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# Existence of viscosity solution for a singular Hamilton–Jacobi equation

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**Abstract.**   In this paper we study the existence of a singular Hamilton–Jacobi equation under the framework of viscosity solutions. The analysis is inspired by the arguments of [8] where a study of a model on dislocation dynamics was considered.

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

In [8], the author analyses a one-dimensional system of partial differential equations modelling the dynamics of dislocation densities in crystals. Dislocations are topological defects within crystal structure that move under the submission of stress fields. Geometrically, each dislocation is characterised by a physical quantity called the Burgers vector, which is responsible for its orientation and magnitude. Dislocations are classified as being positive or negative due to the orientation of its Burgers vector, and they can move in certain crystallographic directions (see [7] for a physical study of dislocations).

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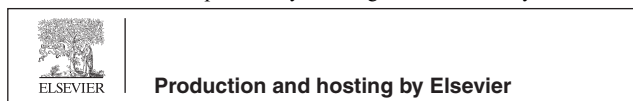
The system in [8] consists of a Hamilton–Jacobi (HJ) equation that is coupled to a diffusion equation, augmented with initial and Dirichlet boundary conditions. The main equation of concern is of the form

$$\kappa_t \kappa_x = \rho_t \rho_x, \tag{1.1}$$

where $\rho$ satisfies (throughout all the papers)

$$\begin{cases} \rho_t = \rho_{xx} \text{ in } \quad \mathbb{R} \times (0, \infty), \\ \rho(\cdot, 0) = \rho^0 \in C_0^\infty(\mathbb{R}). \end{cases}$$

The above system is derived from the dynamics of dislocation densities $\theta^\pm$ that reads (see [6]):

$$\begin{cases} \theta_t^+ = \left[ \left( \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) \theta^+ \right]_x \\ \theta_t^- = - \left[ \left( \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} \right) \theta^- \right]_x, \end{cases} \tag{1.2}$$

after taking an integrated form and making the following assumptions

$$\rho_x^\pm = \theta^\pm, \quad \rho = \rho^+ - \rho^- \quad \text{and} \quad \kappa = \rho^+ + \rho^-.$$

Various results regarding existence and uniqueness of viscosity solutions are established in [8]. Most of the results are obtained by bringing into service the connection between viscosity solutions and entropy solutions of conservation laws. To make this connection clear, we formally state the well-known result that if $u$ is a viscosity solution of the Hamilton–Jacobi equation

$$u_t + H(x, t, u_x) = 0,$$

then the spatial derivative $v = u_x$ is an entropy solution of the scalar conservation laws

$$v_t + [H(x, t, v)]_x = 0.$$

The usual proof of this relation depends strongly on the known results about existence and uniqueness of the solutions of the two problems together with the convergence of the viscosity method (see [3,11,12]). Another proof of this relation could be found in [2] via the definition of viscosity/entropy inequalities, while a direct proof could also be found in [10] using the front tracking method. The case of dealing with a discontinuous Hamiltonian is treated in [13].

The relevant functional class for the Hamilton–Jacobi Eq. (1.1) is singled out to be the Lipschitz continuous functions possessing a suitable lower bound on the spatial gradient. The main complexity was in the singularity of the gradient of $\kappa$ appearing in the Hamiltonian $H(x, t, \kappa_x) = \frac{f(x,t)}{\kappa_x}$ where $f(x, t) = \rho_t(x, t)\rho_x(x, t)$. Such singularity makes it difficult to define a solution of the equation in the case $\kappa_x = 0$. To overcome this difficulty, the author considered a suitable approximated problem:

$$\kappa_t^\epsilon \kappa_x^\epsilon = \rho_t \rho_x,$$

where it was shown that $\kappa_x^\epsilon > 0$ and therefore, by passing to the limit $\epsilon \to 0$, he obtained a solution of (1.1).

In this paper, we show how we can apply similar technics to show the existence of a viscosity solution of the following generalised power HJ equation:

$$\begin{cases} \kappa_t = \frac{F}{\kappa_x^n}, & x \in \mathbb{R}, \, t > 0, \\ \kappa(x,0) = \kappa^0(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $F = \rho_t \rho_x^n$, $n$ is a positive integer, and $\rho$ satisfies the heat equation $\rho_t = \rho_{xx}$. The goal here is not to delve into too much technicality but rather to show the new ideas to solve (1.3).

**Remark 1.1.** *Eq. (1.3) is only inspired from (1.1) with a gradient power, and it does not refer to any dislocation model. The special structure of the Hamiltonian $H = \frac{F}{\kappa_x^n}$, and particularly $F$ is essential as it allows the validity of the computation done in Lemma 2.2. The adaptation of the method used in the present paper to general functions $F$ remains open.*

A viscosity solution for (1.3) is defined as follows:

**Definition 1.2.** Fix $T > 0$, and let $\kappa : \mathbb{R} \times [0, T) \to \mathbb{R}$ be a continuous function. We call $\kappa$ a viscosity sub-solution of (1.3) if $\kappa(\cdot, 0) = \kappa^0$, and if for all $C^1$ functions $\phi$ such that $\kappa - \phi$ has a local maximum at $(x_0, t_0) \in \mathbb{R} \times (0, T)$, then

$$\phi_t(x_0, t_0)\phi_x^n(x_0, t_0) - F(x_0, t_0) \leqslant 0.$$

Analogously, we call $\kappa$ a viscosity super-solution if $\kappa(\cdot, 0) = \kappa^0$, and if for every $C^1$ function $\phi$ such that $\kappa - \phi$ has a local minimum at $(x_0, t_0)$, then

$$\phi_t(x_0, t_0)\phi_x^n(x_0, t_0) - F(x_0, t_0) \geqslant 0.$$

Finally, we say that $\kappa$ is a viscosity solution of (1.3) if it is both a viscosity sub-solution and a super-solution.

We refer the reader to [1] for a general overview on viscosity solutions.

Denoted by $Lip(\mathbb{R})$ the class of all Lipschitz continuous functions on $\mathbb{R}$, and let $Q_T = \mathbb{R} \times (0, T)$. The main results of this paper are summarised by:

**Theorem 1.3.** (Existence of a viscosity solution; case $n$ is odd)*Let $T > 0$, $n = 2p + 1$ with $p \in \mathbb{N}$, and let $\kappa^0 \in Lip(\mathbb{R})$. If*

$$\kappa_x^0 \geqslant |\rho_x^0| \quad \text{a.e. in} \quad \mathbb{R}, \tag{1.4}$$

*then there exists a viscosity solution $\kappa \in Lip(\overline{Q}_T)$ of (1.3).*

**Theorem 1.4.** (Existence of a viscosity solution; case $n$ is even)*Let $T > 0$, $n = 2p$ with $p \in \mathbb{N}^*$, and let $\kappa^0 \in Lip(\mathbb{R})$. Assume $\rho^0$ is monotone. If*

$$(\kappa_x^0 - \rho_x^0)\rho_x^0 \geqslant 0 \quad \text{a.e. in} \quad \mathbb{R}, \tag{1.5}$$

*then there exists a viscosity solution $\kappa \in Lip(\overline{Q}_T)$ of (1.3).*

It is worth mentioning that since $T > 0$ is assumed arbitrary then the solution defined for $\overline{Q}_T$ is indeed a solution for all $t \geqslant 0$.

The remaining of the paper is organised as follows. In Section 2 we present the proof of Theorem 1.3 while in Section 3 we present the proof of Theorem 1.4.

## 2. Existence of a viscosity solution if $N$ is odd

This section is devoted to the proof of Theorem 1.3. We write Eq. (1.3) as

$$\kappa_t = Ff(\kappa_x^n),$$

where $f(x) = \frac{1}{x}$. Remark that the condition (1.4) does not prevent $\kappa_x^0$ from being zero and therefore we have a difficulty in defining a solution of (1.3). To overcome this, we let $0 < \epsilon \leqslant 1$ and we take a particular extension $f_\epsilon$ of $f$ defined as follows (see Fig. 1 below):

$$f_\epsilon(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \geqslant \epsilon^n, \\ \dfrac{2\epsilon^n - x}{\epsilon^{2n} + (x - \epsilon^n)^2} & \text{otherwise.} \end{cases} \tag{2.1}$$

The function $f_\epsilon$ is constructed so that $f_\epsilon \in C_b^1(\mathbb{R})$ ($C^1$ and bounded) and therefore, by [4, Section 4] and [9, Theorem 3], there exists a unique viscosity solution $\kappa^\epsilon \in Lip(\overline{Q}_T)$ of

$$\begin{cases} \kappa_t = Ff_\epsilon(\kappa_x^n), & (x,t) \in Q_T, \\ \kappa(x,0) = \kappa^{0,\epsilon}(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$
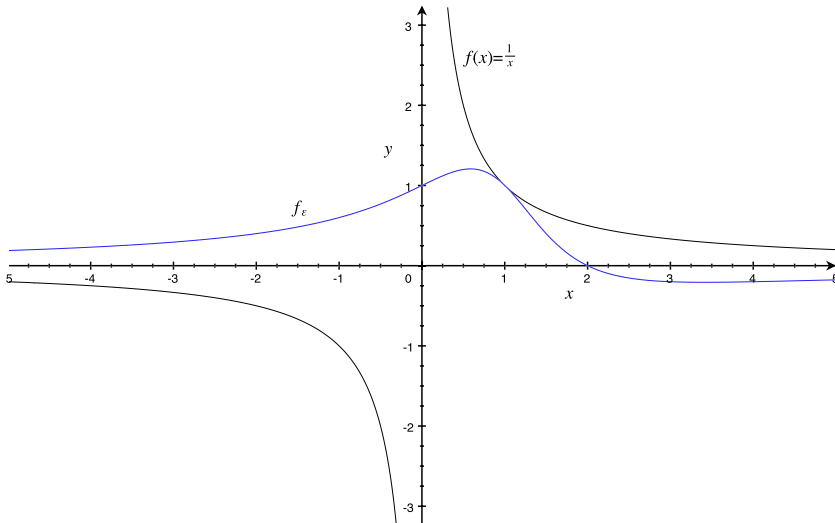


**Fig. 1**    Approximation of the function $\frac{1}{x}$ for $\epsilon = 1$.

where

$$\kappa^{0,\epsilon}(x) = \kappa^0(x) + \epsilon x.$$

Let

$$G_\epsilon(x) = (x^{n+1} + \epsilon^{n+1})^{\frac{1}{n+1}}.$$

The approximation of the initial data shows that

$$\kappa_x^{0,\epsilon} \geqslant |\rho_x^0| + \epsilon \geqslant G_\epsilon(\rho_x^0). \tag{2.3}$$

We have the following lemma:

**Lemma 2.1.** *If $\kappa^\epsilon$ is a Lipschitz viscosity solution of (2.2) then $v = \kappa_x^\epsilon \in L^\infty(\overline{Q}_T)$ is an entropy solution of*

$$\begin{cases} v_t = (Ff_\epsilon(v^n))_x, & (x,t) \in Q_T, \\ v(x,0) = v^0(x), & x \in \mathbb{R}, \end{cases} \tag{2.4}$$

*with $v^0 = \kappa_x^{0,\epsilon}$.*

**Proof.** See [8, Theorem 2.12].   □

The next lemma is of crucial importance. It shows that $G_\epsilon(\rho_x)$ is a strong sub-solution of (2.4) and hence, by Lemma 2.1 and the initial condition (2.3), we obtain

$$\kappa_x^\epsilon \geqslant G_\epsilon(\rho_x) \quad \text{a.e. in} \quad \bar{Q}_T. \tag{2.5}$$

This inequality follows from the comparison principle of entropy solutions (see [5, Theorem 5]).

**Lemma 2.2.** *The function $G_\epsilon(\rho_x)$ is a strong sub-solution of (2.4).*

**Proof.** First remark that for every $x \in \mathbb{R}$ we have $G_\epsilon^n(x) \geqslant \epsilon^n$ and therefore (by (2.1))

$$f_\epsilon(G_\epsilon^n(\rho_x)) = \frac{1}{G_\epsilon^n(\rho_x)}.$$

Remark also that

$$G_\epsilon'(x) = \frac{x^n}{G_\epsilon^n(x)}, \quad \forall x \in \mathbb{R}.$$

Following the above remarks, we plug $G_\epsilon(\rho_x)$ into (2.4) and we compute

$$\begin{aligned}
(G_\epsilon(\rho_x))_t - \left(\frac{\rho_t \rho_x^n}{G_\epsilon^n(\rho_x)}\right)_x &= G_\epsilon'(\rho_x)\rho_{xt} - \left[\frac{\rho_{xt}\rho_x^n + n\rho_x^{n-1}\rho_t\rho_{xx}}{G_\epsilon^n(\rho_x)} - \frac{nG_\epsilon^{n-1}(\rho_x)G_\epsilon'(\rho_x)\rho_t\rho_x^n\rho_{xx}}{G_\epsilon^{2n}(\rho_x)}\right], \\
&= \frac{G_\epsilon^n(\rho_x)\rho_{xt}\left(G_\epsilon^n(\rho_x)G_\epsilon'(\rho_x) - \rho_x^n\right) - nG_\epsilon^{n-1}(\rho_x)\rho_x^{n-1}\rho_{xx}^2\left(G_\epsilon(\rho_x) - \rho_x G_\epsilon'(\rho_x)\right)}{G_\epsilon^{2n}(\rho_x)}, \\
&= -\frac{n\rho_x^{n-1}\rho_{xx}^2\epsilon^{n+1}}{G_\epsilon^{2n+1}(\rho_x)} \leqslant 0,
\end{aligned}$$

where for the second equality we have used the fact that $\rho_t = \rho_{xx}$, and for the last inequality we have used the fact that $G_\epsilon$ is positive and $n$ is odd hence $\rho_x^{n-1}$ is also positive.  $\square$

Now let

$$S(t) = [(n+1)c_1 t + c_2]^{\frac{1}{n+1}}, \quad 0 \leqslant t \leqslant T,$$

with

$$c_1 = \|\rho_{xt}\|_{L^\infty(Q_T)}\|\rho_x\|_{L^\infty(Q_T)}^n + n\|\rho_x\|_{L^\infty(Q_T)}^{n-1}\|\rho_{xx}\|_{L^\infty(Q_T)}^2 \quad \text{and}$$
$$c_2 = (\|\kappa_x^0\|_{L^\infty(\mathbb{R})} + 1)^{n+1}.$$

We have

**Lemma 2.3.** *The function $S$ is a strong super-solution of* (2.4)*.*

**Proof.** We notice that

$$S^n \geqslant c_2^{\frac{n}{n+1}} \geqslant (\|\kappa_x^0\|_{L^\infty(\mathbb{R})} + 1)^n \geqslant \epsilon^n,$$

hence

$$f_\epsilon(S^n) = \frac{1}{S^n}.$$

Also we notice that

$$S\prime = \frac{c_1}{S^n}.$$

All these arguments infer that, upon plugging $S$ into (2.4):

$$(S)_t - \left(\frac{\rho_t \rho_x^n}{S^n}\right)_x = \frac{c_1 - (\rho_{xt}\rho_x^n + n\rho_x^{n-1}\rho_{xx}^2)}{S^n} \geqslant 0,$$

and the result follows.  $\square$

**Remark 2.4.** *By* (2.5)*, we know that $\kappa_x^\epsilon \geqslant \epsilon$ and then the solution $\kappa^\epsilon$ of* (2.2) *is indeed a solution of the equation*

$$\kappa_t^\epsilon(\kappa_x^\epsilon)^n - \rho_t\rho_x^n = 0. \tag{2.6}$$

Therefore, in order to prove Theorem 1.3, it suffices to pass to the limit $\epsilon \to \infty$ in (2.6) using the stability theorem for viscosity solution (see [1]).

The following lemma reflects the passage to the limit mentioned in the previous remark.

**Lemma 2.5.** *There exists a function $\kappa \in Lip(\overline{Q}_T)$ satisfying: for every compact $K \subseteq \overline{Q}_T$, there exists a subsequence of $(\kappa^\epsilon)_\epsilon$ converging uniformly to $\kappa$ over $K$.*

**Proof.** It suffices to obtain $\epsilon$-uniform upper bounds for $|\kappa_x^\epsilon|$ and $|\kappa_t^\epsilon|$ then we apply Ascoli's Theorem on compact domains to conclude.

*Uniform bound on $|\kappa_x^\epsilon|$.* Since

$$\kappa_x^{0,\epsilon} = \kappa_x^0 + \epsilon \leqslant \|\kappa_x^0\|_{L^\infty(\mathbb{R})} + 1 = S(0),$$

then by Lemma 2.1 and Lemma 2.3, we obtain (using comparison principle for entropy solutions):

$$0 \leqslant \kappa_x^\epsilon \leqslant S \quad \text{a.e. in} \quad \overline{Q}_T.$$

But $S$ is increasing over the interval $[0, T]$ therefore

$$|\kappa_x^\epsilon| \leqslant S(T) \quad \text{a.e. in} \quad \overline{Q}_T.$$

*Uniform bound on $|\kappa_t^\epsilon|$.* Using (2.5), (2.6) and the fact that $\rho_t = \rho_{xx}$, we get

$$|\kappa_t^\epsilon| \leqslant \|\rho_{xx}^0\|_{L^\infty(\mathbb{R})} \quad \text{a.e. in} \quad \overline{Q}_T.$$

The result follows by applying Ascoli's Theorem. $\quad\square$

We now arrive to the Proof of Theorem 1.3.

*Proof of Theorem 1.3* Since $(\kappa^\epsilon)_\epsilon$ converge (up to a subsequence) uniformly on compacts to $\kappa$, and since the Hamiltonian of (2.6) is independent of $\epsilon$; indeed, for $X = (x, t)$, the Hamiltonian can be written as:

$$H^\epsilon(X, \nabla u) = u_t u_x^n - \rho_t(X)\rho_x^n(X), \quad \nabla u = (u_t, u_x),$$

we use the stability theorem for viscosity solution (see [1, Theorem 2.3]) to conclude that $\kappa$ is a viscosity solution of the limit equation

$$\kappa_t \kappa_x^n - F = 0.$$

Moreover, it is easily checked that $\kappa^{0,\epsilon} \to \kappa^0$ uniformly in $\mathbb{R}$ and the proof follows. $\quad\square$

## 3. Existence of a viscosity solution if $N$ is even

In this section we present the proof of Theorem 1.4 only in the case $\rho^0$ is decreasing. The case where $\rho^0$ is increasing follows from Theorem 1.3 since, by condition (1.5), we have

$$\kappa_x^0 \geqslant \rho_x^0 \geqslant 0,$$

therefore condition (1.4) is satisfied and we can apply the result of Theorem 1.3.

The difficult case arises when $\rho^0$ is decreasing and we need to slightly deal with the problem differently. Comparison principle for heat equation shows that

$$\rho_x \leqslant 0 \quad \text{in} \quad \overline{Q}_T. \tag{3.1}$$

Again, let $\kappa^\epsilon \in Lip(\overline{Q}_T)$ be the unique viscosity solution of (2.2) with a different approximation on the initial data given by:

$$\kappa^{0,\epsilon}(x) = \kappa^0(x) - \epsilon x, \quad 0 < \epsilon \leqslant 1.$$

Using Lemma 2.1, we obtain that $\kappa_x^\epsilon \in L^\infty(\overline{Q}_T)$ is an entropy solution of (2.4) with $v^0 = \kappa_x^{0,\epsilon}$. Let

$$G_\epsilon(x) = (x^{n+1} - \epsilon^{n+1})^{\frac{1}{n+1}}.$$

Since $\rho_x^0 \leqslant 0$ then, by (1.5), we get

$$\kappa_x^0 \leqslant \rho_x^0.$$

and therefore, the approximation of the initial data shows that

$$\kappa_x^{0,\epsilon} = \kappa_x^0 - \epsilon \leqslant \rho_x^0 - \epsilon \leqslant G_\epsilon(\rho_x^0). \tag{3.2}$$

Analogous to Lemma 2.2, we now show that $G_\epsilon(\rho_x)$ is a super-solution of (2.4).

**Lemma 3.1.** *The function $G_\epsilon(\rho_x)$ is a strong super-solution of (2.4).*

**Proof.** Remark that for every $x \leqslant 0$,

$$G_\epsilon(x) \leqslant -\epsilon.$$

As $\rho_x \leqslant 0$ and $n$ is even, we then obtain

$$G_\epsilon^n(\rho_x) \geqslant \epsilon^n,$$

therefore

$$f_\epsilon(G_\epsilon^n(\rho_x)) = \frac{1}{G_\epsilon^n(\rho_x)}.$$

Remark also that

$$G_\epsilon'(x) = \frac{x^n}{G_\epsilon^n(x)}, \quad \forall x \in \mathbb{R}.$$

Following the above remarks, we plug $G_\epsilon(\rho_x)$ into (2.4) and we compute

$$(G_\epsilon(\rho_x))_t - \left(\frac{\rho_t \rho_x^n}{G_\epsilon^n(\rho_x)}\right)_x = G_\epsilon'(\rho_x)\rho_{xt} - \left[\frac{\rho_{xt}\rho_x^n + n\rho_x^{n-1}\rho_t\rho_{xx}}{G_\epsilon^n(\rho_x)} - \frac{nG_\epsilon^{n-1}(\rho_x)G_\epsilon'(\rho_x)\rho_t\rho_x^n\rho_{xx}}{G_\epsilon^{2n}(\rho_x)}\right],$$

$$= \frac{G_\epsilon^n(\rho_x)\rho_{xt}\left(G_\epsilon^n(\rho_x)G_\epsilon'(\rho_x) - \rho_x^n\right) - nG_\epsilon^{n-1}(\rho_x)\rho_x^{n-1}\rho_{xx}^2\left(G_\epsilon(\rho_x) - \rho_x G_\epsilon'(\rho_x)\right)}{G_\epsilon^{2n}(\rho_x)},$$

$$= \frac{n\rho_x^{n-1}\rho_{xx}^2\epsilon^{n+1}}{G_\epsilon^{2n+1}(\rho_x)} \geqslant 0,$$

where for the second equality we have used the fact that $\rho_t = \rho_{xx}$, and for the last inequality we have used the fact that both $\rho_x^{n-1}$ and $G_\epsilon^{2n+1}(\rho_x)$ are negative terms since $n - 1$ and $2n + 1$ are both odd. This leads to the final positive sign. The proof then follows.    $\square$

This lemma together with (3.2) enable us (using comparison principle for entropy solutions) to conclude that:

$$\kappa_x^\epsilon \leqslant G_\epsilon(\rho_x) \leqslant \rho_x \leqslant 0 \quad \text{a.e.} \quad \overline{Q}_T. \tag{3.3}$$

Now let

$$S(t) = -[(n+1)c_1 t + c_2]^{\frac{1}{n+1}}, \quad 0 \leqslant t \leqslant T,$$

with

$$c_1 = \|\rho_{xt}\|_{L^\infty(Q_T)} \|\rho_x\|^n_{L^\infty(Q_T)} + n\|\rho_x\|^{n-1}_{L^\infty(Q_T)} \|\rho_{xx}\|^2_{L^\infty(Q_T)} \quad \text{and}$$
$$c_2 = (\|\kappa^0_x\|_{L^\infty(\mathbb{R})} + 1)^{n+1}.$$

We have

**Lemma 3.2.** *The function S is a strong sub-solution of* (2.4).

**Proof.** Since $n+1$ is odd, we notice that

$$S \leqslant -c_2^{\frac{1}{n+1}} \leqslant 0,$$

then

$$S^n \geqslant c_2^{\frac{n}{n+1}} \geqslant (\|\kappa^0_x\|_{L^\infty(\mathbb{R})} + 1)^n \geqslant \epsilon^n,$$

hence

$$f_\epsilon(S^n) = \frac{1}{S^n}.$$

Also we notice that

$$S' = -\frac{c_1}{S^n}.$$

All these arguments infer that, upon plugging $S$ into (2.4):

$$(S)_t - \left(\frac{\rho_t \rho_x^n}{S^n}\right)_x = \frac{-c_1 - (\rho_{xt}\rho_x^n + n\rho_x^{n-1}\rho_{xx}^2)}{S^n} \leqslant 0,$$

and the result follows.  $\square$

**Remark 3.3.** *By* (3.3)*, we know that $\kappa^\epsilon_x \leqslant -\epsilon$ then, since n is even, $(\kappa^\epsilon_x)^n \geqslant \epsilon^n$ hence the solution $\kappa^\epsilon$ of* (2.2) *is indeed a solution of* (2.6)*. Therefore, in order to prove Theorem 1.4, it suffices to pass to the limit $\epsilon \to \infty$ in* (2.6) *using the stability theorem for viscosity solution (see* [1]*).*

Similar to Lemma 2.5, we have:

**Lemma 3.4.** *There exists a function $\kappa \in Lip(\overline{Q}_T)$ satisfying: for every compact $K \subseteq \overline{Q}_T$, there exists a subsequence of $(\kappa^\epsilon)_\epsilon$ converging uniformly to $\kappa$ over K.*

**Proof.** It suffices to obtain $\epsilon$-uniform upper bounds for $|\kappa^\epsilon_x|$ and $|\kappa^\epsilon_t|$ then we apply Ascoli's Theorem on compact domains to conclude.

*Uniform bound on $|\kappa^\epsilon_x|$.* Since

$$\kappa_x^{0,\epsilon} = \kappa_x^0 - \epsilon \geqslant -\|\kappa_x^0\|_{L^\infty(\mathbb{R})} - 1 = S(0),$$

then by Lemma 2.1 and Lemma 3.2, we obtain (using comparison principle for entropy solutions):

$$S \leqslant \kappa_x^\epsilon \leqslant 0 \quad \text{a.e. in} \quad \overline{Q}_T.$$

But $S$ is decreasing over the interval $[0, T]$ therefore

$$|\kappa_x^\epsilon| \leqslant |S(T)| \quad \text{a.e. in} \quad \overline{Q}_T.$$

*Uniform bound on* $|\kappa_t^\epsilon|$. Using (3.3), (2.6) and the fact that $\rho_t = \rho_{xx}$, we get

$$|\kappa_t^\epsilon| \leqslant \|\rho_{xx}^0\|_{L^\infty(\mathbb{R})} \quad \text{a.e. in} \quad \overline{Q}_T.$$

The result follows by applying Ascoli's Theorem. $\square$

We now arrive to the proof of Theorem 1.4.

*Proof of Theorem 1.4.* Since $(\kappa^\epsilon)_\epsilon$ converge (up to a subsequence) uniformly on compacts to $\kappa$, and since the Hamiltonian of (2.6) is independent of $\epsilon$; indeed, for $X = (x, t)$, the Hamiltonian can be written as:

$$H^\epsilon(X, \nabla u) = u_t u_x^n - \rho_t(X)\rho_x^n(X), \quad \nabla u = (u_t, u_x),$$

we use the stability theorem for viscosity solution (see [1, Theorem 2.3]) to conclude that $\kappa$ is a viscosity solution of the limit equation

$$\kappa_t \kappa_x^n - F = 0.$$

Moreover, it is easily checked that $\kappa^{0,\epsilon} \to \kappa^0$ uniformly in $\mathbb{R}$ and the proof follows. $\square$

Finally, it is worth mentioning that the stability theorem for viscosity solutions only provides existence of the limit equation. However, it is not evident to us how to show uniqueness and this raises the following:

*Open question:* The question of uniqueness of the singular HJ Eq. (1.3) remains open.

## Acknowledgment

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