ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 21 (2015) 127–135

Existence of positive solutions for a variational inequality of Kirchhoff type

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Received 9 October 2014; received in revised form 18 February 2015; accepted 19 February 2015
Available online 2 March 2015

Abstract. In the present paper, we study existence of nontrivial positive solutions for a Kirchhoff type variational inequality. Our approach is based on the non-smooth critical point theory for Szulkin-type functionals.

Keywords: Variational inequality; Critical point; Mountain pass theorem; Minimization; Szulkin-type functionals

1. INTRODUCTION

Variational inequalities describe phenomena from mathematical physics. They have applications in physics, mechanics, engineering, optimization, and elliptic inequalities, see, for example, [1–4] and [5].

The aim of this work is to study a Kirchhoff type variational inequality which is defined on a bounded interval $(0, 1)$ by using a non-smooth critical point theory due to Szulkin. In [7], the author has proved a number of existence theorems for critical points of functionals which are not smooth. He has generalized some minimization and minimax methods in critical point theory to a class of functionals which are not necessarily continuous and has introduced a new concept of compactness which is suitable to study these kinds of problems.

In the present paper, by using a minimization principle and the Mountain pass theorem of Szulkin-type, we prove existence of positive solutions to a variational inequality of Kirchhoff-type in a closed convex set.

Let $K = \{u \in H_0^1(0, 1) : u \geq 0\}$ be the closed convex set in the Sobolev space $H_0^1(0, 1)$ and we consider the problem, denoted by (P):

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Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2015.02.004

Given $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function and $a, b > 0$, find $u \in K$ such that

$$\Big(a + b\int_0^1 |u'(x)|^2 dx\Big)\int_0^1 u'(x)(v'(x) - u'(x))\mathrm{d}x - \int_0^1 f(x, u(x))(v(x) - u(x))\mathrm{d}x \geq 0, \quad \forall v \in K.$$

Such kind of problems are called obstacle problems and they have been largely studied due to its physical applications. See, for example, the classical books Kinderlehrer and Stampacchia [4], Rodrigues [6] and Troianiello [8] and the references therein.

2. SZULKIN-TYPE FUNCTIONALS

Let X be a real Banach space and X^* its dual. Let E be a functional which is of class C^1 and let $\psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper (i.e. $\psi \neq +\infty$), convex, lower semicontinuous functional. We say that $I = E + \psi$ is a Szulkin-type functional, see [7]. An element $u \in X$ is called a critical point of $I = E + \psi$ if

$$E'(u)(v - u) + \psi(v) - \psi(u) \geq 0 \quad \text{for all } v \in X, \tag{1}$$

which is equivalent to

$$0 \in E'(u) + \partial\psi(u) \quad \text{in } X^*,$$

where $\partial\psi(u)$ is the subdifferential of the convex functional ψ at $u \in X$.

Definition 2.1. The functional $I = E+\psi$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$, denoted by $(PSZ)_c$ if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} I(u_n) = c$ and

$$\langle E'(u_n), v - u_n\rangle_X + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\| \quad \text{for all } v \in \mathrm{X},$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Theorem 2.1 *([7]). Let X be a Banach space, $I = E+\psi : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ a Szulkin-type functional which is bounded below. If I satisfies the $(PSZ)_c$-condition for*

$$c = \inf_{u\in X} I(u),$$

then c is a critical value.

Szulkin has proved the following version of the Mountain Pass theorem.

Theorem 2.2 *([7]). Let X be a Banach space, $I = E+\psi : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ a Szulkin-type functional and assume that*

(i) *$I(u) \geq \alpha$ for all $\|u\| = \rho$ for some $\alpha, \rho > 0$, and $I(0) = 0$;*

(ii) *there is $e \in X$ with $\|e\| > \rho$ and $I(e) \leq 0$.*

If I satisfies the $(PSZ)_c$-condition for

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of I *and* $c \geq \alpha$*, i.e., there exists* u^* *in* X *such that* $I'(u^*) = 0$ *and* $I(u^*) = c \geq \alpha$*.*

3. Main results

We now formulate the main results of this paper. We denote by F the function defined by $F(x,s) = \int_0^s f(x,t)\mathrm{d}t$.

Theorem 3.1. *Let* $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function which satisfies the following condition:*

(f_1) *there exists* $\beta > 0$ *such that*

$$\limsup_{|t|\to\infty} \frac{F(x,t)}{t^2} \leq \beta, \quad \text{uniformly with respect to } x \in [0,1].$$

Then the problem (P) *has at least one solution* $u \in K$*.*

Theorem 3.2. *Let* $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function which satisfies the following conditions:*

(h_1) *There exists* $\nu > 4$ *and* $M > 0$ *such that*

$$0 < \nu F(x,t) \leq t f(x,t) \quad \text{for } |t| \geq M, \ \forall x \in [0,1].$$

(h_2) $\limsup_{|t|\to 0} \frac{F(x,t)}{|t|^2} < \frac{a}{2}$, *uniformly with respect to* $x \in [0,1]$.

Then the problem (P) *has at least one nontrivial solution* $u \in K$*.*

Remark 3.1. The hypotheses in Theorems 3.1 and 3.2 are respectively of sublinear and superlinear types, so they are natural conditions.

We define the functional $E : H_0^1(0,1) \longrightarrow \mathbb{R}$ by

$$E(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \int_0^1 F(x, u(x))\mathrm{d}x.$$

Because $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, by using the Lebesgue theorem on dominated convergence and the compact embedding of $H_0^1(0,1)$ in $C([0,1])$, we can prove easily that $E \in C^1(H_0^1(0,1), \mathbb{R})$.

We define the indicator functional of the set K by

$$\psi_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{if } u \notin K. \end{cases}$$

We remark that the functional ψ_K is convex, proper, and lower semicontinuous. So, $I = E + \psi_K$ is a Szulkin-type functional.

Proposition 3.1. *Every critical point* $u \in H_0^1(0,1)$ *of* $I = E + \psi_K$ *is a solution of (P).*

Proof. Since $u \in H_0^1(0,1)$ is a critical point of $I = E + \psi_K$, we have

$$E'(u)(v-u) + \psi_K(v) - \psi_K(u) \geq 0, \quad \forall v \in H_0^1(0,1).$$

Note that u belongs to K. For if this were not true we had $\psi_K(u) = +\infty$ and taking $v = 0 \in K$ in the above inequality, we obtain a contradiction. We fix $v \in K$. Since

$$\begin{aligned} 0 \leq E'(u)(v-u) &= (a + b\|u\|^2) \int_0^1 u'(x)(v'(x) - u'(x))\mathrm{d}x \\ &\quad - \int_0^1 f(x, u(x))(v(x) - u(x))\mathrm{d}x, \end{aligned}$$

the inequality is proved. □

4. Proof of Theorem 3.1

We assume that the hypothesis of Theorem 3.1 is satisfied and prove the existence of a solution for the problem (P) by using Theorem 2.1.

Proposition 4.1. *If the function f satisfies the hypothesis (f_1), then $I = E + \psi_K$ is coercive and bounded from below in $H_0^1(0,1)$.*

Proof. We have

$$I(u) = E(u) = \frac{1}{2}\left(a\|u\|^2 + \frac{1}{2}b\|u\|^4\right) - \int_0^1 F(x, u(x))\mathrm{d}x$$

for every $u \in K$. By the hypothesis (f_1), there exists $A > 0$ such that $F(x,t) \leq \beta t^2$ for every $|t| > A$ and $x \in [0,1]$. By using the compactness embedding of $H_0^1(0,1)$ in $L^2[0,1]$, we obtain that $\|u\|_{L^2(0,1)} \leq \|u\|_{H_0^1(0,1)}$. Hence

$$\begin{aligned} I(u) &\geq \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \beta\int_0^1 u^2(x)\mathrm{d}x \\ &= \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \beta\|u\|_{L^2}^2 \\ &\geq \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \beta\|u\|^2 \\ &= \left(\frac{1}{2}a - \beta\right)\|u\|^2 + \frac{1}{4}\|u\|^4, \end{aligned}$$

which implies that the functional $I = E + \psi_K$ is coercive. Therefore I is bounded from below in $H_0^1(0,1)$. If this is not true, there exists a sequence $\{u_n\}$ in $H_0^1(0,1)$ such that $\|u_n\| \to +\infty$ and $I(u_n) \to -\infty$, which is a contradiction with the coerciveness of I. □

Proposition 4.2. *If the function f satisfies (f_1), then $I = E + \psi_K$ satisfies $(PSZ)_c$ for every $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ be fixed. Let $\{u_n\}$ be a sequence in $H_0^1(0,1)$ such that

$$I(u_n) = E(u_n) + \psi_K(u_n) \to c; \tag{2}$$

and

$$E'(u_n)(v-u_n)+\psi_K(v)-\psi_K(u_n)\geq -\varepsilon_n\|v-u_n\|, \quad \forall v\in H_0^1(0,1), \tag{3}$$

$\{\varepsilon_n\}$ a sequence in $[0,\infty)$ with $\varepsilon_n\to 0$. By (2), we obtain that the sequence $\{u_n\}$ is in K. By Proposition 4.1, since I is coercive on $H_0^1(0,1)$, the sequence $\{u_n\}$ is bounded in K. Because the sequence $\{u_n\}$ is bounded in $H_0^1(0,1)$. Hence there exists a subsequence still denoted by $\{u_n\}$ which converges weakly in $H_0^1(0,1)$. So there exists $u\in H_0^1(0,1)$ such that

$$u_n\rightharpoonup u \quad \text{in } H_0^1(0,1); \tag{4}$$

$$u_n\to u \quad \text{in } L^2(0,1), \tag{5}$$

$$u_n\to u \quad \text{in } C([0,1]). \tag{6}$$

As K is weakly closed, $u\in K$. Setting $v=u$ in (3), we obtain that

$$\begin{aligned}&(a+b\|u_n\|^2)\int_0^1 u_n'(x)(u'(x)-u_n'(x))\mathrm{d}x\\&\quad+\int_0^1 f(x,u_n(x))(u_n(x)-u(x))\mathrm{d}x\geq -\varepsilon_n\|u-u_n\|.\end{aligned}$$

Therefore, for large $n\in\mathbb{N}$, we have

$$\begin{aligned}(a+b\|u_n\|^2)\|u-u_n\|^2&\leq (a+b\|u_n\|^2)\int_0^1 u'(x)(u'(x)-u_n'(x))\mathrm{d}x\\&\quad+\int_0^1 f(x,u_n(x))(u_n(x)-u(x))\mathrm{d}x+\varepsilon_n\|u-u_n\|\\&\leq (a+b\|u_n\|^2)(u,u-u_n)_{H_0^1}\\&\quad+\|u-u_n\|_{L^2}\left(\int_0^1|f(x,u_n(x))|^2\mathrm{d}x\right)^{\frac{1}{2}}+\varepsilon_n\|u-u_n\|.\end{aligned}$$

Since $\{u_n\}$ is bounded in $H_0^1(0,1)$, it is also bounded in $C([0,1])$. Therefore, there exists a constant $M>0$ such that $\|u_n\|_\infty\leq M$, which together with the continuity of f implies that $|f(x,u_n(x))|\leq M_1$ for some $M_1>0$. We obtain that

$$\begin{aligned}(a+b\|u_n\|^2)\|u-u_n\|^2&\leq (a+b\|u_n\|^2)(u,u-u_n)_{H_0^1}\\&\quad+M_1\|u-u_n\|_{L^2}+\varepsilon_n\|u-u_n\|.\end{aligned} \tag{7}$$

By (4) and the fact that$\{u_n\}$ is bounded in $H_0^1(0,1)$, we have

$$\lim_n (a+b\|u_n\|^2)(u,u-u_n)_{H_0^1}=0.$$

We conclude by (5) that the second term in (7) also converges to 0. Since $\varepsilon_n\to 0^+$, $\{u_n\}$ converges strongly to u in $H_0^1(0,1)$. This completes the proof. □

By Proposition 4.2, the functional I satisfies the $(PSZ)_c$ condition, and by Proposition 4.1, the functional I is bounded from below. Therefore, the number

$$c_1 = \inf_{u \in H_0^1(0,1)} I(u)$$

is a critical value of I by Theorem 2.1. It remains to apply Proposition 3.1 which concludes that the critical point $u_1 \in H_0^1(0,1)$ which corresponds to c_1, is actually an element of K and a solution of the problem (P).

Example 4.1. Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x,t) = x(|t|^{\frac{1}{2}} + 1)$. It satisfies (f_1). Indeed, we have $F(x,t) = x(\frac{2}{3}|t|^{\frac{3}{2}} + t)$ and

$$\frac{F(x,t)}{t^2} = \frac{x(\frac{2}{3}|t|^{\frac{3}{2}} + t)}{t^2} = x\Big(\frac{2}{3|t|^{\frac{1}{2}}} + \frac{1}{t}\Big),$$

so

$$\limsup_{|t| \to \infty} \frac{F(x,t)}{t^2} = 0.$$

5. Proof of Theorem 3.2

We assume that all the hypotheses of Theorem 3.2 are satisfied. Now we prove the existence of a nontrivial solution for the problem (P) by using the Mountain Pass theorem of Szulkin type (see Theorem 2.2).

Proposition 5.1. *If the function f satisfies (h_1), then the functional $I = E + \psi_K$ satisfies $(PSZ)_c$ for every $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ be a fixed number. Let $\{u_n\}$ be a sequence in $H_0^1(0,1)$ such that

$$I(u_n) = E(u_n) + \psi_K(u_n) \to c; \tag{8}$$

and

$$E'(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in H_0^1(0,1), \tag{9}$$

where $\{\varepsilon_n\}$ is a sequence in $[0, \infty)$ with $\varepsilon_n \to 0$. By (8), we see that the sequence $\{u_n\}$ belongs to K. We put $v = 2u_n$ in (9) and obtain

$$E'(u_n)(u_n) \geq -\varepsilon_n \|u_n\|.$$

Therefore, we obtain that

$$a\|u_n\|^2 + b\|u_n\|^4 - \int_0^1 f(x, u_n(x))u_n(x)\mathrm{d}x \geq -\varepsilon_n \|u_n\|. \tag{10}$$

Because (8) is satisfied for large $n \in \mathbb{N}$

$$c + 1 \geq \frac{1}{2}a\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \int_0^1 F(x, u_n(x))\mathrm{d}x. \tag{11}$$

By (h_1), we have

$$\nu F(x,t) - tf(x,t) \le c_1 \quad \text{for } x \in [0,1], t \in \mathbb{R}.$$

Multiplying (10) by $\nu^{-1,}$ and by adding this to (11) and by using (h_1), for large $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
c + 1 + \frac{1}{\nu}\|u_n\| &\ge a\left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{\nu}\right)\|u_n\|^4 \\
&\quad - \int_0^1 F(x, u_n(x)) - \frac{1}{\nu} f(x, u_n(x)) u_n(x) \mathrm{d}x \\
&\ge a\left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{\nu}\right)\|u_n\|^4 \\
&\quad - \frac{1}{\nu}\int_0^1 \nu F(x, u_n(x)) - f(x, u_n(x)) u_n(x) \mathrm{d}x \\
&\ge a\left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{\nu}\right)\|u_n\|^4 - \frac{c_1}{\nu}.
\end{aligned}$$

Since $\nu > 4$ we deduce that the sequence $\{u_n\}$ is bounded in K. So there exists a subsequence which converges weakly in $H_0^1(0,1)$. We can assume that there exists $u \in H_0^1(0,1)$ such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(0,1); \tag{12}$$

$$u_n \to u \quad \text{in } C([0,1]). \tag{13}$$

As K is weakly closed, $u \in K$. When we put $v = u$ in (9), we obtain that

$$\begin{aligned}
&(a + b\|u_n\|^2)\int_0^1 u_n'(x)(u'(x) - u_n'(x))\mathrm{d}x \\
&\quad + \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\mathrm{d}x \ge -\varepsilon_n\|u - u_n\|.
\end{aligned}$$

Hence, for large $n \in \mathbb{N}$, we have

$$\begin{aligned}
(a + b\|u_n\|^2)\|u - u_n\|^2 &\le (a + b\|u_n\|^2)\int_0^1 u'(x)(u'(x) - u_n'(x))\mathrm{d}x \\
&\quad + \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\mathrm{d}x + \varepsilon_n\|u - u_n\| \\
&\le (a + b\|u_n\|^2)(u, u - u_n)_{H_0^1} + \|u - u_n\|_{C([0,1])} \\
&\quad \times \int_0^1 \max_{s \in [-R,R]} |f(x,s)|\mathrm{d}x + \varepsilon_n\|u - u_n\|,
\end{aligned}$$

where $R = \|u\|_{C([0,1])} + 1$. By (12) and the fact that $\{u_n\}$ is bounded in $H_0^1(0,1)$, we have

$$\lim_n (a + b\|u_n\|^2)(u, u - u_n)_{H_0^1} = 0.$$

By using (13), the second term in the last expression also tends to 0. Since $\varepsilon_n \to 0^+$, $\{u_n\}$ converges strongly to u in $H_0^1(0,1)$. This completes the proof. $\square$

Proposition 5.2. *If the function f satisfies (h_1) and (h_2), then the following assertions are true:*

(i) *there exist constants $\alpha > 0$ and $\rho > 0$ such that $I(u) \geq \alpha$ for all $\|u\| = \rho$;*
(ii) *there exists $e \in H_0^1(0,1)$ with $\|e\| > \rho$ and $I(e) \leq 0$.*

Proof. (i) By condition (h_2), there exist $\varepsilon > 0$ and $\rho > 0$ such that

$$\frac{F(x,t)}{|t|^2} \leq \frac{a}{2} - \varepsilon \quad \text{for } |t| \leq \rho.$$

Therefore, by using the compactness embedding of $H_0^1(0,1)$ in $L^2(0,1)$ with $\|u\|_{L^2(0,1)} \leq \|u\|_{H^1(0,1)}$, we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_0^1 F(x,u(x))\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_0^1 \left(\frac{a}{2} - \varepsilon\right)|u(x)|^2\mathrm{d}x \\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \left(\frac{a}{2} - \varepsilon\right)\|u\|_{L^2}^2 \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \left(\frac{a}{2} - \varepsilon\right)\|u\|^2 \\ &= \varepsilon\|u\|^2 + \frac{b}{4}\|u\|^4. \end{aligned}$$

For $\|u\| = \rho$ we have $\alpha = \varepsilon\rho^2 + \frac{b}{4}\rho^4 > 0$, and the assertion of (i) holds true.

(ii) The condition (h_1) implies that the function $t \to \frac{F(x,t)}{|t|^\nu}$ is increasing for $t \geq M$ and decreasing for $t \leq -M$ as one can see by differentiation, so there exists $r_1 > 0$ such that $F(x,t) \geq r_1|t|^\nu$, for $x \in [0,1], |t| \geq M$. Also the function $t \to F(x,t)$ is continuous on the compact $[0,1] \times [-M,M]$, then there exists $r_2 > 0$ such that $F(x,t) \geq -r_2$, for $x \in [0,1], |t| \leq M$, so

$$F(x,t) \geq r_1|t|^\nu - r_2, \quad \text{for } x \in [0,1], t \in \mathbb{R}.$$

Fix $u_0 \in K \setminus \{0\}$. Letting $u = su_0$ $(s > 0)$, we have that

$$\begin{aligned} I(su_0) &= \frac{a}{2}s^2\|u_0\|^2 + \frac{b}{4}s^4\|u_0\|^4 - \int_0^1 F(x,su_0(x))\mathrm{d}x \\ &\leq \frac{a}{2}s^2\|u_0\|^2 + \frac{b}{4}s^4\|u_0\|^4 - \int_0^1 (r_1 s^\nu|u_0|^\nu - r_2)\mathrm{d}x \\ &= \frac{a}{2}s^2\|u_0\|^2 + \frac{b}{4}s^4\|u_0\|^4 - r_1 s^\nu\|u_0\|_{L^\nu}^\nu + r_2. \end{aligned}$$

Since $\nu > 4$ we obtain that $I(su_0) \to -\infty$ as $s \to +\infty$. Thus, it is possible to take s so large such that for $e = su_0$, we have $\|e\| > \rho$ and $I(e) \leq 0$. The proof of the proposition is achieved. $\square$

By Proposition 5.1, the functional I satisfies the $(PSZ)_c$-condition for every $c \in \mathbb{R}$, and $I(0) = 0$. By Proposition 5.2 it follows that there exist constants $\alpha, \rho > 0$ and $e \in H_0^1(0,1)$ such that I satisfies all the conditions of Theorem 2.2. Therefore,

$$c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

is a critical value of I with $c_2 \geq \alpha > 0$, where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

We remark that the critical point $u_2 \in H_0^1(0,1)$ associated to the critical value c_2 cannot be trivial because $I(u_2) = c_2 > 0 = I(0)$. By Proposition 3.1, we conclude that u_2 is a solution of (P).

Example 5.1. Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x,t) = \frac{1}{1+x^2}\frac{a}{2}t(1+t^2)e^{t^2}$. As we will show, it satisfies (h_1) and (h_2). We have $F(x,t) = \frac{1}{1+x^2}\frac{a}{4}t^2e^{t^2}$, and

$$6F(x,t) - tf(x,t) = \frac{1}{1+x^2}\frac{a}{2}t^2(2-t^2)e^{t^2} \leq 0,$$

for all $|t| \geq \sqrt{2}$. So there exist $\nu = 6 > 4$ and $M = \sqrt{2} > 0$ such that

$$0 < \nu F(x,t) \leq tf(x,t).$$

Moreover

$$\limsup_{|t| \to 0} \frac{F(x,t)}{|t|^2} = \limsup_{|t| \to 0} \frac{1}{1+x^2}\frac{a}{4}e^{t^2} = \frac{a}{4} < \frac{a}{2}.$$

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