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ORIGINAL ARTICLE

Existence of solutions for fractional differential inclusions with nonlocal strip conditions

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Abstract In this paper, we discus the existence of solutions for a nonlocal boundary value problem of fractional differential inclusions concerning a nonlocal strip condition via some fixed point theorems. Our results include the cases when the right-hand side of the inclusion is convex as well as nonconvex valued.

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1. Introduction

In this paper, we discuss the existence of solutions for a boundary value problem of nonlinear fractional differential inclusions of order $q \in (1,2)$ with nonlocal strip conditions given by

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \ 0 < t < 1, 1 < q \leqslant 2, \\ x(0) = 0, \ x(1) = \eta \int_v^\tau x(s)ds, \ 0 < v < \tau < 1 (v \neq \tau), \end{cases} \tag{1.1}$$

where ${}^c\mathrm{D}^q$ denotes the Caputo fractional derivative of order q, $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

The single-valued problem, that is the equation ${}^c\mathrm{D}^q x(t) = f(t, x(t))$, with the boundary conditions in (1.1) was studied recently in [4]. As argued in [4], the nonlocal strip condition ($x(1) = \eta \int_v^\tau x(s)ds, \ 0 < v < \tau < 1$) in (1.1) is an extension of a three-point nonlocal boundary condition of the form $x(1) = \eta x(v)$, $\eta \in \mathbb{R}$, $0 < v < 1$. In fact, this strip condition corresponds to a continuous distribution of the values of the unknown function on an arbitrary finite segment of the interval. In other words, the strip condition in (1.1) can be regarded as a four-point nonlocal boundary condition which reduces to the typical integral boundary conditions in the limit $v \to 0$, $\tau \to 1$. Strip conditions of fixed size appear in the mathematical modeling of real world problems, for example, see [6,12]. Thus, the present idea of nonlocal strip conditions will be quite fruitful in modeling the strip problems as one can choose a strip of arbitrary size according to the requirement by fixing the nonlocal parameters involved in the problem. As a matter of fact, integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [5,16] and references therein. For the basic theory of fractional differential equations and its applications see [24–27], and the recent development on the topic can be found in [1,2,10,3,7–9,11,13,14] and the references cited therein.

Here we extend the results of [4] to cover the multi-valued case. We establish the existence of results for the problem (1.1), when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray–Schauder type. In the second result, we shall combine the nonlinear alternative of Leray–Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

The methods used are standard, however their exposition in the framework of problems (1.1) is new.

2. Preliminaries

2.1. Fractional Calculus

Let us recall some basic definitions of fractional calculus [24,27].

Definition 2.1. For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \; n-1 < q < n, \; n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann–Liouville fractional integral of order q is defined as

$$I^{q}g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \; q > 0,$$

provided the integral exists.

To define the solution for the inclusion problem we need the following lemma.

Lemma 2.3 (*[4]*). *For a given $g \in C([0, 1], \mathbb{R})$ the unique solution of the boundary value problem*

$$\begin{cases} ^{c}D^{q}x(t) = g(t), \; 0 < t < 1, \; 1 < q \leqslant 2, \\ x(0) = 0, \; x(1) = \eta \int_{v}^{\tau} x(s)ds, \; 0 < v < \tau < 1 (v \neq \tau), \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)ds - \frac{2t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \\ & \times \int_0^1 (1-s)^{q-1} g(s)ds + \frac{2\eta t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \\ & \times \int_v^\tau \left(\int_0^s (s-m)^{q-1} g(m)dm \right) ds. \end{aligned} \tag{2.2}$$

2.2. Multivalued analysis

Let us recall some basic definitions on multi-valued maps [19,21].

For a normed space $(X, \|.\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to$

$\mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0; 1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

Definition 2.4. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$; Further a Carathéodory function F is called L^1-Carathéodory if
(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leqslant \varphi_\alpha(t)$$

for all $\|x\| \leqslant \alpha$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t, y(t)) \text{for a.e. } t \in [0,1]\}.$$

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes B$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.5. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0, 1] \times \mathbb{R}) \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0, 1], \mathbb{R}) : w(t) \in F(t, x(t)) \text{for a.e. } t \in [0, 1]\},$$

which is called the Nemytskii operator associated with F.

Definition 2.6. Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X, d) be a metric space induced from the normed space $(X; \|.\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \ \sup_{b \in B} d(A, b)\},$$

where $d(A,b) = \inf_{a \in A} d(a;b)$ and $d(a,B) = \inf_{b \in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [22]).

Definition 2.7. A multivalued operator $N: X \to P_{cl}(X)$ is called:

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leqslant \gamma d(x, y) \text{ for each } x, y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.8 (Nonlinear alternative for Kakutani maps [20]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is a upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Lemma 2.9 [23]. *Let X be a Banach space. Let $F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let* H *be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator*

$$\Theta \circ S_F : C([0,1], X) \to P_{cp,c}(C([0,1], X)), \ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Lemma 2.10 [15]. *Let* Y *be a separable metric space and let* $N : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ *be a multivalued operator satisfying the property (BC). Then* N *has a continuous selection, that is, there exists a continuous function (single-valued)* $g : Y \to L^1([0,1], \mathbb{R})$ *such that* $g(x) \in N(x)$ *for every* $x \in Y$.

Lemma 2.11 [18]. *Let* (X,d) *be a complete metric space. If* $N : X \to P_{cl}(X)$ *is a contraction, then* $Fix\ N \neq \emptyset$.

Definition 2.12. A function $x \in C^2([0,1], \mathbb{R})$ is a solution of the problem (1.1) if $x(0) = 0, \ x(1) = \eta \int_v^\tau x(s)ds$, and there exists a function $f \in L^1([0,1], \mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds - \frac{2t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds + \frac{2\eta t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_v^\tau \left(\int_0^s (s-m)^{q-1} f(m)dm \right) ds. \quad (2.3)$$

3. Existence results

3.1. The Carathéodory case

Theorem 3.1. *Assume that:*

- (H_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact and convex values;*
- (H_2) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,1], \mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leqslant p(t)\psi(\|x\|) \text{for each } (t,x) \in [0,1] \times \mathbb{R}.$$

- (H_3) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\frac{\psi(M)}{\Gamma(q)} \left\{ (1+\delta_1) \int_0^1 (1-s)^{q-1} p(s)ds + |\eta|\delta_1 \int_v^\tau \left(\int_0^s (s-m)^{q-1} p(m)dm \right) ds \right\}} > 1$$

where

$$\delta_1 = \frac{2}{|2-\eta(\tau^2 - v^2)|}.$$

Then the boundary value problem (1.1) has at least one solution on [0,1].

Proof. Define the operator $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds \\ -\frac{2t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ +\frac{2\eta t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \int_v^\tau \left(\int_0^s (s-m)^{q-1} f(m)dm \right) ds, \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof consists of several steps. As a first step, we show that Ω *is convex for each* $x \in C([0,1],\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Next, we show that Ω maps bounded sets (balls) into bounded sets in $C([0,1],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \leqslant \rho\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds - \frac{2t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ & + \frac{2\eta t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \int_v^\tau \left(\int_0^s (s-m)^{q-1} f(m)dm \right) ds, \end{aligned}$$

and,

$$\begin{aligned} |h(t)| \leqslant & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s))|ds \\ & + \left| \frac{2t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \right| \int_0^1 (1-s)^{q-1} |f(s,x(s))|ds \\ & + \left| \frac{2\eta t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \right| \int_v^\tau \left(\int_0^s (s-m)^{q-1} |f(m,x(m))|dm \right) ds \\ \leqslant & \psi(\|x\|) \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s)ds + \left| \frac{2t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \right| \right. \\ & \times \int_0^1 (1-s)^{q-1} p(s)ds + \left| \frac{2\eta t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \right| \\ & \left. \times \int_v^\tau \left(\int_0^s (s-m)^{q-1} p(m)dm \right) ds \right] \\ \leqslant & \frac{\psi(\|x\|)}{\Gamma(q)} \left\{ (1+\delta_1) \int_0^1 (1-s)^{q-1} p(s)ds + |\eta|\delta_1 \int_v^\tau \left(\int_0^s (s-m)^{q-1} p(m)dm \right) ds \right\}. \end{aligned}$$

Thus,

$$\|h\| \leqslant \frac{\psi(\rho)}{\Gamma(q)}\left\{(1+\delta_1)\int_0^1 (1-s)^{q-1}p(s)ds + |\eta|\delta_1 \int_\nu^\tau \left(\int_0^s (s-m)^{q-1}p(m)dm\right)ds\right\}.$$

Now we show that Ω maps bounded sets into equicontinuous sets of $C([0, 1], \mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_\rho$, where B_ρ is a bounded set of $C([0, 1], \mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned} |h(t'') - h(t')| \leqslant & \left|\int_0^{t'} \left[\frac{(t''-s)^{q-1} - (t'-s)^{q-1}}{\Gamma(q)}\right] f(s)ds + \int_{t'}^{t''} \frac{(t''-s)^{q-1}}{\Gamma(q)} f(s)ds\right| \\ & - \frac{2|t''-t'|}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ & + \frac{2\eta|t''-t'|}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} f(m)dm\right)ds. \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t'' - t' \to 0$. As Ω satisfies the above three assumptions, therefore it follows by the Ascoli–Arzelá theorem that $\Omega : C([0, 1], \mathbb{R}) \to \mathcal{P}(C([0, 1], \mathbb{R}))$ is completely continuous.

In our next step, we show that Ω has a closed graph. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_n(s)ds - \frac{2t}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f_n(s)ds \\ & + \frac{2\eta t}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} f_n(m)dm\right)ds. \end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s)ds - \frac{2t}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f_*(s)ds \\ & + \frac{2\eta t}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} f_*(m)dm\right)ds. \end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds - \frac{2t}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ & + \frac{2\eta t}{[2-\eta(\tau^2-\nu^2)]\Gamma(q)} \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} f(m)dm\right)ds. \end{aligned}$$

Observe that

$$\|h_n(t) - h_*(t)\| = \Bigg\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(f_n(s) - f_*(s))ds \\ - \frac{2t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1}(f_n(s) - f_*(s))ds \\ + \frac{2\eta t}{[2-\eta(\tau^2-v^2)]\Gamma(q)} \int_v^\tau \left(\int_0^s (s-m)^{q-1}(f_n(m) - f_*(m))dm \right) ds$$

as $n \to \infty$.

Thus, it follows by Lemma 2.9 that $\mathrm{H} \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s)ds - \frac{2t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f_*(s)ds \\ + \frac{2\eta t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_v^\tau \left(\int_0^s (s-m)^{q-1} f_*(m)dm \right) ds,$$

for some $f_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq C([0,\ 1], \mathbb{R})$ with $x \notin \Omega(x)$ for $\lambda \in (0,1)$ and $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega(x)$. Then there exists $f \in L^1([0,\ 1], \mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds - \frac{2t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ + \frac{2\eta t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_v^\tau \left(\int_0^s (s-m)^{q-1} f(m)dm \right) ds,$$

Using the computations of the second step above we have

$$\|h\| \leqslant \frac{\psi(\|x\|)}{\Gamma(q)} \left\{ (1+\delta_1) \int_0^1 (1-s)^{q-1} p(s)ds + |\eta|\delta_1 \int_v^\tau \left(\int_0^s (s-m)^{q-1} p(m)dm \right) ds \right\}.$$

Consequently, we have

$$\frac{\|x\|}{\frac{\psi(\|x\|)}{\Gamma(q)} \left\{ (1+\delta_1) \int_0^1 (1-s)^{q-1} p(s)ds + |\eta|\delta_1 \int_v^\tau \left(\int_0^s (s-m)^{q-1} p(m)dm \right) ds \right\}} \leqslant 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,\ 1], \mathbb{R}) : \|x\| < M + 1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,\ 1], \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\Omega(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.8), we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. $\square$

Example 3.2. Consider the following strip fractional boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t, x(t)), \ 0 < t < 1, \\ x(0) = 0, \ x(1) = \int_{1/4}^{3/4} x(s)ds. \end{cases} \tag{3.1}$$

Here, $q = 3/2$, $v = 1/4$, $\tau = 3/4$, $\eta = 1$, and $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map given by

$$x \to F(t, x) = \left[\frac{|x|^3}{|x|^3 + 3} + 3t^3 + 5, \ \frac{|x|}{|x| + 1} + t + 1\right].$$

For $f \in F$, we have

$$|f| \leqslant \max\left(\frac{|x|^3}{|x|^3 + 3} + 3t^3 + 5, \ \frac{|x|}{|x| + 1} + t + 1\right) \leqslant 9, \ x \in \mathbb{R}.$$

Thus,

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \leqslant 9 = p(t)\psi(\|x\|), \ x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|) = 9$.

Further, using the condition (H_3) we find that $M > 5.8159541$. Clearly, all the conditions of Theorem 3.1 are satisfied. So there exists at least one solution of the problem (3.1) on $[0, 1]$.

3.2. The lower semicontinuous case

Here, we study the case when F is not necessarily convex valued. Our strategy to deal with these problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [15] for lower semi-continuous maps with decomposable values.

Theorem 3.3. *Assume that (H_1), (H_2) and the following condition holds:*

(H_4) $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
  - (a) *$(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
  - (b) *$x \mapsto F(t, x)$ is lower semicontinuous for each $t \in [0, 1]$;*

Then the boundary value problem (1.1) has at least one solution on [0,1].

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 2.10, there exists a continuous function $f : C([0, 1], \mathbb{R}) \to L^1([0, 1], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0, 1], \mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x)(t), \ t \in [0,1], \ T > 0, \ 1 < q \leqslant 2, \\ x(0) = 0, \ x(1) = \eta \int_v^\tau x(s)ds, \ 0 < v < \tau < 1 (v \neq \tau). \end{cases} \tag{3.2}$$

Observe that if $x \in C^2([0, 1], \mathbb{R})$ is a solution of (3.2), then x is a solution to the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\overline{\Omega}x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(x)(s)ds - \frac{2t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} f(x) \\ \times (s)ds + \frac{2\eta t}{[2-\eta(\tau^2 - v^2)]\Gamma(q)} \int_v^\tau \left(\int_0^s (s-m)^{q-1} f(x)(m)dm \right) ds.$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. □

3.3. The Lipschitz case

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [27].

Theorem 3.4. *Assume that the following conditions hold:*

(H_5) *$F : [0, 1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0, 1] \to P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.*

(H_6) *$H_d(F(t, x), F(t, \bar{x})) \leqslant m(t)|x - \bar{x}|$ for almost all $t \in [0,1]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0, 1], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leqslant m(t)$ for almost all $t \in [0,1]$. Then the boundary value problem (1.1) has at least one solution on [0,1] if*

$$\frac{1}{\Gamma(q)} \left\{ (1+\delta_1) \int_0^1 (1-s)^{q-1} m(s)ds + |\eta|\delta_1 \int_v^\tau \left(\int_0^s (s-r)^{q-1} m(r)dr \right) ds \right\} < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0, 1], \mathbb{R})$ by the assumption (H_5), so F has a measurable selection (see Theorem III.6 [17]). Now we show that the operator Ω, defined in the beginning of proof of Theorem 3.1, satisfies the assumptions of Lemma 2.11. To show that $\Omega(x) \in P_{cl}((C[0, 1], \mathbb{R}))$ for each $x \in C([0, 1], \mathbb{R})$, let $\{u_n\}_{n \geqslant 0} \in \Omega(x)$ be such that $u_n \to u (n \to \infty)$ in $C([0, 1], \mathbb{R})$. Then $u \in C([0, 1], \mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_n(s)ds - \frac{2t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} v_n(s)ds + \frac{2\eta t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} v_n(m)dm \right) ds,$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$u_n(t) \to u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds - \frac{2t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \times \int_0^1 (1-s)^{q-1} v(s)ds + \frac{2\eta t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \times \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} v(m)dm \right) ds.$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega(x), \Omega(\bar{x})) \leqslant \gamma \|x - \bar{x}\| \text{ for each } x, \bar{x} \in C([0,1],\mathbb{R}).$$

Let $x, \bar{x} \in C([0,1],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,1]$,

$$h_1(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_1(s)ds - \frac{2t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} v_1(s)ds + \frac{2\eta t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} v_1(m)dm \right) ds,$$

By (H_7), we have

$$H_d(F(t,x), F(t,\bar{x})) \leqslant m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leqslant m(t)|x(t) - \bar{x}(t)|, \ t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leqslant m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [17]), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leqslant m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$h_2(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_2(s)ds - \frac{2t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \int_0^1 (1-s)^{q-1} v_2(s)ds + \frac{2\eta t}{[2-\eta(\tau^2 - \nu^2)]\Gamma(q)} \int_\nu^\tau \left(\int_0^s (s-m)^{q-1} v_2(m)dm \right) ds,$$

Thus,

$$
\begin{aligned}
|h_1(t) - h_2(t)| &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |v_1(s) - v_2(s)| ds + \left| \frac{2t}{[2 - \eta(\tau^2 - \nu^2)]\Gamma(q)} \right| \\
&\times \int_0^1 (1-s)^{q-1} |v_1(s) - v_2(s)| ds + \left| \frac{2\eta t}{[2 - \eta(\tau^2 - \nu^2)]\Gamma(q)} \right| \\
&\times \int_\nu^\tau \left(\int_0^s (s-r)^{q-1} |v_1(r) - v_2(r)| dr \right) ds \\
&\leqslant \frac{\|x - \bar{x}\|)}{\Gamma(q)} \left\{ (1 + \delta_1) \int_0^1 (1-s)^{q-1} m(s) ds + |\eta| \delta_1 \right. \\
&\left. \times \int_\nu^\tau \left(\int_0^s (s-r)^{q-1} m(r) dr \right) ds \right\}.
\end{aligned}
$$

Hence,

$$
\|h_1 - h_2\| \leqslant \frac{\|x - \bar{x}\|)}{\Gamma(q)} \left\{ (1 + \delta_1) \int_0^1 (1-s)^{q-1} m(s) ds + |\eta| \delta_1 \int_\nu^\tau \left(\int_0^s (s-r)^{q-1} m(r) dr \right) ds \right\}.
$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$
\begin{aligned}
H_d(\Omega(x), \Omega(\bar{x})) \leqslant \gamma \|x - \bar{x}\| \leqslant \frac{\|x - \bar{x}\|)}{\Gamma(q)} &\left\{ (1 + \delta_1) \int_0^1 (1-s)^{q-1} m(s) ds + |\eta| \delta_1 \right. \\
&\left. \times \int_\nu^\tau \left(\int_0^s (s-r)^{q-1} m(r) dr \right) ds \right\}.
\end{aligned}
$$

Since Ω is a contraction, it follows by Lemma 2.11 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. □

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