho(t))\| = M(\rho(t))$, there exists $\delta > 0$ such that, for every $u \in \mathbb{R}^n$ such that $\|u - x\| < \delta$, and $\|u - v(\rho(t))\| \geq M(\rho(t))$, there exists $y \in F(t, u)$ such that

$$\langle u - v(\rho(t)), y - v^{\nabla}(t)\rangle \geq M^{\nabla}(t)\|u - v(\rho(t))\|.$$

(2) $v^{\nabla}(t) \in F(t, v(\rho(t)))$ ∇-a.e. $t \in \mathbb{T}_0$ such that $M(\rho(t)) = 0$,

(3) $M(t) = 0$, for every $t \in \mathbb{T}_0$ such that $M(\rho(t)) = 0$,

(4) – If (BC) denotes (2), then $\|x_0 - v(b)\| \leq M(b)$.
– If (BC) denotes (3), then $\|v(a) - v(b)\| \leq M(b) - M(a)$.

We denote

$$T(v, M) = \{x \in W_{\nabla}^{1,1}(\mathbb{T}, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t) \ for\ every\ t \in \mathbb{T}\}.$$

We need the following auxiliary Lemmas.

Lemma 3.2. *Let $g \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$. The function $x : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$x(t) = \hat{e}_{-1}(b, t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s, b)\nabla s\Big) \tag{8}$$

is a solution of the problem

$$\begin{cases} x^{\nabla}(t) - x(\rho(t)) = g(t), & \nabla\text{-a.e. } t \in \mathbb{T}_0 \\ x(b) = x_0. \end{cases} \tag{9}$$

Proof. We check (9) for each pair $(x_i,\ g_i),\ i \in \{1, 2, \dots, n\}$, by direct calculation. From Theorem 3.3 in [9] and Proposition 2.19 we have that

$$\begin{aligned} x^{\nabla}(t) - x(\rho(t)) = {} & x_0\big(\hat{e}_{-1}(b, t)\big)^{\nabla} - \big(\hat{e}_{-1}(b, t)\big)^{\nabla} \int_{(\rho(t),b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s, b)\nabla s \\ & - \hat{e}_{-1}(b, t)\Big(\int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s, b)\nabla s\Big)^{\nabla} - \hat{e}_{-1}(b, \rho(t))x_0 \\ & + \hat{e}_{-1}(b, \rho(t)) \int_{(\rho(t),b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s, b)\nabla s \\ = {} & g(t), \end{aligned}$$

for all $t \in \mathbb{T}$. It is easy to verify that $x(b) = x_0$. □

Lemma 3.3. *Let $g \in L^1_{\nabla}(\mathbb{T}_0, \mathbb{R}^n)$. The function $x : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$x(t) = \frac{1}{\hat{e}_{-1}(t, b)}\left(\frac{1}{1 - \hat{e}_{-1}(a, b)} \int_{(a,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s, b)\nabla s - \int_{(t,b]\cap\mathbb{T}} g(s)\hat{e}_{-1}(s, b)\nabla s\right)$$

is a solution of the problem

$$\begin{cases} x^{\nabla}(t) - x(\rho(t)) = g(t), & \nabla\text{-}a.e.\ t \in \mathbb{T}_0, \\ x(a) = x(b). \end{cases} \tag{10}$$

Proof. The result follows in a similar way to the proof of Lemma 3.2. □

The following lemma can be proved analogously to Lemma 2.24 in [16].

Lemma 3.4. *Let $r \in W^{1,1}_{\nabla}(\mathbb{T})$ such that $r^{\nabla}(t) > 0$ ∇-a.e. $t \in \{t \in \mathbb{T}_0 : r(\rho(t)) > 0\}$. If one of the following conditions holds,*

(i) $r(b) \leq 0$;

(ii) $r(b) \leq r(a)$;

then $r(t) \leq 0$, for every $t \in \mathbb{T}$.

We assume the following hypothesis

(H_1) $F : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is a ∇-Carathéodory multivalued map with compact and convex values.

(H_2) There exists $(v, M) \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T}, [0, \infty))$ a solution tube of (1).

To prove our existence theorem, we consider the following modified problem:

$$\begin{cases} x^{\nabla}(t) - x(\rho(t)) \in F_u(t, x(\rho(t))) - \bar{x}(\rho(t)), & \nabla\text{-a.e. } t \in \mathbb{T}_0, \\ x \in (BC), \end{cases} \tag{11}$$

where $F_u : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by:

$$F_u(t, x) = F(t, \bar{x}(\rho(t))) \cap G(t, x); \tag{12}$$

with

$$G(t,x) = \begin{cases} v^{\nabla}(t) & \text{if } M(\rho(t)) = 0, \\ \mathbb{R}^n & \text{if } \|x(\rho(t)) - v(\rho(t))\| \leq M(\rho(t)), \\ & \text{and } M(\rho(t)) > 0, \\ \{z \in \mathbb{R}^n : \langle x - v(\rho(t)), z - v^{\nabla}(t)\rangle & \\ \qquad \geq M^{\nabla}(t)\|x - v(\rho(t))\|\}, & \text{otherwise}, \end{cases}$$

and

$$\bar{x}(t) = \begin{cases} \dfrac{M(t)}{\|x - v(t)\|}(x - v(t)) + v(t), & \text{if } \|x - v(t)\| > M(t), \\ x(t), & otherwise, \end{cases} \tag{13}$$

where (v, M) is a solution tube of (1).

Remark 3.5. For every (t, x) such that $\|x - v(\rho(t))\| > M(\rho(t))$,

$$G(t, x) = G(t, \bar{x}_{\theta}(\rho(t)))\ for\ all\ \theta \in [0, 1[, \tag{14}$$

where

$$\bar{x}_{\theta}(\rho(t)) = \theta\bar{x}(\rho(t)) + (1 - \theta)x.$$

Indeed, for $\theta \in [0, 1[$,

$$\bar{x}_\theta(\rho(t)) - v(\rho(t)) = \left(1 - \theta + \frac{\theta M(\rho(t))}{\|x - v(\rho(t))\|}\right)(x - v(\rho(t))).$$

Thus,

$$\begin{aligned} G(t, x) &= \left\{z \in \mathbb{R}^n : \langle x - v(\rho(t)), z - v^\nabla(t)\rangle \geq M^\nabla(t)\|x - v(\rho(t))\|\right\} \\ &= \left\{z \in \mathbb{R}^n : \langle \bar{x}_\theta(\rho(t)) - v(\rho(t)), z - v^\nabla(t)\rangle \geq M^\nabla(t)\|\bar{x}_\theta(\rho(t)) - v(\rho(t))\|\right\}. \end{aligned}$$

So, for $\theta \in [0, 1[$, $G(t, x) = G(t, \bar{x}_\theta(\rho(t)))$ since $\|\bar{x}_\theta(\rho(t)) - v(\rho(t))\| > M(\rho(t))$.

Similar to the Propositions 3.3 and 3.4 in [14], we give the following propositions.

Proposition 3.6. *The multivalued map $G : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following properties:*

(i) *$G(t, x)$ has nonempty, closed, convex values for all $x \in \mathbb{R}^n$, and for ∇-almost every $t \in \mathbb{T}_0$;*

(ii) *$x \mapsto G(t, x)$ has closed graph for ∇-almost every $t \in \mathbb{T}_0$;*

(iii) *$t \mapsto G(t, x)$ is ∇-measurable for every $x \in \mathbb{R}^n$.*

Proof. (i) It is obvious that G has nonempty, closed, convex values.

(ii) To show that

$$A_t = \left\{(x, y) \in \mathbb{R}^{2n} : y \in G(t, x)\right\}$$

is closed for ∇-a.e. $t \in \mathbb{T}_0$, we just have to check the case where $t \in \mathbb{T}_0$ is such that $M(\rho(t)) \neq 0$. Let $\{(x_k, y_k)\}$ be in A_t such that $x_k \to x$ and $y_k \to y$. If $\|x - v(\rho(t))\| \leq M(\rho(t))$ then $y \in G(t, x) = \mathbb{R}^n$. So, $(x, y) \in A_t$. Otherwise, $\|x - v(\rho(t))\| > M(\rho(t))$ and for k sufficiently large $\|x_k - v(\rho(t))\| > M(\rho(t))$ and

$$\langle x_k - v(\rho(t)), y_k - v^\nabla(t)\rangle \geq M^\nabla(t)\|x_k - v(\rho(t))\|.$$

Therefore,

$$\langle x - v(\rho(t)), y - v^\nabla(t)\rangle \geq M^\nabla(t)\|x - v(\rho(t))\|, \text{ and hence } (x, y) \in A_t.$$

(iii) Let C be a nonempty, closed subset of $\mathbb{R}^n$, and fix $x \in \mathbb{R}^n$. Let $\{y_m : m \in \mathbb{N}\}$ be a countable, dense subset of C. Observe that

$$\mathcal{B}_x = \left\{t \in \mathbb{T}_0 : G(t, x) \cap C \neq \emptyset\right\} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup (\mathcal{B}_3 \cap \mathcal{B}_4),$$

where

$$\begin{aligned} \mathcal{B}_1 &= \left\{t \in \mathbb{T}_0 : v^\nabla(t) \in C\right\} \cap \left\{t \in \mathbb{T}_0 : M(\rho(t)) = 0\right\}, \\ \mathcal{B}_2 &= \left\{t \in \mathbb{T}_0 : \|x - v(\rho(t))\| - M(\rho(t)) \leq 0\right\} \cap \left\{t \in \mathbb{T}_0 : M(\rho(t)) > 0\right\}, \\ \mathcal{B}_3 &= \left\{t \in \mathbb{T}_0 : \|x - v(\rho(t))\| - M(\rho(t)) > 0\right\} \cap \left\{t \in \mathbb{T}_0 : M(\rho(t)) > 0\right\}, \\ \mathcal{B}_4 &= \bigcap_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \left\{t \in \mathbb{T}_0 : \left\langle x - v(\rho(t)), y_m - v^\nabla(t)\right\rangle \geq M^\nabla(t)\|x - v(\rho(t))\| - \frac{1}{k}\right\}. \end{aligned}$$

The ∇-measurability of the maps $t \mapsto v(\rho(t))$, $t \mapsto M(\rho(t))$, $t \mapsto v^\nabla(t)$, and $t \mapsto M^\nabla(t)$ imply that $\mathcal{B}_x$ is ∇-measurable, and so is $t \mapsto G(t, x)$. $\square$

We now define the multivalued map $\mathcal{H} : C(\mathbb{T}, \mathbb{R}^n) \to L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ by

$$\mathcal{H}(x) = \Big\{ w \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n) : w(t) \in F_u(t, x(\rho(t))) \ \nabla\text{-a.e. } t \in \mathbb{T}_0 \Big\}.$$

Proposition 3.7. *Assume* (H_1) *and* (H_2)*. Then,* $\mathcal{H}$ *has nonempty, convex values, and there exists* $h \in L^1_\nabla(\mathbb{T}_0, [0, \infty)$ *such that*

$$\|w(t)\| \le h(t) \ \ \nabla\text{-a.e. on } \mathbb{T}_0 \ \text{for all } w \in \mathcal{H}(x) \text{ and all } x \in C(\mathbb{T}, \mathbb{R}^n). \tag{15}$$

Proof. First of all, we want to show that $\mathcal{H}$ has nonempty values. Let $x \in C(\mathbb{T}, \mathbb{R}^n)$. There exists a sequence of simple functions $\{x_m\}_{m\in\mathbb{N}}$ such that

$$\|x_m(\rho(t)) - v((\rho(t)))\| > M(\rho(t))$$
$$\nabla\text{-a.e. } on \ \Big\{ t : \|x(\rho(t)) - v((\rho(t)))\| > M(\rho(t)) \Big\},$$

and such that $x_m \to \bar{x}$ in $C(\mathbb{T}, \mathbb{R}^n)$. Since the multivalued maps $t \mapsto F(t, y)$ and $t \mapsto G(t, y)$ are ∇-measurable for every $y \in \mathbb{R}^n$, the maps $t \mapsto F(t, x_m(\rho(t)))$ and $t \mapsto G(t, x_m(\rho(t)))$ are also ∇-measurable for every $m \in \mathbb{N}$. Proposition 2.24 implies that, for every $m \in \mathbb{N}$,

$$t \mapsto F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))$$

is ∇-measurable, and for every $k \in \mathbb{N}$,

$$t \mapsto \bigcup_{m \ge k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t))) \Big)$$

is ∇-measurable. Again, Propositions 2.23 and 2.24 imply that

$$t \mapsto \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{m \ge k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t))) \Big)}$$

is ∇-measurable. Definition 3.1(1) guarantees that this map has nonempty values ∇-almost everywhere on $\{t : M(\rho(t)) \neq 0\}$. Indeed, ∇-almost everywhere on

$$\{t : M(\rho(t)) \neq 0 \ and \ \|\bar{x}(\rho(t)) - v(\rho(t))\| < M(\rho(t))\},$$

for $m \ge k$ sufficiently large, $\|x_m(\rho(t)) - v((\rho(t)))\| < M(\rho(t))$ and

$$F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t))) = F(t, x_m(\rho(t))) \cap \mathbb{R}^n \neq \emptyset.$$

On the other hand, for ∇-almost every

$$t \in \{t : \|\bar{x}(\rho(t)) - v(\rho(t))\| = M(\rho(t)) > 0\},$$

if there exists $m \ge k$ such that $\|x_m(\rho(t)) - v((\rho(t)))\| \le M(\rho(t))$, then as before, $F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t))) \neq \emptyset$. Otherwise, there exists a $\delta > 0$ given by Definition 3.1(1) and $m \ge k$ sufficiently large such that

$$\|x_m(\rho(t)) - \bar{x}((\rho(t)))\| < \delta, \quad \|x_m(\rho(t)) - v((\rho(t)))\| > M(\rho(t)),$$

and there exists $z \in F(t, x_m(\rho(t)))$ such that

$$\langle x_m(\rho(t)) - v(\rho(t)), z - v^\nabla(t)\rangle \ge \|x_m(\rho(t)) - v((\rho(t)))\| M^\nabla(t),$$

i.e. $z \in F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))$.

Thus, the multivalued map $\Phi : \mathbb{T}_0 \to L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ defined by

$$\Phi(t) = \begin{cases} \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{m\geq k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))\Big)}, & \text{if } t \in \{t : M(\rho(t)) \neq 0\}, \\ v^\nabla(t), & \text{if } t \in \{t : M(\rho(t)) = 0\}, \end{cases}$$

is ∇-measurable and has nonempty and compact values. Finally, Theorem 2.25 guarantees the existence of a ∇-measurable selection w of Φ.

We must show that $w \in \mathcal{H}(x)$. Since $w(t) \in \Phi(t)$ ∇-a.e., we have,

$$w(t) \in \overline{\bigcup_{m\geq k} \Big(F(t, x_m(\rho(t))) \cap G(t, x_m(\rho(t)))\Big)} \;\; \nabla\text{-a.e. in}\{t : M(\rho(t)) \neq 0\},$$

for every $k \in \mathbb{N}$. So, for ∇-almost every $t \in \{t : M(\rho(t)) \neq 0\}$, there exists a subsequence

$$u_{m_l}(t) \in F(t, x_{m_l}(\rho(t))) \cap G(t, x_{m_l}(\rho(t)))$$

such that $u_{m_l}(t) \to w(t)$. If $\|x(\rho(t)) - v((\rho(t)))\| \leq M(\rho(t))$, since $y \mapsto F(t, y)$ and $y \mapsto G(t, y)$ have closed graph and since $x_{m_l}(\rho(t)) \to \bar{x}(\rho(t)) = x(\rho(t))$, we deduce that

$$w(t) \in F(t, \bar{x}(\rho(t))) \cap G(t, x(\rho(t))) = F_u(t, x(\rho(t))).$$

On the other hand, if $\|x(\rho(t)) - v((\rho(t)))\| > M(\rho(t))$, since $x_{m_l}(\rho(t)) \to \bar{x}(\rho(t))$, there exists a sequence $\{y_{m_l}\}$ such that $y_{m_l} \to x(\rho(t))$ and

$$x_{m_l}(\rho(t)) = \theta_{m_l}\bar{x}_{m_l}(\rho(t)) + (1 - \theta_{m_l})y_{m_l} = \overline{(y_{m_l})}_{\theta_{m_l}}(\rho(t)) \;\; for\ some\ \theta_{m_l} \in [0, 1[.$$

By (14),

$$u_{m_l}(t) \in F(t, x_{m_l}(\rho(t))) \cap G(t, x_{m_l}(\rho(t))) = F(t, x_{m_l}(\rho(t))) \cap G(t, y_{m_l}).$$

Again, since $y \in F(t, y)$ and $y \in G(t, y)$ have closed graph and since $x_{m_l}(\rho(t)) \to \bar{x}(\rho(t))$ and $y_{m_l} \to x(\rho(t))$, we can deduce that

$$w(t) \in F(t, \bar{x}(\rho(t))) \cap G(t, x(\rho(t))) = F_u(t, x(\rho(t))).$$

Moreover, Definition 3.1(2) implies that ∇-a.e. on $\{t : M(\rho(t)) = 0\}$,

$$w(t) = v^\nabla(t) \in F(t, \bar{x}(\rho(t))) \cap G(t, x(\rho(t))) = F_u(t, x(\rho(t))).$$

Hence, we can conclude that $w \in \mathcal{H}(x)$ since by hypothesis (H_1), $w \in L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$.

The convexity of $\mathcal{H}(x)$ follows from convexity of the values of F and G.

Finally, hypothesis (H_1) guarantees the existence of $h := h_q \in L^1_\nabla(\mathbb{T}_0, [0, \infty))$ with $q = max\{\|v(t)\| + M(t) : t \in \mathbb{T}\}$, such that for every $x \in C(\mathbb{T}, \mathbb{R}^n)$ and every $w \in \mathcal{H}(x)$,

$$\|w(t)\| \leq h(t) \quad \nabla\text{-a.e. } t \in \mathbb{T}_0. \quad \square$$

Now, we give the main result on the existence of solutions for the nonlinear problem (1), (2).

Theorem 3.8. *Assume (H_1) and (H_2). The problem (1), (2) has a solution $x \in W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n) \cap T(v, M)$.*

Proof. Transform the problem (1), (2) into a fixed point problem. From Lemma 3.2, a solution to (11) is fixed point of the multivalued operator $\mathcal{N}_I : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ defined by

$$\mathcal{N}_I(x)(t) = \Big\{u : u(t) = \hat{e}_{-1}(b,t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s,b)\Big(w(s) - \bar{x}(\rho(s))\Big)\nabla s\Big), \quad where\ w \in \mathcal{H}(x)\Big\}.$$

First, we shall show that $\mathcal{N}_I$ satisfies the assumptions of Theorem 2.26. The proof will be given in several steps.

Step 1. The previous proposition insures that $\mathcal{N}_I$ has nonempty, convex values, and guarantees the existence of $h \in L^1_\nabla(\mathbb{T}_0, [0,\infty))$ satisfying (15).

Set $K := \max\left\{\left|\hat{e}_{-1}(t,s)\right|, s,t \in \mathbb{T}\right\}$ and $q := \max\left\{\|v(t)\| + M(t) : t \in \mathbb{T}\right\}$. To show that $\mathcal{N}_I(C(\mathbb{T}, \mathbb{R}^n))$ is bounded, we just have to remark that for every $u \in \mathcal{N}_I(C(\mathbb{T}, \mathbb{R}^n))$,

$$\begin{aligned}\|u(t)\| &\le K\Big(\|x_0\| + \int_{(a,b]\cap\mathbb{T}} K\,\|w(s) - \bar{x}(\rho(s))\|\,\nabla s\Big)\\ &\le K\Big(\|x_0\| + \int_{(a,b]\cap\mathbb{T}} K\,(h(s)+q)\,\nabla s\Big)\ for\ all\ t \in \mathbb{T}.\end{aligned}$$

On the other hand, for every $t_2 > t_1 \in \mathbb{T}$,

$$\begin{aligned}&\|u(t_2) - u(t_1)\|\\ &\le \left|\hat{e}_{-1}(b,t_2) - \hat{e}_{-1}(b,t_1)\right|\left(\|x_0\| + \int_{(t_2,b]\cap\mathbb{T}} |\hat{e}_{-1}(s,b)|\,\|w(s) - \bar{x}(\rho(s))\|\nabla s\right)\\ &\quad + \int_{(t_1,t_2]\cap\mathbb{T}} \left|\hat{e}_{-1}(s,t_1)\right|\,\|w(s) - \bar{x}(\rho(s))\|\,\nabla s\\ &\le \left|\hat{e}_{-1}(b,t_2) - \hat{e}_{-1}(b,t_1)\right|\left(\|x_0\| + \int_{(a,b]\cap\mathbb{T}} K(q + h(s))\nabla s\right)\\ &\quad + K^2\int_{(t_1,t_2]\cap\mathbb{T}} (q + h(s))\nabla s.\end{aligned}$$

Thus, $\mathcal{N}_I(C(\mathbb{T}, \mathbb{R}^n))$ is equicontinuous since

$$t \to \hat{e}_{-1}(b,t) \quad and \quad t \to \int_{(t,b]\cap\mathbb{T}} (q + h(s))\nabla s$$

are continuous on $\mathbb{T}$. By an analogous version of the Arzelà – Ascoli Theorem adapted to our context, we conclude that $\mathcal{N}_I(C(\mathbb{T}, \mathbb{R}^n))$ is relatively compact in $C(\mathbb{T}, \mathbb{R}^n)$.

Step 2. $\mathcal{N}_I$ has closed graph.

Let $\{x_m\}$ and $\{u_m\}$ be convergent sequences in $C(\mathbb{T}, \mathbb{R}^n)$ such that $x_m \to x$, $u_m \to u$ and $u_m \in \mathcal{N}_I(x_m)$. Let $w_m \in \mathcal{H}(x_m)$ be such that

$$u_m(t) = \hat{e}_{-1}(b,t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s,b)\Big(w_m(s) - \bar{x}_m(\rho(s))\Big)\nabla s\Big).$$

Let h be the function given in (15). Considering the extensions $\widetilde{w}_m$ and $\widetilde{h}$ in $L^1([a,b])$, we have

$$\|\widetilde{w}_m(t)\| \le \widetilde{h}(t)\ for\ almost\ every\ t \in [a,b].$$

By Dunford–Pettis Theorem [13], there exist $g \in L^1([a,b], \mathbb{R}^n)$ and a subsequence still denoted $\{\widetilde{w}_m\}$ such that $\widetilde{w}_m \to g$ in $L^1([a,b], \mathbb{R}^n)$. Since a closed convex set is weakly closed, there exist $\widetilde{z}_m \in co\{\widetilde{w}_m, \widetilde{w}_{m+1}, \ldots\}$ such that $\widetilde{z}_m \to g$ in $L^1([a,b], \mathbb{R}^n)$.

Thus, there exists a subsequence again noted $\{\widetilde{z}_m\}$ such that, $\widetilde{z}_m(t) \to g(t)$ for almost every $t \in [a, b]$. Therefore, for almost every $t \in [a, b]$,

$$\widetilde{z}_m(t) \in co\Big\{\bigcup_{l\geq m} \widetilde{w}_l(t)\Big\} \subset co\Big\{\bigcup_{l\geq m} \widetilde{F}(t, \bar{x}_l(\rho(t))) \cap \widetilde{G}(t, x_l(\rho(t)))\Big\}.$$

where the multivalued maps $\widetilde{F}$ and $\widetilde{G}$ are respectively extensions of the multivalued maps F and G in the sense of (4). Taking the limit, we get

$$\begin{aligned} g(t) \in \bigcap_{m\in\mathbb{N}} \overline{co}\Big\{\bigcup_{l\geq m} \widetilde{F}(t, \bar{x}_l(\rho(t))) \cap \widetilde{G}(t, x_l(\rho(t)))\Big\} \\ \subset \widetilde{F}(t, \bar{x}(\rho(t))) \cap \widetilde{G}(t, x(\rho(t))) = \widetilde{F}_u(t, x(\rho(t))), \end{aligned}$$

since $x_m \to x$ in $C(\mathbb{T}, \mathbb{R}^n)$ and since $y \to \widetilde{F}(t, y)$ and $y \to \widetilde{G}(t, y)$ have closed graph and closed, convex values. By Theorem 2.16, there exists a function $w : \mathbb{T}_0 \to \mathbb{R}^n$ such that $g = \widetilde{w}$. So,

$$w(t) \in \widetilde{F}_u(t, x(\rho(t))) = F_u(t, x(\rho(t))) \ \ \nabla\text{-a.e. } t \in \mathbb{T}_0.$$

Thus, $w \in \mathcal{H}(x)$.

Finally, since $\widetilde{w}_m \to \widetilde{w}$ in $L^1([a, b], \mathbb{R}^n)$ and $x_m \to x$ in $C(\mathbb{T}, \mathbb{R}^n)$, again by Theorem 2.16, we deduce that for every $t \in \mathbb{T}$,

$$\int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s, b)\Big(w_m(s) - \bar{x}_m(\rho\,(s))\Big)\nabla s \longrightarrow \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s, b)\Big(w(s) - \bar{x}(\rho\,(s))\Big)\nabla s.$$

Moreover, since $u_m \to u$ in $C(\mathbb{T}, \mathbb{R}^n)$, we get that for every $t \in \mathbb{T}$,

$$u(t) = \hat{e}_{-1}(b, t)\Big(x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_{-1}(s, b)\Big(w(s) - \bar{x}(\rho\,(s))\Big)\nabla s\Big).$$

Thus, $u \in \mathcal{N}_I(x)$ and hence, $\mathcal{N}_I$ has closed graph. Since $\mathcal{N}_I$ is compact and has closed graph, N_I has compact values.

Step 3. $\mathcal{N}_I$ is upper semi-continuous.

Let $B \subset C(\mathbb{T}, \mathbb{R}^n)$ be a closed set and $\mathcal{A} = \{x \in C(\mathbb{T}, \mathbb{R}^n) : \mathcal{N}_I(x) \cap B \neq \emptyset\}$. Let $\{x_m\}$ be a sequence in $\mathcal{A}$ converging to x in $C(\mathbb{T}, \mathbb{R}^n)$. There exists $u_m \in \mathcal{N}_I(x_m) \cap B$. The compacity of $\mathcal{N}_I$ guarantees the existence of a subsequence still denoted $\{u_m\}$ converging to u in $C(\mathbb{T}, \mathbb{R}^n)$. Since B is closed and $\mathcal{N}_I$ has closed graph, we deduce that $u \in \mathcal{N}_I(x) \cap B$. Thus $x \in \mathcal{A}$.

$\mathcal{N}_I$ is compact and upper semi-continuous with nonempty, convex, and compact values. It has a fixed point by the Kakutani fixed point Theorem. Lemma 3.2 implies that, x, this fixed point of $\mathcal{N}_I$ is a solution of Problem (11), (2).

For the second part of the proof, we have to show that the solution x of (11), (2) satisfies $x \in T(v, M)$.

Consider the set $A = \{t \in \mathbb{T}_0 : \|x(\rho(t)) - v(\rho(t))\| > M(\rho(t))\}$. By Example 2.3, ∇-a.e. on the set $\{t \in A : t = \rho(t)\}$, we have

$$(\|x(t) - v(t)\| - M(t))^{\nabla} = \frac{\langle x(\rho(t)) - v(\rho(t)), x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t). \tag{16}$$

If $t \in A$ is left scattered, then $\nu(t) = t - \rho(t) > 0$ and

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^{\nabla} \\
&\quad = \frac{\|x(\rho(t)) - v(\rho(t))\| \|x(t) - v(t)\| - \|x(\rho(t)) - v(\rho(t))\|^2}{\nu(t)\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t) \\
&\quad \geq \frac{\langle x(\rho(t)) - v(\rho(t)),\ (x(t) - v(t)) - (x(\rho(t)) - v(\rho(t)))\rangle}{\nu(t)\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t) \\
&\quad = \frac{\langle x(\rho(t)) - v(\rho(t)),\ x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t).
\end{aligned} \tag{17}$$

Let us denote $y(t) = \left(x^{\nabla}(t) - x(\rho(t)) + \bar{x}(\rho(t))\right) \in F_u(t, x(\rho(t)))$ ∇-a.e. on $\mathbb{T}_0$. Since (v, M) is a solution-tube of (1) and from (12), (16), (17) and Remark 3.5, we deduce that ∇-a.e. on $\{t \in A : M(\rho(t)) > 0\}$,

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^{\nabla} \\
&\geq \frac{\langle x(\rho(t)) - v(\rho(t)), y(t) - \left(\bar{x}(\rho(t)) + x(\rho(t))\right) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t) \\
&= \frac{\langle \bar{x}(\rho(t)) - v(\rho(t)), y(t) - v^{\nabla}(t)\rangle}{M(\rho(t))} \\
&\quad - \left(M(\rho(t)) - \|x(\rho(t)) - v(\rho(t))\|\right) - M^{\nabla}(t) \\
&> \frac{M(\rho(t))M^{\nabla}(t)}{M(\rho(t))} - M^{\nabla}(t) = 0.
\end{aligned}$$

On the other hand, if $M(\rho(t)) = 0$, then $F_u(t, x(\rho(t))) = \{v^{\nabla}(t)\}$ and ∇-a.e. on $\{t \in A : M(\rho(t)) = 0\}$, we have

$$\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^{\nabla} \\
&\geq \frac{\langle x(\rho(t)) - v(\rho(t)), y(t) - \left(\bar{x}(\rho(t)) + x(\rho(t))\right) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t) \\
&= \frac{\langle x(\rho(t)) - v(\rho(t)), v^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} + \|x(\rho(t)) - v(\rho(t))\| - M^{\nabla}(t) \\
&> -M^{\nabla}(t) = 0.
\end{aligned}$$

This last equality follows from Definition 3.1(3) and Proposition 2.20.

If we set $r(t) = \|x(t) - v(t)\| - M(t)$, then $r^{\nabla}(t) > 0$ ∇-a.e. on $A = \{t \in \mathbb{T}_0 : r(\rho(t)) > 0\}$. Moreover, since (v, M) is a solution tube of (1) and x satisfies (2), then $r(b) < 0$. Lemma 3.4 implies that $A = \emptyset$. So, $x \in T(v, M)$ and the theorem is proved. □

Next, we give the main result on the existence of solutions for the problem (1), (3).

Theorem 3.9. *Assume (H_1) and (H_2). The problem (1), (3) has a solution $x \in W_{\nabla}^{1,1}(\mathbb{T}, \mathbb{R}^n) \cap T(v, M)$.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.8. In this case, we consider the multivalued operator $\mathcal{N}_p : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ defined by

$$\mathcal{N}_p(x)(t) = \Big\{ u \in C(\mathbb{T}, \mathbb{R}^n) : u(t) = \frac{1}{\hat{e}_{-1}(t,b)} \Big[\frac{1}{1-\hat{e}_{-1}(a,b)} \int_{(a,b]\cap\mathbb{T}} \Big(w(s) - \bar{x}(\rho(s)) \Big) \hat{e}_{-1}(s,b) \nabla s - \int_{(t,b]\cap\mathbb{T}} \Big(w(s) - \bar{x}(\rho(s)) \Big) \hat{e}_{-1}(s,b) \nabla s \Big], \ \textit{where } w \in \mathcal{H}(x) \Big\}. \qquad \square$$

References

[1] R.P. Agarwal, V. Otero-Espinar, K. Perera, D.R. Vivero, Basic properties of Sobolev's spaces on time scales, Adv. Differential Equations 2006 (2006) 38121.

[2] D. Anderson, J. Bullock, L. Erbe, A. Peterson, H. Tran, Nabla dynamic equations on time scales, Panamer. Math. J. 13 (1) (2003) 1–47.

[3] M.F. Atici, D.C. Biles, First order dynamic inclusions on time scales, J. Math. Anal. Appl. 292 (1) (2004) 222–237.

[4] M.F. Atici, G.Sh. Guseinov, On Green's functions and positive solutions for boundary value problems on time scales, J. Comput. Appl. Math. 141 (1–2) (2002) 75–99.

[5] J.P. Aubin, A. Cellina, Differential Inclusions, Springer-Verlag, Berlin-Heidelberg, New York, 1984.

[6] B. Bayour, A. Hammoudi, D.F.M. Torres, Existence of solution to a nonlinear first-order dynamic equation on time scales, J. Math. Anal. 1 (7) (2016) 2217–3412.

[7] A. Belarbi, M. Benchohra, A. Ouahab, Existence results for impulsive dynamic inclusions on time scales, Electron J. Qual. Theory Differ. Equ. 13 (2005) 1–22.

[8] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, Birkhäuser Boston, Boston, MA, 2001.

[9] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser Boston, Boston, MA, 2003.

[10] Y.G. Borisovich, B.D. Gel'man, A.D. Myshkis, V.V. Obukhovskii, Multivalued mappings, J. Sov. Math. 24 (1982) 719–791. Transl. from Itogi Nauki i Tekhn. Ser. Mat. Anal. 19 (1982) 127–230.

[11] A. Cabada, D.R. Vivero, Criterions for absolute continuity on time scales, J. Difference. Equ. Appl. 11 (2005) 1013–1028.

[12] A. Cabada, D.R. Vivero, Expression of the lebesgue Δ-integral on time scales as a usual lebesgue integral: Application to the calculus of Δ-antiderivatives, Math. Comput. Modelling 43 (2006) 194–207.

[13] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic Publishers, Boston, 2003.

[14] M. Frigon, H. Gilbert, Systems of first order inclusions on time scales, Topol. Methods Nonlinear Anal. 37 (2011) 147–163.

[15] M. Frigon, M. Lotfipour, Multiplicity results for systems of first order dynamic inclusions on time scales, Adv. Difference Equ. 2015 (2015) 320.

[16] H. Gilbert, Existence theorems for first-order equations on time scales with Δ-Carathédory functions, Adv. Difference Equ. 2010 (2010) 650827. 20 pp.

[17] E. Girejko, D.F.M. Torres, The existence of solutions for dynamic inclusions on time scales via duality, Appl. Math. Lett. 25 (2012) 1632–1637.

[18] Sh. Guseinov, Integration on time scales, J. Math. Anal. Appl. 285 (1) (2003) 107–127.

[19] S. Hilger, Ein Maβtkettenkalkül Ein Matkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten (Ph.D. thesis), Universtät Wüurzburg, Wüurzburg Germany, 1988.

[20] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.

[21] C.J. Himmelberg, Measurable relations, Fund. Math. 87 (1975) 53–72.

[22] Sh. Hu, N.S. Papageorgiou, Handbook of Multi-Valued Analysis, Volume I: Theory, Kluwer, Dordrecht, 1997.

[23] B. Mirandette, Résultats d'existence pour des systèmes d'équations différentielles du premier ordre avec tube-solution (Master thesis), Université de Montréal, Montréal, 1996.

[24] M. Rafaqat, R. Ahmed, T. Donchev, V. Lupulescu, Filippov-Pliss lemma for dynamical inclusions on a time scale, Adv. Difference Equ. 2017 (2017) 302.

[25] I.LD. Santos, G.N. Silva, Absolute continuity and existence of solutions to dynamic inclusions in time scales, Math. Ann. 356 (2013) 373–399.
[26] B.-R. Satco, C.-O. Turcu, Hybrid systems by methods of time scales analysis, Adv. Dyn. Sys. Appl. 8 (2) (2013) 317–326.
[27] B. Satco, C. Turcu, First order multivalued problems on time scales, Fixed Point Theory. 14 (2) (2013) 483–496.