Arab Journal of Mathematical Sciences (2012) **18**, 165–175

King Saud University
Arab Journal of Mathematical Sciences
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

Existence of solutions for multi point boundary value problems for fractional differential equations

Nemat Nyamoradi *

Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran

Received 12 September 2011; revised 25 December 2011; accepted 5 January 2012
Available online 18 January 2012

KEYWORDS
Cone;
Multi point boundary value problem;
Fixed point theorem;
Riemann–Liouville fractional derivative

Abstract In this paper, by employing the Leggett–Williams fixed point theorem, we study the existence of three solutions in the multi point fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t), u'(t)), \quad t \in [0, 1], \\ u(0) = u'(0) = 0, \quad u(1) - \sum_{i=1}^{m} a_i u(\xi_i) = \lambda \end{cases}$$

where $2 < \alpha \leqslant 3$ and $m \geqslant 1$ are integers, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$ are constants, $\lambda \in (0,\infty)$ is a parameter, $a_i > 0$ for $1 \leqslant i \leqslant m$ and $\sum_{i=1}^{m} a_i \xi_i^{\alpha-1} < 1$, $f \in C([0, 1] \times [0, \infty) \times [0, \infty); [0, \infty))$.

* Tel.: +98 08314274569.
E-mail address: nyamoradi@razi.ac.ir

Peer review under responsibility of King Saud University.
doi:10.1016/j.ajmsc.2012.01.002

Production and hosting by Elsevier

1. Introduction

In this paper, we will study the existence of three solutions for multi point boundary value problems for fractional differential equations of the form

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t), u'(t)), \quad t \in [0, 1], \\ u(0) = u'(0) = 0, \quad u(1) - \sum_{i=1}^{m} a_i u(\xi_i) = \lambda \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative of order $2 < \alpha \leqslant 3$ and $m \geqslant 1$ is integer, $\lambda \in (0, \infty)$ is a parameter, and a_i, ξ_i, f satisfying

(H1) $a_i > 0$ for $1 \leqslant i \leqslant m$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$ and $\sum_{i=1}^{m} a_i \xi_i^{\alpha-1} < 1$;
(H2) f: $[0,1] \times [0,\infty) \times [0,\infty) \leftarrow [0,\infty)$ is continuous.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [5,7,8] and the references therein.

The solution of differential equations of fractional order is quite involved. Some analytical methods are presented, such as the popular Laplace transform method [18,19], the Fourier transform method [13], the iteration method [20] and the Green function method [21,12]. Numerical schemes for solving fractional differential equations are introduced, for example, in [3,4,15]. Recently, a great deal of effort has been expended over the last years in attempting to find robust and stable numerical as well as analytical methods for solving fractional differential equations of physical interest. The Adomian decomposition method [16], the homotopy perturbation method [17], the homotopy analysis method [2], the differential transform method [14] and the variational method [6] are relatively new approaches to provide an analytical approximate solution to linear and nonlinear fractional differential equations. The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature [20,18,1,10] and the references therein.

The basic space used in this paper is a real Banach space $C^1([0,1])$ with the norm $\|\cdot\|$ defined by $\|u\| = \max_{0 \leqslant t \leqslant 1} | u(t)|$. For convenience, we present here the Leggett–Williams fixed point theorem [11].

Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp. convex) functional on K provided that α:K arrow $[0. + \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y),$$
$$(resp.\alpha(tx + (1-t)y) \leqslant t\alpha(x) + (1-t)\alpha(y)),$$

for all $x, y \in K$ and $t \in [0,1]$.

Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \{x \in K | \|x\| < r\},$$

and

$$P(\alpha, a, b) = \{x \in K | a \leqslant \alpha(x), \ \|x\| \leqslant b\}.$$

Theorem 1 (Leggett–Williams fixed point theorem). *Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \leqslant \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \leqslant c$ such that*

(A1) *$\{x \in P(\alpha, b, d) | \ \alpha(x) > b\} \neq \emptyset$, and $\alpha(Ax) > b$ for $x \in P(\alpha, b, d)$,*
(A2) *$\|Ax\| < a$ for $\|x\| \leqslant a$, and*
(A3) *$\alpha(Ax) > b$ for $x \in P(\alpha, b, c)$ with $\|Ax\| > d$.*

Then A has at least three fixed points x_1, x_2, and x_3 and such that $\|x_1\| < a$, $b < \alpha(x_2)$ and $\|x_3\| > a$, with $\alpha(x_3) < b$.

In this paper, we will consider the existence of positive solutions to the problem (1). We will firstly give a new form of the solution, and then determine the properties of the Green's function for associated fractional boundary value problems; finally, by employing the Leggett–Williams fixed point theorem, some sufficient conditions guaranteeing the existence of three positive solutions.

The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main result and proofs will be given in Section 3. Finally, in Section 4, an example is given to demonstrate the application of our main result.

2. Preliminaries

This section devotes to present some notation and preliminary lemmas that will be used in the proofs of the main results.

Definition 1. Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:

(1) $x \in P, \mu \geqslant 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 3. The Riemann–Liouville fractional derivative of order $\alpha > 0,\ n-1 < \alpha < n,\ n \in \mathbb{N}$ is defined as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 1. ([9]) *The equality* $D_{0^+}^{\gamma} I_{0^+}^{\gamma} f(t) = f(t),\ \gamma > 0$ *holds for* $f \in L(0,1)$.

Lemma 2. ([9]) *Let* $\alpha > 0$. *Then the differential equation*

$$D_{0^+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},\ c_i \in \mathbb{R},\ i = 1, \ldots, n,$ *there* $n-1 < \alpha \leqslant n$.

Lemma 3. ([9]) *Let* $\alpha > 0$. *Then the following equality holds for* $u \in L(0,1),\ D_{0^+}^{\alpha} u \in L(0,1)$;

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},\ c_i \in \mathbb{R},\ i = 1, \ldots, n,\ \text{there } n-1 < \alpha \leqslant n.$$

Lemma 4. *Suppose that* $\Delta := \sum_{i=1}^{m} a_i \xi_i^{\alpha-1} \neq 0$, *then for* $h(t) \in C([0,1])$, *the following boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) = h(t), & 2 < \alpha \leqslant 3, \quad t \in [0,1], \\ u(0) = u'(0) = 0, & u(1) - \sum\limits_{i=1}^{m} a_i u(\xi_i) = \lambda \end{cases} \tag{2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\sum_{i=1}^{m} a_i t^{\alpha-1}}{1-\Delta} \int_0^1 G(\xi_i, s)h(s)ds + \frac{\lambda t^{\alpha-1}}{1-\Delta}, \tag{3}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{4}$$

Proof. According to Lemma 3, we can obtain that

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

By the boundary conditions of (2), there are $c_2 = c_3 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)(1-\Delta)}\left[\int_0^1 (1-s)^{\alpha-1}h(s)ds - \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}h(s)ds\right] + \frac{\lambda}{1-\Delta}.$$

Therefore, problem (2) has a unique solution

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)(1-\Delta)}\left[\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds\right.\\
&\quad\left. - \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}t^{\alpha-1}h(s)ds\right] + \frac{\lambda t^{\alpha-1}}{1-\Delta}\\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds\\
&\quad + \frac{\Delta}{\Gamma(\alpha)(1-\Delta)}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds\\
&\quad - \frac{\sum\limits_{i=1}^m a_i}{\Gamma(\alpha)(1-\Delta)}\int_0^{\xi_i} (\xi_i - s)^{\alpha-1}t^{\alpha-1}h(s)ds + \frac{\lambda t^{\alpha-1}}{1-\Delta}\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left[(1-s)^{\alpha-1}t^{\alpha-1} - (t-s)^{\alpha-1}\right]h(s)ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds + \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)}\int_0^1 (1-s)^{\alpha-1}\xi_i^{\alpha-1}h(s)ds\\
&\quad - \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)}\int_0^{\xi_i} (\xi_i - s)^{\alpha-1}h(s)ds + \frac{\lambda t^{\alpha-1}}{1-\Delta}\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left[(1-s)^{\alpha-1}t^{\alpha-1} - (t-s)^{\alpha-1}\right]h(s)ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds + \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)}\\
&\quad \times \left(\int_0^{\xi_i}\left[(1-s)^{\alpha-1}\xi_i^{\alpha-1} - (\xi_i - s)^{\alpha-1}\right]h(s)ds\right.\\
&\quad \left. + \int_{\xi_i}^1 (1-s)^{\alpha-1}\xi_i^{\alpha-1}h(s)ds\right) + \frac{\lambda t^{\alpha-1}}{1-\Delta}\\
&= \int_0^1 G(t,s)h(s)ds + \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{1-\Delta}\int_0^1 G(\xi_i, s)h(s)ds + \frac{\lambda t^{\alpha-1}}{1-\Delta}.
\end{aligned}$$

Therefore, the proof is completed. □

Lemma 5. *The function $G(t,s)$ defined by (4) satisfies the following conditions:*

(i) *$G(t,s) \geqslant 0$, $G(t,s) \leqslant G(s,s)$ for all $s, t \in [0,1]$;*

(ii) *there exists a positive function* $g \in C(0,1)$ *such that* $\min_{\gamma \leqslant t \leqslant \delta} G(t,s) \geqslant g(s)$ $G(s,s)$, $s \in (0,1)$, *where* $0 < \gamma < \delta < 1$ *and*

$$g(s) = \begin{cases} \frac{\delta^{\alpha-1}(1-s)^{\alpha-1}-(\delta-s)^{\alpha-1}}{s^{\alpha-1}(1-s)^{\alpha-1}}, & s \in (0, m_1], \\ \left(\frac{\gamma}{s}\right)^{\alpha-1}, & s \in [m_1, 1), \end{cases} \tag{5}$$

where $\gamma < m_1 < \delta$;

(iii) $\max_{0 \leqslant t \leqslant 1} \int_0^1 G(t, s) = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}$.

Proof.

(i) By definition of G, for all $(t,s) \in [0,1] \times [0,1]$ if $s \leqslant t$, it can be written

$$\begin{aligned} G(t, s) = \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] &\geqslant \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1} \\ - (t-ts)^{\alpha-1}] = \frac{t^{\alpha-1}}{\Gamma(\alpha)}[(1-s)^{\alpha-1} &- (1-s)^{\alpha-1}] = 0, \end{aligned}$$

and if $t \leqslant s$, it is obvious that $G(t,s) \geqslant 0$. Therefore, one can canclude

$$G(t, s) \geqslant 0 \quad \text{for all} \quad (t, s) \in [0, 1] \times [0, 1].$$

Let $L(t,s) := t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}$, $0 \leqslant s \leqslant t \leqslant 1$. Then

$$\begin{aligned} \frac{dL(t, s)}{dt} &= (\alpha-1)[t^{\alpha-2}(1-s)^{\alpha-1} - (t-s)^{\alpha-2}] \\ &= (\alpha-1)t^{\alpha-2}[(1-s)^{\alpha-1} - (1-\frac{s}{t})^{\alpha-2}] \\ &\leqslant (\alpha-1)t^{\alpha-2}[(1-s)^{\alpha-1} - (1-s)^{\alpha-2}] \leqslant 0, \end{aligned}$$

which implies that $L(\cdot,s)$ is non-increasing for all $s \in (0,1]$, hence, we obtain that

$$L(t, s) \leqslant L(s, s) \quad \text{for all} \quad 0 \leqslant s \leqslant t \leqslant 1. \tag{6}$$

Thus, by definition of G and (6), we Know that $G(t,s) \leqslant G(s,s)$ for all $s,t \in [0,1]$

(ii) Let $J(t,s) = t^{\alpha-1}(1-s)^{\alpha-1}$, $0 \leqslant t \leqslant s \leqslant 1$. Since $L(\cdot,s)$ is non-increasing, $J(\cdot,s)$ is nondecreasing, for all $s \in (0,1)$. Then, one can give

$$\min_{\gamma \leqslant t \leqslant \delta} G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} L(\delta,s), & s \in (0,\gamma], \\ \min\{L(\delta,s), J(\gamma,s)\}, & s \in [\gamma,\delta], \\ J(\gamma,s), & s \in [\delta,1), \end{cases} = \begin{cases} L(\delta,s), & s \in (0,m_1], \\ J(\gamma,s), & s \in [m_1,1), \end{cases} = \begin{cases} \delta^{\alpha-1}(1-s)^{\alpha-1} - (\delta-s)^{\alpha-1}, & s \in (0,m_1], \\ \gamma^{\alpha-1}(1-s)^{\alpha-1}, & s \in [m_1,1). \end{cases}$$

where $\gamma < m_1 < \delta$ is the solution of equation

$$\delta^{\alpha-1}(1-m_1)^{\alpha-1} - (\delta-m_1)^{\alpha-1} = \gamma^{\alpha-1}(1-m_1)^{\alpha-1}.$$

It follows from the monotonicity of L and J that

$$\max_{0 \leqslant t \leqslant 1} G(t, s) = G(s, s) = \frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{7}$$

Thus, we set $g(s)$ as in (5).

(iii) By (7) and Beta function, we have

$$\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)=\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}.$$

Therefore, the proof is complete. □

We now define

$$\eta := \min_{\gamma\leqslant t\leqslant\delta} g(t),$$

and let

$$\sigma := \min\{\eta, \gamma^{\alpha-1}\}. \tag{8}$$

Then, choose a cone K is $C^1([0,1])$, by

$$K=\left\{u\in C[0,1]|u(t)\geqslant 0,\ \min_{t\in[\gamma,\delta]} u(t)\geqslant\frac{\sigma}{3}\|u\|\right\}.$$

It is obvious that K is cone.

Define an operator T by

$$\begin{aligned}(Tu)(t)&=\int_0^1 G(t,s)f(s,u(s),u'(s))ds+\frac{\sum_{i=1}^m a_i t^{\alpha-1}}{1-\Delta}\\&\quad\times\int_0^1 G(\xi_i,s)f(s,u(s),u'(s))ds+\frac{\lambda t^{\alpha-1}}{1-\Delta}.\end{aligned}\tag{9}$$

It is clear that the existence of a positive solution for the system (1) is equivalent to the existence of nontrivial fixed point of T in K.

Lemma 6. *$T{:}K\to K$ is a completely continuous operator.*

Proof. Let $u\in K$. Then, it follows from Lemma 5(i) that

$$\begin{aligned}\|Tu\|&\leqslant\int_0^1 G(s,s)f(s,u(s),u'(s))ds+\frac{\sum_{i=1}^m a_i}{1-\Delta}\int_0^1 G(s,s)f(s,u(s),u'(s))ds\\&\quad+\frac{\lambda}{1-\Delta}=\left(\int_0^\gamma+\int_\gamma^\delta+\int_\delta^1\right)(G(s,s)f(s,u(s),u'(s)))ds\\&\quad+\frac{\sum_{i=1}^m a_i}{1-\Delta}\left(\int_0^\gamma+\int_\gamma^\delta+\int_\delta^1\right)(G(s,s)f(s,u(s),u'(s)))ds\\&\quad+\frac{\lambda}{1-\Delta}\leqslant 3\int_\gamma^\delta G(s,s)f(s,u(s),u'(s))ds\\&\quad+\frac{\sum_{i=1}^m a_i}{1-\Delta}\int_\gamma^\delta G(s,s)f(s,u(s),u'(s))ds+\frac{\lambda}{1-\Delta}.\end{aligned}$$

On the other hand, (8) and (9) and Lemma 5(ii) imply that, for any $t \in [\gamma, \delta]$,

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)f(s,u(s),u'(s))ds + \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{1-\Delta} \int_0^1 G(\xi_i,s)f(s,u(s),u'(s))ds \\ &+ \frac{\lambda t^{\alpha-1}}{1-\Delta} \geqslant \int_\gamma^\delta G(t,s)f(s,u(s),u'(s))ds + \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{1-\Delta} \\ &\times \int_\gamma^\delta G(\xi_i,s)f(s,u(s),u'(s))ds + \frac{\lambda t^{\alpha-1}}{1-\Delta} \\ &\geqslant \int_\gamma^\delta g(s)G(s,s)f(s,u(s),u'(s))ds + \frac{\sum_{i=1}^m a_i \gamma^{\alpha-1}}{1-\Delta} \\ &\times \int_\gamma^\delta g(s)G(s,s)f(s,u(s),u'(s))ds + \frac{\lambda}{1-\Delta} \\ &\geqslant \lambda \Bigg[\int_\gamma^\delta G(s,s)f(s,u(s),u'(s))ds + \frac{\sum_{i=1}^m a_i \gamma^{\alpha-1}}{1-\Delta} \\ &\times \int_\gamma^\delta G(s,s)f(s,u(s),u'(s))ds + \frac{\lambda}{1-\Delta}\Bigg] \geqslant \frac{\sigma}{3}\|Tu\|.\end{aligned}$$

This fact directly implies that $T{:}K \to K$ is well defined. Now we show that T is a completely continuous operator. □

3. Main results

In this section, we discuss the existence of a positive solution of problem (1). We define the nonnegative continuous concave functional on K by

$$\alpha(u) = \min_{\gamma \leqslant t \leqslant \delta}(u(t)).$$

It is obvious that, for each $u \in K, \alpha(u) \leqslant \|u\|$.

We use the following notations. Let

$$M = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left(1 + \frac{\sum_{i=1}^n a_i}{\Gamma(\alpha)(1-\Delta)}\right)$$

$$R = \min_{\gamma \leqslant t \leqslant \delta}\left\{\int_\gamma^\delta G(t,s)ds + \frac{\sum_{i=1}^n a_i}{\Gamma(\alpha)(1-\Delta)}\int_\gamma^\delta G(\xi_i,s)ds\right\}.$$

We are now ready to state our main results.

Theorem 2. *Assume that there exist nonnegative numbers a,b,c such that $0 < a < b \leqslant \sigma c$, and $f(t,u,u')$, satisfy the following conditions:*

H3) $f(t,u,u') \leqslant \frac{c}{M}$, *for all* $(t,u,u') \in [0,1] \times [0,c] \times [0,c]$;
H4) $f(t,u,u') \leqslant \frac{a}{M}$, *for all* $(t,u,u') \in [0,1] \times [0,a] \times [0,a]$;
H5) $f(t,u,u') > \frac{b}{R}$, *for all* $(t,u,u') \in [\gamma,\delta] \times \left[b, \frac{b}{\sigma}\right] \times \left[b, \frac{b}{\sigma}\right]$.

In addition, suppose that λ satisfy

$$0 < \lambda < \frac{c(1-\Delta)}{2}. \tag{10}$$

Then the problem (1) has at least three positive solutions u_1, u_2, u_3 such that $\|u_1\| < a, b < \alpha(u_2(t))$, and $\|u_3\| > a$, with $\alpha(u_3(t)) < b$.

Proof. Firstly, we show that $T : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator. In fact, if $u \in \overline{P_c}$, by condition (H3) and (10), we have

$$\begin{aligned}
\|Tu\| &= \max_{0\leqslant t\leqslant 1} |(Tu)(t)| \\
&= \max_{0\leqslant t\leqslant 1} \left(\int_0^1 G(t,s)f(s,u(s),u'(s))ds + \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{1-\Delta} \right. \\
&\quad \left. \times \int_0^1 G(\xi_i,s)f(s,u(s),u'(s))ds + \frac{\lambda t^{\alpha-1}}{1-\Delta} \right) \\
&\leqslant \frac{c}{M}\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{\sum_{i=1}^n a_i}{\Gamma(\alpha)(1-\Delta)} \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \right) + \frac{\lambda}{1-\Delta} \leqslant \frac{c}{2} + \frac{c}{2} = c.
\end{aligned}$$

Therefore, $\|Tu\| \leqslant c$, that is, $T : \overline{P_c} \to \overline{P_c}$. The operator T is completely continuous by an application of the Ascoli–Arzela theorem.

In a completely analogous way, the condition (H4) implies that the condition (A2) of Theorem 1 is satisfied.

We now show that the condition (A1) of Theorem 1 is satisfied. Clearly, $\{u \in P(\alpha, b, \frac{b}{\sigma}) | \alpha(u) > b\} \neq \emptyset$. If $u \in P(\alpha, b, \frac{b}{\sigma})$, then $b \leqslant u(s) \leqslant \frac{b}{\sigma}$, $s \in [\gamma, \delta]$.

By condition (H5), we get

$$\begin{aligned}
\alpha((Tu)(t)) = \min_{\gamma\leqslant t\leqslant \delta}((Tu)(t)) &\geqslant \min_{\gamma\leqslant t\leqslant \delta}\left(\int_0^1 G(t,s)f(s,u(s),u'(s))ds \right. \\
&\quad \left. + \frac{\sum_{i=1}^m a_i t^{\alpha-1}}{1-\Delta} \int_0^1 G(\xi_i,s)f(s,u(s),u'(s))ds + \frac{\lambda t^{\alpha-1}}{1-\Delta} \right) \\
&\geqslant \frac{b}{R} \min_{\gamma\leqslant t\leqslant \delta} \left\{ \int_\gamma^\delta G(t,s)ds + \frac{\sum_{i=1}^n a_i}{\Gamma(\alpha)(1-\Delta)} \int_\gamma^\delta G(\xi_i,s)ds \right\} = b.
\end{aligned}$$

Therefore, the condition (A1) of Theorem 1 is satisfied.

Finally, we show that the condition (A3) of Theorem 1 is also satisfied.

If $u \in P(\alpha, b, c)$, and $\|Tu\| > \frac{b}{\sigma}$, then

$$\alpha((Tu)(t)) = \min_{\gamma\leqslant t\leqslant \delta}(Tu)(t) \geqslant \sigma\|Tu\| > b.$$

Therefore, the condition (A3) of Theorem 1 is also satisfied. By Theorem 1, there exist three positive solutions u_1, u_2, u_3 such that $\|u_1\| < a, b < \alpha(u_2(t))$, and $\|u_3\| > a$, with $\alpha(u_3(t)) < b$. Therefore, we have the conclusion. □

4. Application

Example 3. Consider the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{5}{2}} u(t) = f(t, u(t), u'(t)), & t \in [0, 1], \\ u(0) = u'(0) = 0, & u(1) - \frac{1}{4} u\left(\frac{1}{3}\right) - \frac{3}{4} u\left(\frac{2}{3}\right) = \lambda \end{cases} \tag{11}$$

where

$$f(t, u, v) = \begin{cases} \sin(\pi t) + u^6 + \frac{\sqrt{v}}{30}, & t \in [0, 1], \quad 0 \leqslant u < 2, \ v \geqslant 0, \\ \sin(\pi t) + 64 + \frac{15}{2}\sqrt{u-2} + \frac{\sqrt{v}}{30}, & t \in [0, 1], \quad 2 \leqslant u < 18, \ v \geqslant 0, \\ \sin(\pi t) + 94 + \sqrt{u - 18} + \frac{\sqrt{v}}{30}, & t \in [0, 1], \quad u \geqslant 18, \ v \geqslant 0. \end{cases}$$

To show the problem (11) has at least three positive solutions, we apply Theorem 2 with $\alpha = \frac{5}{2}$, $m = 2$, $a_1 = \frac{1}{4}$, $a_2 = \frac{3}{4}$, $\xi_1 = \frac{1}{3}$ and $\xi_2 = \frac{2}{3}$.

We choose $\gamma = \frac{1}{3}$ and $\delta = \frac{2}{3}$. Then, by direct calculations, we can obtain that

$$\Delta = 0.4564 \quad M = 0.264048, \quad R = 0.18297$$

By calculating, we can let $m_1 = \frac{\sqrt[3]{3-2\sqrt{2}}-2}{\sqrt[3]{3-2\sqrt{2}}-3}$ (see Lemma 5) and $\sigma \simeq 0.01437$. If we take $a = 1$, $b = 2$ and $c = 100$, we obtain

$$\begin{aligned} f(t, u, v) \leqslant 104.3833 \leqslant \frac{c}{M} = 378.719, \quad \text{for all} \quad 0 \leqslant t \leqslant 1, \ 0 \leqslant u \leqslant 100, \ 0 \\ \leqslant v \leqslant 100, f(t, u, v) \leqslant 2.0333 < \frac{a}{M} = 3.677, \quad \text{for all} \quad 0 \leqslant t \leqslant 1, \ 0 \\ \leqslant u \leqslant 1, \ 0 \leqslant v \leqslant 1, f(t, u, v) \geqslant 64.5471 > \frac{b}{R} \\ = 10.9307, \quad \text{for all} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \ 2 \leqslant u \leqslant 4.559, \ 2 \leqslant v \leqslant 4.559. \end{aligned}$$

Thus for $0 < \lambda \leqslant \frac{c(1-\Delta)}{2} = 27.18$ by Theorem 2, the problem (1) has at least three positive solutions $u_i, i = 1,2,3$, such that $u_1\| < 1, 2 < \alpha(u_2)$, and $\|u_3\| > 1$, with $\alpha(u_3) < 2$.

Acknowledgement

N.N. thanks Razi University for support.

References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math. 109 (2010) 973–1033.
[2] J. Cang, Y. Tan, H. Xu, S.J. Liao, Series solutions of non-linear Riccati differential equations with fractional order, Chaos Solitons Fractals 40 (1) (2009) 1–9.
[3] K. Diethelm, N. Ford, A. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dynam. 29 (2002) 3–22.
[4] K. Diethelm, N. Ford, A. Freed, Detailed error analysis for a fractional Adams Method, Numer. Algorithms 36 (2004) 31–52.
[5] A.M.A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Anal. 33 (1998) 181–186.
[6] J. Feng, Z. Yong, Existence of solutions for a class of fractional boundary value problems via critical point theory, Comput. Math. Appl. 62 (2011) 1181–1199.
[7] A.A. Kilbas, J.J. Trujillo, Differential equations of fractional order: methods, results and problems I, Appl. Anal. 78 (2001) 153–192.
[8] A.A. Kilbas, J.J. Trujillo, Differential equations of fractional order: methods, results and problems II, Appl. Anal. 81 (2002) 435–493.
[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential EquationsNorth-Holland Mathematics Studies, Vol. 204, Elsevier Science, B.V Amsterdam, 2006.
[10] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. TMA 69 (8) (2008) 2677–2682.
[11] R.W. Leggett, L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana Univ. Math. J. 28 (1979) 673–688.
[12] F. Mainardi, Y. Luchko, G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, Fract. Calc. Appl. Anal. 4 (2001) 153–192.
[13] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[14] S. Momani, Z. Odibat, A novel method for nonlinear fractional partial differential equations: combination of DTM and generalized Taylor's formula, J. Comput. Appl. Math. 220 (1–2) (2008) 85–95.
[15] Z. Odibat, S. Momani, An algorithm for the numerical solution of differential equations of fractional order, J. Appl. Math. Inform. 26 (1-2) (2008) 15–27.
[16] Z. Odibat, S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, Appl. Math. Modelling 32 (12) (2008) 28–39.
[17] Z. Odibat, S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, Chaos Solitons Fractals 36 (1) (2008) 167–174.
[18] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[19] I. Podlubny, The Laplace Transform Method for Linear Differential Equations of Fractional Order, Slovac Academy of Science, Slovak Republic, 1994.
[20] G. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Amsterdam, 1993.
[21] W. Schneider, W. Wyss, Fractional diffusion and wave equations, J. Math. Phys. 30 (1989) 134–144.