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Existence of a solution for a class of parabolic equations with three unbounded nonlinearities, natural growth terms and L^1 data

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Abstract. We give an existence result of a renormalized solution for a class of nonlinear parabolic equations $\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + g(u)|\nabla u|^p = f$, where the right side belongs to $L^1(\Omega\times(0,T))$, $b(x,u)$ is an unbounded function of u and $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u, but without any growth assumption on u. The function g is just assumed to be continuous on $\mathbb{R}$ and satisfying a sign condition.

Keywords: Nonlinear parabolic equations; Existence; Renormalized solutions

Mathematics Subject Classification: Primary 47A15; Secondary 46A32, 47D20

1. INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N \geqslant 1)$, $T > 0$ and let $Q{:=}\Omega\times(0,T)$. We prove the existence of a renormalized solution for a class of nonlinear parabolic equations of the type:

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$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + g(u)|\nabla u|^p = f \text{ in } Q, \tag{1.1}$$

$$b(x,u)(t=0) = b(x,u_0) \text{ in } \Omega, \tag{1.2}$$

$$u = 0 \text{ on } \partial\Omega \times (0,T). \tag{1.3}$$

In Problem 1.1, 1.2 and 1.3, the framework is the following: the data f and $b(x,u_0)$ are respectively in $L^1(Q)$ and $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator which is coercive and which grows like $|\nabla u|^{p-1}$ with respect to ∇u, but which is not restricted by any growth condition with respect to u (see assumptions 2.4, 2.5 and 2.6 of Section 2). The function g is just assumed to be continuous on $\mathbb{R}$ and satisfying a sign condition.

We use in this paper the framework of renormalized solutions. This notion was introduced by Lions and Di Perna [25] for the study of Boltzmann equation (see also Lions [21] for a few applications to fluid mechanics models), (see also [7,24] for nonlinear parabolic equations with natural growth). For elliptic versions of 1.1, 1.2 and 1.3 we refer to [12] and [22,23]. The equivalent notion of entropy solutions has been developed independently by [1] for the study of nonlinear elliptic problems.

The existence and uniqueness of renormalized solution of 1.1, 1.2 and 1.3 have been proved in [26,27] in the case where $g = 0$ and $a(x,t,s,\xi)$ is replaced by $a(x,t,s,\xi) + \Phi(s)$. Where $b(x,u) = u$, $g = 0$ and f is replaced by $f + \operatorname{div}(F)$, the existence and uniqueness of renormalized solution have been proved in [5,24]. In the case where $a(x,t,s,\xi)$ is independent of s and $g = 0$, existence and uniqueness of renormalized solution have been established in [3,4]. In the case where $b(x,u) = b(u)$, $g = 0$ (where $b(r)$ is strictly increasing function of r that can possibly blow up for some finite r_0) and $a(x,t,s,\xi)$ is independent of s and linear with respect to ξ, existence and uniqueness of renormalized solution have been established in [9], and in the case where $b(x,u) = b(u)$ (where b is a maximal monotone graph on $\mathbb{R}$) and $a(x,t,s,\xi)$ is independent of t, existence and uniqueness of renormalized solution have been established in [8], (see also [7,15–17]).

The paper is organized as follows : Section 2 is devoted to specify the assumptions on b, $a(x,t,s,\xi)$, g, f and u_0 needed in the present study and to give the definition of a renormalized solution of 1.1, 1.2 and 1.3. In Section 3 (Theorem 3.1) we establish the existence of such a solution.

2. Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ $(N \geqslant 1)$ $T > 0$ is given and we set $Q = \Omega \times (0,T)$.

Hypothesis. $[H_1]$

$$b, \frac{\partial b}{\partial s} : \Omega \times \mathbb{R} \to \mathbb{R} \tag{2.1}$$

are Carathéodory functions such that, for almost every $x \in \Omega$, $b(x,s)$ is a strictly increasing C^1-function with $b(x,0) = 0$.

For any $K > 0$, there exist $\lambda_K > 0$ and a function A_K in $L^\infty(\Omega)$, such that

$$\lambda_K \leqslant \frac{\partial b(x,s)}{\partial s} \leqslant A_K(x), \tag{2.2}$$

for almost every $x \in \Omega$, for every s such that $|s| \leqslant K$.

For any $s \in \mathbb{R}$, the function $\frac{\partial b(x,s)}{\partial s}$ belongs to $L^1_{loc}(\Omega)$ and for any $K > 0$, there exists a function B_K in $L^p(\Omega)$ such that

$$\left|\nabla_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leqslant B_K(x), \tag{2.3}$$

for almost every $x \in \Omega$, for every s such that $|s| \leqslant K$.

Hypothesis. $[H_2]$

$$a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \tag{2.4}$$

is a Carathéodory function such that: for any $K > 0$, there exist $\beta_K > 0$ and a function C_K in $L^{p'}(\Omega)$ such that

$$a(x,t,s,\xi)\xi \geqslant \alpha|\xi|^p, \tag{2.5}$$

$$|a(x,t,s,\xi)| \leqslant C_K(x,t) + \beta_K|\xi|^{p-1}, \tag{2.6}$$

for almost every $(t,x) \in Q$, for every s such that $|s| \leqslant K$, and for every $\xi \in \mathbb{R}^N$.

$$[a(x,t,s,\xi) - a(x,t,s,\xi')][\xi - \xi'] > 0, \tag{2.7}$$

for any $s \in \mathbb{R}$, for any $(\xi, \xi') \in \mathbb{R}^{2N}$ and for almost every $(x,t) \in Q$.

Hypothesis. $[H_3]$

$$g : \mathbb{R} \to \mathbb{R} \text{ is a continuous function such that } \quad s\,g(s) \geqslant 0 \ \forall s \in \mathbb{R}. \tag{2.8}$$

$$f \text{ is an element of } L^1(Q) \text{ and } f \geqslant 0. \tag{2.9}$$

$$u_0 \text{ is an element of } L^1(\Omega) \text{ such that } u_0 \geqslant 0 \text{ and } b(x,u_0) \in L^1(\Omega). \tag{2.10}$$

Remark 2.1. As already mentioned in the introduction, Problem 1.1, 1.2 and 1.3 does not admit, in general a weak solution under assumptions (2.1)–(2.9) and (2.10) (even when $b(x,u) = u$), since the growth of $a(x,t,u,\nabla u)$ with respect to u is not controlled (so that the term $a(x,t,u,\nabla u)$ is not defined as a distribution in general, even when u belongs $L^p(0,T;W_0^{1,p}(\Omega))$.

Throughout the paper, T_K denotes the truncation function at height $K \geqslant 0$, $T_K(r) = min(K,max(r,-K))$. We denote by: $\theta_n(s) = T_{n+1}(s) - T_n(s)$.

The definition of a renormalized solution for Problem 1.1, 1.2 and 1.3 is given below.

Definition 2.2. A measurable function u defined on Q is a renormalized solution of Problem 1.1, 1.2 and 1.3 if

$$T_K(u) \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)\ \forall K \geqslant 0,\ u \geqslant 0 \text{ a.e., and } b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{2.11}$$

$$g(u)|\nabla u|^p \in L^1(Q), \tag{2.12}$$

$$\int_{\{(t,x)\in Q;\ n\leqslant u(x,t)\leqslant n+1\}} a(x,t,u,\nabla u)\nabla u dx\, dt \to 0 \text{ as } n \to +\infty, \tag{2.13}$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support, we have

$$\begin{aligned}&\frac{\partial b_S(x,u)}{\partial t} - \operatorname{div}(S'(u)a(x,t,u,\nabla u)) + S''(u)a(x,t,u,\nabla u)\nabla u\\ &\quad + S'(u)g(u)|\nabla u|^p\\ &\quad = fS'(u) \text{ in } D'(Q),\end{aligned} \tag{2.14}$$

$$b_S(x,u)(t=0) = b_S(x,u_0) \text{ in } \Omega, \tag{2.15}$$

where $b_S(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} S'(s)ds$.

Eq. (2.14) is formally obtained through pointwise multiplication of Eq. (1.1) by $S'(u)$. Recall that for a renormalized solution, due to (2.11), each term in (2.14) has a meaning in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$ (see e.g. [4–8]...).

We have

$$\frac{\partial b_S(x,u)}{\partial t} \text{ belongs to } L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q). \tag{2.16}$$

The properties of S, assumptions (2.2) and (2.3) imply that if K is such that *supp* $S' \subset [-K,K]$

$$|\nabla b_S(x,u)| \leqslant \|A_K\|_{L^\infty(\Omega)}|DT_K(u)|\|S'\|_{L^\infty(\mathbb{R})} + K\|S'\|_{L^\infty(\mathbb{R})}B_K(x) \tag{2.17}$$

and

$$b_S(x,u) \text{ belongs to } L^p\big(0,T;W_0^{1,p}(\Omega)\big). \tag{2.18}$$

Then (2.16) and (2.18) imply that $b_S(x,u)$ belongs to $C^0([0,T];L^1(\Omega))$ (for a proof of this trace result see [24]), so that the initial condition (2.15) makes sense.

Remark 2.3. For every $S \in W^{1,\infty}(\mathbb{R})$, nondecreasing function such that supp $S' \subset [-K,K]$, in view of (2.2) we have

$$\lambda_K|S(r) - S(r')| \leqslant |b_S(x,r) - b_S(x,r')| \leqslant \|A_K\|_{L^\infty(\Omega)}|S(r) - S(r')|, \tag{2.19}$$

for almost every $x \in \Omega$ and for every $r,\ r' \in \mathbb{R}$.

3. Existence result

This section is devoted to establish the following existence theorem.

Theorem 3.1. *Under assumptions 2.5,2.6,2.7,2.8,2.9 and 2.10 there exists at least a renormalized solution u of Problem 1.1,1.2 and 1.3.*

Proof. The proof is divided into 8 steps. In Step 1, we introduce an approximate problem. Step 2 is devoted to establish a few *a priori* estimates. In Step 3, the limit u of the approximate solutions u^ε is introduced with $b(x,u)$ belonging to $L^\infty(0,T;L^1(\Omega))$ and (2.11) is established. Then the main argument consists in proving the strong convergence of the truncations $T_K(u^\varepsilon)$ and this is done through a monotonicity method (as in [6–8,24],...). To this end, we define a time regularization $T_K(u)_\mu$ of the field $T_K(u)$ in Step 4 and we also state Lemma 3.2 which allows us to control the parabolic contribution that arises in this method. In Step 5, we deal with the elliptic terms by treating separately the positive part of $T_K(u^\varepsilon) - T_K(u)_\mu$. We prove an energy estimate (Lemma 3.3) which is a key point for the monotonicity arguments which are developed in Step 6. In Step 6 we prove the monotonicity estimate and the strong L^p convergence of $\nabla T_K(u^\varepsilon)$. In Step 7, we prove that u satisfies (2.13). At last, Step 8 is devoted to prove that u satisfies (2.14) and (2.15) of Definition 2.2

Step 1. For $\varepsilon > 0$ fixed, let us introduce the following regularizations of the data

$$b_\varepsilon(x,s) = b(x, T_{\frac{1}{\varepsilon}}(s))) + \varepsilon\, s \quad \text{a.e. in } \Omega,\ \forall s \in \mathbb{R}, \tag{3.1}$$

$$a_\varepsilon(x,t,s,\xi) = a(x,t,T_{\frac{1}{\varepsilon}}(s),\xi) \text{ a.e. in } Q,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{3.2}$$

$$g_\varepsilon(s) = g\left(T_{\frac{1}{\varepsilon}}(s)\right), \tag{3.3}$$

$$f^\varepsilon \in L^{p'}(Q),\ f^\varepsilon \geqslant 0 \text{ satisfies } f^\varepsilon \to f \text{ in } L^1(Q) \text{ as } \varepsilon \text{ tends to } 0, \tag{3.4}$$

$$u_0^\varepsilon \in C_0^\infty(\Omega),\ u_0^\varepsilon \geqslant 0 \text{ satisfies } b_\varepsilon\left(x,u_0^\varepsilon\right) \to b(x,u_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0. \tag{3.5}$$

Let us now consider the following regularized problem.

$$\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t} - \operatorname{div}(a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)) + g(u^\varepsilon)|\nabla u^\varepsilon|^p = f^\varepsilon \text{ in } Q, \tag{3.6}$$

$$u^\varepsilon = 0 \text{ on } (0,T) \times \partial\Omega, \tag{3.7}$$

$$b_\varepsilon(x,u^\varepsilon)(t=0) = b_\varepsilon\left(x,u_0^\varepsilon\right) \text{ in } \Omega. \tag{3.8}$$

In view of (3.1), b_ε satisfies (2.1) and due to (2.2), we have for $\varepsilon > 0$

$$\varepsilon \leqslant \frac{\partial b_\varepsilon(x,s)}{\partial s} \leqslant A_{\frac{1}{\varepsilon}}(x) + \varepsilon \qquad \text{and} \qquad \left|\nabla_x\left(\frac{\partial b_\varepsilon(x,s)}{\partial s}\right)\right| \leqslant B_{\frac{1}{\varepsilon}}(x) \text{ a.e. in } \Omega,$$
$$\forall s \in \mathbb{R}.$$

In view of (3.2), a_ε satisfies (2.5) and (2.7), and due to (2.6) we have

$$|a_\varepsilon(x,t,s,\xi)| \leqslant C_{\frac{1}{\varepsilon}}(x,t) + \beta_{\frac{1}{\varepsilon}}|\xi|^{p-1} \qquad \text{a.e. in } Q,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

As a consequence, with the above regularizations, proving existence of a weak solution $u^\varepsilon \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ and $u^\varepsilon \geqslant 0$ a.e. of 3.6, 3.7 and 3.8 follows from a straightforward adaptation of the techniques developed in [2,14].

Step 2. The estimates derived in this step rely on usual techniques for problems of type 3.6, 3.7 and 3.8 and we just sketch the proof of them (the reader is referred to [3–6,9,11] or to [12,22,23] for elliptic versions of 3.6, 3.7 and 3.8).

Using $T_K(u^\varepsilon)$ as a test function in (3.6) leads to

$$\begin{aligned}\int_\Omega b_K^\varepsilon(x,u^\varepsilon)(t)dx + \int_0^t \int_\Omega a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u^\varepsilon)dx\,ds \\ + \int_0^t \int_\Omega g_\varepsilon(u^\varepsilon)|\nabla u^\varepsilon|^p T_K(u^\varepsilon)dx\,ds \\ = \int_0^t \int_\Omega f^\varepsilon T_K(u^\varepsilon)dx\,ds + \int_\Omega b_K^\varepsilon(x,u_0^\varepsilon)dx,\end{aligned} \tag{3.9}$$

for almost every t in $(0,T)$, and where $b_K^\varepsilon(x,r) = \int_0^r T_K(s)\frac{\partial b_\varepsilon(x,s)}{\partial s}ds$. Since g_ε satisfies the sign condition, we have

$$\int_0^t \int_\Omega g_\varepsilon(u^\varepsilon)|\nabla u^\epsilon|^p T_K(u^\varepsilon)dx\,ds \geqslant 0, \tag{3.10}$$

for almost any $t \in (0,T)$.

Due to the definition of b_K^ε we have

$$0 \leqslant \int_\Omega b_K^\varepsilon\big(x,u_0^\varepsilon\big)dx \leqslant K\int_\Omega b_\varepsilon\big(x,u_0^\varepsilon\big)dx.$$

Since a_ε satisfies (2.5), the behaviors of f^ε and u_0^ε permit to deduce from (3.9) that

$$T_K(u^\varepsilon) \text{ is bounded in } L^p\big(0,T;W_0^{1,p}(\Omega)\big) \tag{3.11}$$

independently of ε for any $K \geqslant 0$.

Proceeding as in [4,9,5], we have for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support (supp $S' \subset [-K,K]$)

$$b_S^\varepsilon(x,u^\varepsilon) \text{ is bounded in } L^p\big(0,T;W_0^{1,p}(\Omega)\big) \tag{3.12}$$

and

$$\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega)) \tag{3.13}$$

independently of ε. Indeed, we have first

$$\begin{aligned}\big|\nabla b_S^\varepsilon(x,u^\varepsilon)\big| \leqslant \|A_K\|_{L^\infty(\Omega)}|DT_K(u^\varepsilon)|\|S'\|_{L^\infty(\mathbb{R})} \\ + K\|S'\|_{L^\infty(\mathbb{R})}B_K(x) \text{ a.e. in } Q.\end{aligned} \tag{3.14}$$

As a consequence of (3.11) and (3.14) we then obtain (3.12). To show that (3.13) holds true, we multiply the equation for u^ε in (3.6) by $S'(u^\varepsilon)$ to obtain

$$\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t} = \operatorname{div}(S'(u^\varepsilon)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)) - S''(u^\varepsilon)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon \\ - g_\varepsilon(u^\varepsilon)|\nabla u^\varepsilon|^p S'(u^\varepsilon) + f^\varepsilon S'(u^\varepsilon) \text{ in } D'(Q), \tag{3.15}$$

where $b_S^\varepsilon(x,r) = \int_0^r S'(s)\frac{\partial b_\varepsilon(x,s)}{\partial s}ds$. Since supp S' and supp S'' are both included in $[-K,K]$, u^ε may be replaced by $T_K(u^\varepsilon)$ in each of these terms. As a consequence, each term in the right hand side of (3.15) is bounded either in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ or in $L^1(Q)$ which shows that (3.13) holds true.

Now for fixed $K>0$: $a_\varepsilon(x,t,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) = a(x,t,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon))$ a.e. in Q as soon as $\varepsilon < \frac{1}{K}$, while assumption (2.6) gives

$$|a_\varepsilon(x,t,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon))| \leqslant C_K(t,x) + \beta_K|\nabla T_K(u^\varepsilon)|^{p-1},$$

where $\beta_K > 0$ and $C_K \in L^{p'}(Q)$. In view (3.11), we deduce that,

$$a_\varepsilon(x,t,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) \text{ is bounded in } (L^{p'}(Q))^N, \tag{3.16}$$

independently of ε for $\varepsilon < \frac{1}{K}$.

Step 3. Arguing again as in [4–7,9,24] estimates (3.12) and (3.13) imply that, for a subsequence still indexed by ε,

$$b_\varepsilon(x,u^\varepsilon) \text{ converges to } \chi \text{ almost every where in } Q. \tag{3.17}$$

Since b^{-1} is continuous (3.17) shows that (3.16),

$$u^\varepsilon \text{ converges to } u = b^{-1}(\chi) \text{ almost every where in } Q, \tag{3.18}$$

and (3.11) and (3.16) then give

$$T_K(u^\varepsilon) \text{ converges weakly to } T_K(u) \text{ in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.19}$$

$$\theta_n(u^\varepsilon) \rightharpoonup \theta_n(u) \text{ weaklyin } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.20}$$

$$a_\varepsilon(x,t,T_K(u^\varepsilon),DT_K(u^\varepsilon)) \rightharpoonup \sigma_K \text{ weakly in } (L^{p'}(Q))^N, \tag{3.21}$$

Under the sign condition on the function g, the fact that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$ is very standard as well as the following behavior of the energy (using the admissible test function $(T_{n+1}-T_n)(u^\varepsilon)$ in (3.6))

$$\lim_{n\to+\infty}\overline{\lim_{\varepsilon\to0}}\int_{\{n\leqslant u^\varepsilon\leqslant n+1\}} a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon dx\,dt = 0. \tag{3.22}$$

Step 4. This step is devoted to establish Lemma 3.2 below which is the original part of the present article. The estimates given in this lemma allows us to perform the monotonicity method which will be developed in Step 5 and Step 6. Let us notice that similar lemmas have been established in [8] (see Lemmas 2.1 and 2.3), where Stefan's type problems are investigated, but that here they cannot be used as such because of the term $g(u)|\nabla u|^p$ in Eq. (1.1).

For $K \geqslant 0$ fixed, we will use the now usual time regularization of the function $T_K(u)$ introduced in [20] (see Lemma 6 and Propositions 3 and 4) and more recently extensively exploited to solve a few nonlinear evolution problems with L^1 or measure data (see e.g. [10,18]).

Let $(v_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$v_0^\mu \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \text{ for all } \mu > 0, \tag{3.23}$$

$$\|v_0^\mu\|_{L^\infty(\Omega)} \leqslant K \ \forall \mu > 0, \tag{3.24}$$

$$v_0^\mu \to T_K(u_0) \text{ a.e. in } \Omega \text{ and } \frac{1}{\mu}\|v_0^\mu\|_{L^p(\Omega)} \to 0, \text{ as } \mu \to +\infty. \tag{3.25}$$

Existence of such a subsequence $(v_0^\mu)_\mu$ is easy to establish (see e.g., [19]). For fixed $K \geqslant 0$ and $\mu > 0$, let us consider the unique solution $T_K(u)_\mu \in L^\infty(Q) \cap L^p(0, T; W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial T_K(u)_\mu}{\partial t} + \mu(T_K(u)_\mu - T_K(u)) = 0 \text{ in } D'(Q), \tag{3.26}$$

$$T_K(u)_\mu(t=0) = v_0^\mu \text{ in } \Omega. \tag{3.27}$$

Remark that due to (3.26), we have for $\mu > 0$ and $K \geqslant 0$,

$$\frac{\partial T_K(u)_\mu}{\partial t} \in L^p(0, T; W_0^{1,p}(\Omega)). \tag{3.28}$$

The behavior of $T_K(u)_\mu$ as $\mu \to +\infty$ is investigated in [20] (see also [18,19]) and we just recall here that (3.26) and (3.27) imply that

$$T_K(u)_\mu \to T_K(u) \text{ a.e. in } Q \text{ in } L^\infty(Q) \text{ weak } \star \text{ and strongly in } L^p(0, T; W_0^{1,p}(\Omega)), \tag{3.29}$$

as $\mu \to +\infty$ with

$$\|T_K(u)_\mu\|_{L^\infty(Q)} \leqslant max\left(\|T_K(u)\|_{L^\infty(Q)};\ \|v_0^\mu\|_{L^\infty(\Omega)}\right) \leqslant K \tag{3.30}$$

for any μ and any $K \geqslant 0$.

We also introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_n such that

$$S_n(r) = r \text{ for } |r| \leqslant n,\ supp(S_n') \subset [-(n+1), n+1],\ \|S_n''\|_{L^\infty(\mathbb{R})} \leqslant 1,$$

for any $n \geqslant 1$. We recall that

$$b_{\varepsilon,S_n}(x,r) = \int_0^r \frac{\partial b_\varepsilon(x,s)}{\partial s} S_n'(s)ds \tag{3.31}$$

At last we will use the function $\Phi_\lambda(s) = se^{\lambda s^2}$ for $\lambda > 0$ which has been introduced in [13] in order to deal with term $g(u)|\nabla u|^p$. In what follows, we denote by $w(\varepsilon,\mu)$ and $w(\varepsilon,\mu,n)$ quantities such that

$$\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0} w(\varepsilon,\mu) = 0; \limsup_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0} w(\varepsilon,\mu,n) = 0.$$

The main estimates are given in the following lemma.

Lemma 3.2.

We have, for any $K > 0$ and any integer n such that $k < n + 1$

$$\int_0^T \left\langle \frac{\partial b_{\varepsilon,S_n}(x,u^\varepsilon)}{\partial t},\ (T-t)(T_K(u^\varepsilon) - T_K(u)_\mu)^+ \right\rangle dt \geqslant w(\varepsilon,\mu) \tag{3.32}$$

and for any $\lambda > 0$,

$$\int_0^T \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, (T-t)\Phi_\lambda(u^\varepsilon - T_K(u)_\mu)^- \right\rangle dt \leqslant w(\varepsilon,\mu) \tag{3.33}$$

where $\langle\ ,\ \rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$.

Proof. Let n and K be fixed such that $K < n + 1$. Defining $u^\varepsilon(t) = u_0^\varepsilon$ for $t < 0$ (see (3.5)), we have

$$\begin{aligned} X_{\varepsilon,\mu} &= \textstyle\int_0^T \left\langle \frac{b_{\varepsilon,S_n}(x,u^\varepsilon)}{\partial t}, (T-t)(T_K(u^\varepsilon) - T_K(u)_\mu)^+ \right\rangle dt \\ &= \lim_{l\to 0} \textstyle\int_0^T (T-t)\int_\Omega \left[\frac{b_{\varepsilon,S_n}(x,u^\varepsilon(t)) - b_{\varepsilon,S_n}(x,u^\varepsilon(t-l))}{l}\right] \times (T_K(u^\varepsilon) - T_K(u)_\mu)^+ dx\,dt. \end{aligned} \tag{3.34}$$

Now since $b_{\varepsilon,S_n}(x,.)$ is nondecreasing the following inequality holds true for all real numbers $z_1 \geqslant 0$ and $z_2 \geqslant 0$ and for a.e. t and x

$$\begin{aligned} &\textstyle\int_{z_1}^{z_2} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z} (T_K(z) - T_K(u)_\mu(t,x))^+ dz \\ &\quad \leqslant (b_{\varepsilon,S_n}(x,z_2) - b_{\varepsilon,S_n}(x,z_1))(T_K(z_2) - T_K(u)_\mu(t,x))^+. \end{aligned} \tag{3.35}$$

Then (3.34) gives

$$\begin{aligned} X_{\varepsilon,\mu} &\geqslant \frac{1}{l}\int_Q (T-t)\int_{u^\varepsilon(t-l)}^{u^\varepsilon(t)} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}(T_K(z) - T_K(u)_\mu(t,x))^+ dz\,dx\,dt + \omega(l) \\ &= Y_{\varepsilon,\mu,l} + \omega(l), \end{aligned} \tag{3.36}$$

where $\lim_{l\to 0}\omega(l) = 0$. In what follows we pass to the limsup in the right hand side of (3.36) as tends to 0; ε tends to 0 and μ tends to $+\infty$. To this end let us set for $t \in [0,T]$, $z \in \mathbb{R}$ and almost any $x \in \Omega$

$$F^{\varepsilon,\mu}(t,x,z)=\frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\left(T_K(z)-T_K(u)_\mu(t,x)\right)^+,$$

$$G^{\varepsilon,\mu}(s,t,x)=\int_0^{u^\varepsilon(s)}F^{\varepsilon,\mu}(t,x,z)dz,$$

and

$$H^{\varepsilon,\mu}(t,l,x)=\int_{t-l}^{t}G^{\varepsilon,\mu}(s,t)ds.$$

Remark that $F^{\varepsilon,\mu}(t,x,z)\geqslant 0$ so that since u^ε is nonnegative, $G^{\varepsilon,\mu}(s,t,x)\geqslant 0$ and $H^{\varepsilon,\mu}(t,l,x)\geqslant 0$. With these notations the definition of $Y_{\varepsilon,\mu,l}$ (see (3.36)) leads to

$$\begin{aligned}Y_{\varepsilon,\mu,l}&=\frac{1}{l}\int_\Omega (T-t)H^{\varepsilon,\mu}(t,l,x)dx-\frac{1}{l}\int_\Omega TH^{\varepsilon,\mu}(0,l,x)dx\\&\quad-\frac{1}{l}\int_Q\int_{t-l}^{t}(T-t)\frac{\partial G^{\varepsilon,\mu}}{\partial t}(s,t)ds\,dx\,dt\\&\geqslant-\frac{1}{l}\int_\Omega TH^{\varepsilon,\mu}(0,l,x)dx-\frac{1}{l}\int_Q\int_{t-l}^{t}(T-t)\frac{\partial G^{\varepsilon,\mu}}{\partial t}(s,t)ds\,dx\,dt.\end{aligned}\tag{3.37}$$

Now

$$H^{\varepsilon,\mu}(0,l,x)=\int_{-l}^{0}G^{\varepsilon,\mu}(s,t)ds=l\int_0^{u_0^\varepsilon}F^{\varepsilon,\mu}(t,x,z)dz$$

so that using the definition of F^ε and (3.35)

$$H^{\varepsilon,\mu}(0,l,x)\leqslant l\int_\Omega\left(b_{\varepsilon,S_n}(x,u_0^\varepsilon)-b_{\varepsilon,S_n}(x,0)\right)\left(T_K\left(u_0^\varepsilon\right)-T_K\left(v_0^\mu\right)\right)^+dx.\tag{3.38}$$

Due to the convergences of u_0^ε and v_0^μ to u_0 we obtain

$$\limsup_{l\to 0}\ \limsup_{\varepsilon\to 0}\ \limsup_{\mu\to+\infty}\left(-\frac{1}{l}H^{\varepsilon,\mu}(0,l,x)\right)\geqslant 0.\tag{3.39}$$

As far as the last term in (3.37) is concerned, we have

$$\begin{aligned}Z^{\varepsilon,\mu,l}&=\frac{1}{l}\int_Q\int_{t-l}^{t}(T-t)\frac{\partial G^{\varepsilon,\mu}}{\partial t}(s,t)ds\,dx\,dt\\&=\frac{1}{l}\int_Q(T-t)\int_{t-l}^{t}\int_0^{u^\varepsilon(s)}\frac{\partial F^{\varepsilon,\mu}}{\partial t}(t,x,z)dz\,dx\,dt\\&=\int_Q(T-t)\int_0^{u^\varepsilon(t)}\frac{\partial F^{\varepsilon,\mu}}{\partial t}(t,x,z)dz\,dx\,dt+\omega(l).\end{aligned}\tag{3.40}$$

Using the definition of $F^{\varepsilon,\mu}$ gives

$$Z^{\varepsilon,\mu,l} = -\int_Q (T-t)\int_0^{u^\varepsilon(t)} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\mathrm{sg}^+(T_K(z)-T_K(u)_\mu) \times \frac{\partial T_K(u)_\mu}{\partial t}dz\,dx\,dt + \omega(l), \tag{3.41}$$

and with the definition of $T_K(u)_\mu$ it follows that

$$Z^{\varepsilon,\mu,l} = \mu\int_Q (T-t)\int_0^{u^\varepsilon(t)} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\mathrm{sg}^+(T_K(z)-T_K(u)_\mu)(T_K(u)_\mu - T_K(u))dz\,dx\,dt + \omega(l). \tag{3.42}$$

Now since indeed $\frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z} = 0$ for $z > n+1$, $u^\varepsilon(t)$ can be replaced by $T_{n+1}(u^\varepsilon)(t)$ in the above expression of $Z^{\varepsilon,\mu,l}$ and this allows to pass to the limit as ε tends to 0. It gives

$$Z^{\varepsilon,\mu,l} = \mu\int_Q \int_0^{T_{n+1}(u(t))} \frac{\partial b_{S_n}(x,z)}{\partial z}\mathrm{sg}^+(T_K(z)-T_K(u)_\mu)(T_K(u)_\mu - T_K(u))dz\,dx\,dt + \omega(l,\varepsilon). \tag{3.43}$$

For $n+1 > K$, in the above integral the integrand is equal to 0 except for the (z,x,t)'s such that $T_K(u)_\mu(t,x) < T_K(z) \leqslant T_K(T_{n+1}(u(x,t))) = T_K((u(x,t))$ in which case it is negative. As a consequence, we get for $n+1 > K$ and for any $\mu > 0$

$$\limsup_{l\to 0}\limsup_{\varepsilon\to 0}\left(-\frac{1}{l}\int_Q (T-t)\int_{t-l}^{t} \frac{\partial G^{\varepsilon,\mu}}{\partial t}(s,t)ds\,dx\,dt\right) \geqslant 0. \tag{3.44}$$

The definition (3.34) of $X^{\varepsilon,\mu}$ together with the inequalities 3.36, 3.37, 3.39 and 3.44 show that (3.32) holds true.

We carry on by proving (3.33). We have

$$U^{\varepsilon,\mu} = \int_0^T \left\langle \frac{\partial b_{\varepsilon,S_n}(x,u^\varepsilon)}{\partial t}, (T-t)\Phi_\lambda\left(u^\varepsilon - T_K(u)_\mu\right)^-\right\rangle dt = \int_0^T\int_\Omega (T-t)\left[\frac{b_{\varepsilon,S_n}(x,u^\varepsilon(t)) - b_{\varepsilon,S_n}(x,u^\varepsilon(t-l))}{l}\right]\Phi_\lambda(u^\varepsilon - T_K(u)_\mu)^-\,dx\,dt + \omega(l). \tag{3.45}$$

Since the function $\Phi_\lambda(z - T_K(u)_\mu)^-$ is non-increasing with respect to z, the following inequality holds true for all real numbers $z_1 \geqslant 0$ and $z_2 \geqslant 0$ and for a.e. t and x

$$\int_{z_1}^{z_2} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\Phi_\lambda(z - T_K(u)_\mu(x,t))^-\,dz \geqslant (b_{\varepsilon,S_n}(x,z_2) - b_{\varepsilon,S_n}(x,z_1))\Phi_\lambda(z_2 - T_K(u)_\mu(x,t))^-,$$

so that (3.45) gives

$$U^{\varepsilon,\mu} \leqslant \frac{1}{l}\int_Q (T-t)\int_{u^\varepsilon(t-l)}^{u^\varepsilon(t)} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\Phi_\lambda(z - T_K(u)_\mu(x,t))^- \, dz\, dx\, dt$$
$$+\,\omega(l)$$
$$= V_{\varepsilon,\mu,l} + \omega(l), \tag{3.46}$$

In what follows we pass to the limsup in the right hand side of (3.46) as tends to 0, ε tends to 0 and μ tends to $+\infty$. To this end let us set for $t \in [0,T]$, $z \in \mathbb{R}$ and almost any $x \in \Omega$

$$R^{\varepsilon,\mu}(t,x,z) = \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\Phi_\lambda(z - T_K(u)_\mu(x,t))^-,$$

$$T^{\varepsilon,\mu}(s,t,x) = \int_{T_K(u)_\mu(t)}^{u^\varepsilon(s)} R^{\varepsilon,\mu}(t,x,z)dz,$$

and

$$Z^{\varepsilon,\mu}(t,l,x) = \int_{t-l}^{t} T^{\varepsilon,\mu}(s,t,x)ds.$$

With these notations the definition of $V_{\varepsilon,\mu,l}$ (see (3.36)) leads to

$$V_{\varepsilon,\mu,l} = \frac{1}{l}\int_0^T (T-t)\int_\Omega \int_{T_K(u)_\mu(t)}^{u^\varepsilon(t)} R^{\varepsilon,\mu}(t,x,z)dz\, dx\, dt$$
$$-\frac{1}{l}\int_0^T (T-t)\int_\Omega \int_{T_K(u)_\mu(t)}^{u^\varepsilon(t-l)} R^{\varepsilon,\mu}(t,x,z)dz\, dx\, dt$$
$$= \frac{1}{l}\int_0^T (T-t)\int_\Omega T^{\varepsilon,\mu}(t,t,x)dx\, dt$$
$$-\frac{1}{l}\int_0^T (T-t)\int_\Omega T^{\varepsilon,\mu}(t-l,t,x)dx\, dt, \tag{3.47}$$

or equivalently

$$V_{\varepsilon,\mu,l} = \frac{1}{l}\int_Q (T-t)\frac{\partial Z^{\varepsilon,\mu}}{\partial t}(t,l,x)dx\, dt - \frac{1}{l}\int_Q \int_{t-l}^{t} \frac{\partial T^{\varepsilon,\mu}}{\partial t}(s,t,x)ds\, dx\, dt. \tag{3.48}$$

Integrating by parts the first term in (3.48) gives

$$V_{\varepsilon,\mu,l} = \frac{1}{l}\int_Q Z^{\varepsilon,\mu}(t,l,x)dx\, dt - \frac{T}{l}\int_Q Z^{\varepsilon,\mu}(0,l,x)dx\, dt$$
$$-\frac{1}{l}\int_Q \int_{t-l}^{t} \frac{\partial T^{\varepsilon,\mu}}{\partial t}(s,t,x)ds\, dx\, dt. \tag{3.49}$$

According to the definition of $R^{\varepsilon,\mu}$, $T^{\varepsilon,\mu}$ and $Z^{\varepsilon,\mu}$ we obtain using also $\Phi_\lambda(0) = 0$,

$$
\begin{aligned}
V_{\varepsilon,\mu,l} = {} & \frac{1}{l}\int_Q \int_{t-l}^{t} T^{\varepsilon,\mu}(s,t,x)ds\,dx\,dt - T\int_\Omega \int_{v_0^\mu}^{u_0^\varepsilon} R^{\varepsilon,\mu}(0,x,z)dz\,dx \\
& - \frac{1}{l}\int_Q \int_{t-l}^{t} \int_{T_K(u)_\mu(t)}^{u^\varepsilon(s)} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\Phi_\lambda(z - T_K(u)_\mu(x,t))^- \chi_{\{z\leqslant T_K(u)_\mu(x,t)\}} \\
& \times \frac{\partial T_K(u)_\mu}{\partial t}(t,x)ds\,dx\,dt
\end{aligned} \tag{3.50}
$$

As far as the first term in (3.50) is concerned, we have (using again $\Phi_\lambda(z - T_K(u)_\mu)^- = 0$ if $z > K$)

$$
\begin{aligned}
\frac{1}{l}\int_Q \int_{t-l}^{t} T^{\varepsilon,\mu}(s,t,x)ds\,dx\,dt = {} & \frac{1}{l}\int_Q \int_{t-l}^{t} \\
& \times \int_{T_K(u)_\mu(t)}^{T_K(u^\varepsilon(s))} \partial b_{\varepsilon,S_n}(x,z)\partial z \Phi_\lambda\Big(z - T_K(u)_\mu(x,t)\Big)^- ds\,dx\,dt \\
= {} & \int_Q \int_{T_K(u)_\mu(t)}^{T_K(u^\varepsilon(t))} \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\Phi_\lambda(z - T_K(u)_\mu(x,t))^- dx\,dt + \omega(l).
\end{aligned}
$$

Due to the strong convergence of $T_K(u^\varepsilon)$ to $T_K(u)$ (i.e. in $L^1(Q)$) as ε tends to 0, to the strong convergence of $T_K(u)_\mu$ to $T_K(u)$ as μ tends to infinity and to the uniform bounded character of $\frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}$ with respect to ε, it follows that

$$
\frac{1}{l}\int_Q \int_{t-l}^{t} T^{\varepsilon,\mu}(s,t,x)ds\,dx\,dt = \omega(l,\varepsilon,\mu). \tag{3.51}
$$

Similarly for the second term in (3.50), the strong convergence of $T_K(u_0^\varepsilon)$ to $T_K(u_0)$ (i.e. in $L^1(Q)$) as ε tends to 0 and the strong convergence of v_0^μ to u_0 in $L^1(Q)$ as μ tends to $+\infty$ gives

$$
\int_\Omega \int_{v_0^\mu}^{u_0^\varepsilon} R^{\varepsilon,\mu}(0,x,z)dz\,dx = \omega(\varepsilon,\mu). \tag{3.52}
$$

In view of the definition of $T_K(u)_\mu$, the third term in (3.50) is equal to

$$
\begin{aligned}
W_{\varepsilon,\mu,l} = {} & -\frac{1}{l}\int_Q \int_{t-l}^{t} \int_{T_K(u)_\mu(t)}^{T_K(u^\varepsilon(s))} \\
& \times \frac{\partial b_{\varepsilon,S_n}(x,z)}{\partial z}\Phi'_\lambda(z - T_K(u)_\mu(x,t))^- \chi_{\{z\leqslant T_K(u)_\mu(x,t)\}}\mu(T_K(u) - T_K(u)_\mu) \\
& \times (t,x)ds\,dx\,dt.
\end{aligned}
$$

Passing to the limit as l tends to 0 and then as ε tends to 0 gives

$$
\begin{aligned}
W_{\varepsilon,\mu,l} = {} & -\int_Q \int_{T_K(u)_\mu(t)}^{T_K(u(t))} \frac{\partial b_{S_n}(x,z)}{\partial z}\Phi'_\lambda(z - T_K(u)_\mu(x,t))^- \chi_{\{z\leqslant T_K(u)_\mu(x,t)\}}\mu(T_K(u) \\
& - T_K(u)_\mu)(t,x)ds\,dx\,dt + \omega(l,\varepsilon).
\end{aligned}
$$

Since $b_{S_n}(x,.)$ and Φ_λ are nondecreasing functions, we have

$$W_{\varepsilon,\mu,l} \leqslant \omega(l,\varepsilon). \tag{3.53}$$

Gathering together 3.45, 3.46, 3.50, 3.51 and 3.52 finally shows that (3.33) holds true.
□

Step 5. In this step we prove the following Lemma which is the key point for the monotonicity arguments that are developed in Step 6.

Lemma 3.3. *The subsequence of u^ε defined in Step 3 satisfies for any $K \geqslant 0$*

$$\int_Q (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla(T_K(u^\varepsilon)-T_K(u)_\mu)^+ dx\,dt \leqslant w(\varepsilon,\mu) \tag{3.54}$$

and

$$\int_{\{u^\varepsilon \leqslant T_K(u)_\mu\}} (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla(u^\varepsilon - T_K(u)_\mu)dx\,dt = w(\varepsilon,\mu) \tag{3.55}$$

Proof. For $K > 0$, we choose $W^\varepsilon = (T-t)(T_K(u^\varepsilon)-T_K(u)_\mu)^+$ as a test function in (3.6), we obtain

$$\int_0^T \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t},\ W^\varepsilon \right\rangle dt + \int_0^T \int_\Omega a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla W^\varepsilon dx\,dt$$
$$+ \int_0^T \int_\Omega g_\varepsilon(u^\varepsilon) W^\varepsilon |\nabla u^\varepsilon|^p\, dx\,dt = \int_0^T \int_\Omega f^\varepsilon W^\varepsilon\, dx\,dt. \tag{3.56}$$

We use (3.32) and since $g_\varepsilon(u^\varepsilon)$ is positive, we easily obtain

$$\int_Q (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla(T_K(u^\varepsilon)-T_K(u)_\mu)^+\, dx\,dt$$
$$\leqslant \int_Q f^\varepsilon W^\varepsilon\, dx\,dt + w(\varepsilon,\mu). \tag{3.57}$$

Thanks to 3.4, 3.18, 3.29 and 3.57 we obtain (3.54).

Let us prove (3.55). Using $(T-t)\Phi_\lambda(u^\varepsilon - T_K(u)_\mu)^-$ as a test function in (3.6) gives

$$\int_0^T \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t},\ (T-t)\Phi_\lambda\Big(u^\varepsilon - T_K(u)_\mu\Big)^- \right\rangle dt$$
$$+ \int_Q (T-t)a_\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\Phi_\lambda(u^\varepsilon - T_K(u)_\mu)^-\, dx\,dt$$
$$+ \int_Q (T-t)g_\varepsilon(u^\varepsilon)\Phi_\lambda\Big(u^\varepsilon - T_K(u)_\mu\Big)^- |\nabla u^\varepsilon|^p\, dx\,dt$$
$$= \int_Q (T-t)f^\varepsilon \Phi_\lambda(u^\varepsilon - T_K(u)_\mu)^-\, dx\,dt. \tag{3.58}$$

Upon applying (3.33) of Lemma 3.2 to the first term of (3.58), we obtain

$$\begin{aligned}
&\int_{\{u^\varepsilon\leqslant T_K(u)_\mu\}}(T-t)a_\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla(u^\varepsilon-T_K(u)_\mu)\Phi'_\lambda(u^\varepsilon-T_K(u)_\mu)^-\,dx\,dt\\
&\qquad\leqslant\int_Q(T-t)g_\varepsilon(u^\varepsilon)\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-|\nabla u^\varepsilon|^p\,dx\,dt\\
&\qquad-\int_Q(T-t)f^\varepsilon\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-\,dx\,dt+w(\varepsilon,\mu)\\
&\qquad\leqslant\frac{\rho_K}{\alpha}\int_Q(T-t)\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-a_\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon\,dx\,dt+w(\varepsilon,\mu),
\end{aligned}\tag{3.59}$$

for $K\leqslant\frac{1}{\varepsilon}$ and where $\rho_K=\max_{0\leqslant s\leqslant K}(g(s))$. It follows that

$$\begin{aligned}
&\int_{\{u^\varepsilon\leqslant T_K(u)_\mu\}}(T-t)(a_\varepsilon(x,\nabla u^\varepsilon)-a_\varepsilon(x,\nabla T_K(u)_\mu))\nabla(u^\varepsilon-T_K(u)_\mu)\Big(\Phi'_\lambda-\frac{\rho_K}{\alpha}\Phi_\lambda\Big)dx\,dt\\
&\qquad=\int_{\{u^\varepsilon\leqslant T_K(u)_\mu\}}(T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla(u^\varepsilon-T_K(u)_\mu)^-\Phi'_\lambda(u^\varepsilon-T_K(u)_\mu)^-\,dx\,dt\\
&\qquad-\frac{\rho_K}{\alpha}\int_Q(T-t)\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon\,dx\,dt\\
&\qquad+\frac{\rho_K}{\alpha}\int_Q(T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u)_\mu\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-\,dx\,dt\\
&\qquad+\int_Q(T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla T_K(u)_\mu)\nabla(u^\varepsilon-T_K(u)_\mu)^-\Big(\Phi'_\lambda-\frac{\rho_K}{\alpha}\Phi_\lambda\Big)dx\,dt.
\end{aligned}\tag{3.60}$$

For fixed μ, the sequence

$$a_\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-=a_\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon))\Phi_\lambda\Big(T_K(u^\varepsilon)-T_K(u)_\mu\Big)^-,$$

weakly converges in $L^{p'}(Q)^N$, as ε tends to zero so that (using also (3.18))

$$\begin{aligned}
&\frac{\rho_K}{\alpha}\int_Q(T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u)_\mu\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-\,dx\,dt\\
&\quad=\frac{\rho_K}{\alpha}\int_Q(T-t)a(x,T_K(u),\nabla T_K(u))\nabla T_K(u)_\mu\Phi_\lambda(T_K(u)-T_K(u)_\mu)^-\,dx\,dt+w(\varepsilon).
\end{aligned}\tag{3.61}$$

Because $T_K(u)_\mu$ converges to $T_K(u)$ strongly in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ and almost everywhere in Q as μ tends to infinity, we obtain

$$\frac{\rho_K}{\alpha}\int_Q(T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u)_\mu\Phi_\lambda(u^\varepsilon-T_K(u)_\mu)^-\,dx\,dt=w(\varepsilon,\mu).\tag{3.62}$$

Observe that

$$a_\varepsilon(x,t,u^\varepsilon,\nabla T_K(u)_\mu)\nabla(u^\varepsilon - T_K(u)_\mu)^- = a_\varepsilon(x,t,T_K(u^\varepsilon),\nabla T_K(u)_\mu)\nabla(T_K(u^\varepsilon) - T_K(u)_\mu)^-,$$

a.e. in Q, and weakly converges in $L^1(Q)$, as ε tends to zero, and the sequence $\left(\Phi'_\lambda - \frac{\rho_K}{\alpha}\Phi_\lambda\right)(T_K(u^\varepsilon) - T_K(u)_\mu)$ is uniformly bounded with respect to ε and converges a.e. in Q to $\left(\Phi'_\lambda - \frac{\rho_K}{\alpha}\Phi_\lambda\right)(T_K(u) - T_K(u)_\mu)$, we have

$$\int_Q (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla T_K(u)_\mu)\nabla(u^\varepsilon - T_K(u)_\mu)^-\left(\Phi'_\lambda - \frac{\rho_K}{\alpha}\Phi_\lambda\right)dx\,dt$$
$$= \int_Q (T-t)a(x,T_K(u),\nabla T_K(u)_\mu)\nabla(T_K(u) - T_K(u)_\mu)^-\left(\Phi'_\lambda - \frac{\rho_K}{\alpha}\Phi_\lambda\right)dx\,dt + w(\varepsilon), \tag{3.63}$$

because $T_K(u)_\mu$ converges to $T_K(u)$ strongly in $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ and almost everywhere in Q as μ tends to infinity, we obtain

$$\int_Q (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla T_K(u)_\mu)\nabla(u^\varepsilon - T_K(u)_\mu)^-\left(\Phi'_\lambda - \frac{\rho_K}{\alpha}\Phi_\lambda\right)dx\,dt = w(\varepsilon,\mu). \tag{3.64}$$

As a consequence of 3.59, 3.60, 3.62 and 3.64 we are in a position to deduce that

$$\int_{\{u^\varepsilon \leqslant T_K(u)_\mu\}} (T-t)(a_\varepsilon(x,\nabla u^\varepsilon) - a_\varepsilon(x,\nabla T_K(u)_\mu))\nabla(u^\varepsilon - T_K(u)_\mu)\left(\Phi'_\lambda - \frac{\rho_K}{\alpha}\Phi_\lambda\right)dx\,dt$$
$$\leqslant w(\varepsilon,\mu), \tag{3.65}$$

Choosing λ large enough so that $\left(\Phi'_\lambda(s) - \frac{\rho_K}{\alpha}\Phi_\lambda(s)\right) \geqslant \frac{1}{2}$ for every $s \in \mathbb{R}$, and we use the weak convergence of $T_K(u^\varepsilon)$, the strong convergence of $T_K(u)_\mu$ in $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$, we obtain

$$\frac{1}{2}\int_{\{u^\varepsilon \leqslant T_K(u)_\mu\}} (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla(u^\varepsilon - T_K(u)_\mu)dx\,dt$$
$$\leqslant \int_{\{u^\varepsilon \leqslant T_K(u)_\mu\}} (T-t)(a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon) - a_\varepsilon(x,u^\varepsilon,\nabla T_K(u)_\mu))\nabla(u^\varepsilon - T_K(u)_\mu)$$
$$\times\left(\Phi'_\lambda - \frac{\rho_K}{\alpha}\Phi_\lambda\right)dx\,dt = w(\varepsilon,\mu) \qquad \square \tag{3.66}$$

Step 6. In this step we prove the following monotonicity estimate and the strong $(L^p(Q))^N$ convergence of $\nabla T_K(u^\varepsilon)$ as ε tends to 0:

Lemma 3.4. *The subsequence of u^ε defined in Step 3 satisfies for any $K \geqslant 0$*

$$\overline{\lim_{\varepsilon\to 0}}\int_Q (T-t)a(x,u^\varepsilon,\nabla T_K(u^\varepsilon))\nabla T_K(u^\varepsilon)dx\,dt \leqslant \int_Q (T-t)\sigma_K\nabla T_K(u)dx\,dt, \tag{3.67}$$

$$\lim_{\varepsilon\to 0}\int_Q (T-t)[a(x,\nabla T_K(u^\varepsilon)) - a(x,\nabla T_K(u))][\nabla T_K(u^\varepsilon) - \nabla T_K(u)]dx\,dt = 0, \tag{3.68}$$

and

$$T_K(u^\varepsilon) \to T_K(u) \text{ strongly in } L^p\big(0,T;W_0^{1,p}(\Omega)\big). \tag{3.69}$$

Proof. Because $(u^\varepsilon - T_K(u)_\mu)^- = (T_K(u^\varepsilon) - T_K(u)_\mu)^-$, we can write

$$\begin{aligned}&\int_Q (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla T_K(u^\varepsilon))\nabla(T_K(u^\varepsilon) - T_K(u)_\mu)dx\,dt\\&\quad= \int_Q (T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla T_K(u^\varepsilon))\nabla(T_K(u^\varepsilon) - T_K(u)_\mu)^+dx\,dt\\&\qquad+\int_{\{u^\varepsilon\leqslant T_K(u)_\mu\}}(T-t)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla(u^\varepsilon - T_K(u)_\mu)dx\,dt\end{aligned} \tag{3.70}$$

Thanks to 3.21, 3.54 and 3.55 and since $T_K(u)_\mu$ strongly converges to $T_K(u)$ in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$, then it is possible to conclude (3.67).

The monotone character of $a(x,t,s,\xi)$ together with the definition of σ_K and (3.67) allow to conclude through the usual monotonicity argument that (3.68) holds true.

From (3.68) and due to the strict monotonicity of $a(x,t,s,\xi)$, then it possible to conclude (3.69) (see Lemma 5 [13] and Lemma 4 [6]). □

Step 7. In this step we prove that u satisfies (2.13). To this end, remark that for any fixed $n \geqslant 0$ one has

$$\begin{aligned}&\int_{\{(t,x)/n\leqslant|u^\varepsilon|\leqslant n+1\}} a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon\,dx\,dt\\&= \int_Q a_\varepsilon(x,t,u^\varepsilon,Du^\varepsilon)[\nabla T_{n+1}(u^\varepsilon) - \nabla T_n(u^\varepsilon)]dx\,dt\\&= \int_Q a(x,t,T_{n+1}(u^\varepsilon),\nabla T_{n+1}(u^\varepsilon))\nabla T_{n+1}(u^\varepsilon)dx\,dt\\&- \int_Q a(x,t,T_n(u^\varepsilon),\nabla T_n(u^\varepsilon))\nabla T_n(u^\varepsilon)dx\,dt,\end{aligned}$$

for $\varepsilon < \frac{1}{(n+1)}$. According to (3.69), one is at liberty to pass to the limit as ε tends to 0 for fixed $n \geqslant 0$ and to obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\{(t,x)/n\leqslant|u^\varepsilon|\leqslant n+1\}} a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon\,dx\,dt\\&\quad= \int_Q a(x,t,T_{n+1}(u),\nabla T_{n+1}(u))\nabla T_{n+1}(u)dx\,dt\\&\qquad- \int_Q a(x,t,T_n(u),\nabla T_n(u))\nabla T_n(u)dx\,dt\\&\quad= \int_{\{(t,x)/n\leqslant|u|\leqslant n+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt.\end{aligned} \tag{3.71}$$

Taking the limit as n tends to $+\infty$ in (3.71) and using the estimate (3.22) show that u satisfies (2.13).

Step 8. In this step, u is shown to satisfy (2.14) and (2.15). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that supp $S' \subset [-K,K]$. Pointwise multiplication of the approximate Eq. (3.6) by $S'(u^\varepsilon)$ leads to

$$\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t} - \operatorname{div}(S'(u^\varepsilon)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)) + S''(u^\varepsilon)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)Du^\varepsilon + S'(u^\varepsilon)g_\varepsilon(u^\varepsilon)|\nabla u^\varepsilon|^p = f^\varepsilon S'(u^\varepsilon) \text{ in } D'(Q), \tag{3.72}$$

where $b_S^\varepsilon(x,r) = \int_0^r \frac{\partial b_\varepsilon(x,s)}{\partial s} S'(s)ds$.

In what follows we pass to the limit as ε tends to 0 in each term of (3.72).

☆*Limit of* $\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}$. Since S is bounded, and $b_S^\varepsilon(x,u^\varepsilon)$ converges to $b_S(x,u)$ a.e. in Q and in $L^\infty(Q)$ weak ☆, then $\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}$ converges to $\frac{\partial b_S(x,u)}{\partial t}$ in $D'(Q)$ as ε tends to 0.

☆*Limit of* $-\operatorname{div}(S'(u^\varepsilon)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon))$. Since supp $S' \subset [-K,K]$, we have for $\varepsilon < \frac{1}{K}$: $S'(u^\varepsilon)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon) = S'(u^\varepsilon)a_\varepsilon(x,t,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon))$ a.e. in Q.

The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (3.69) of Lemma 3.4 imply that $S'(u^\varepsilon)a_\varepsilon(x,t,T_K(u^\varepsilon),\ \nabla T_K(u^\varepsilon))$ converges to $S'(u)a(x,t,T_K(u),\nabla T_K(u))$ weakly in $(L^{p'}(Q))^N$, as ε tends to 0, because $S'(u)=0$ for $|u| \geqslant K$ a.e. in Q, and the term $S'(u)a(x,t,T_K(u),\nabla T_K(u)) = S'(u)a(x,t,u,\nabla u)$ a.e. in Q.

☆*Limit of* $S''(u^\varepsilon)\ a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon$. Since supp $S'' \subset [-K,K]$, we have for $\varepsilon \leqslant \frac{1}{K}$: $S''(u^\varepsilon)a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon = S''(u^\varepsilon)a_\varepsilon(x,t,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon))\nabla T_K(u^\varepsilon)$ a.e. in Q.

The pointwise convergence of $S''(u^\varepsilon)$ to $S''(u)$ as ε tends to 0, the bounded character of S'' and (3.69) of Lemma 3.4 allow to conclude that $S''(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon$ converges to $S''(u)\ a(x,t,T_K(u),\nabla T_K(u))\nabla T_K(u)$ weakly in $L^1(Q)$, as ε tends to 0, and $S''(u)\ a(x,t,T_K(u),\nabla T_K(u))\nabla T_K(u) = S''(u)a(x,t,u,\nabla u)\nabla u$ a.e. in Q.

☆*Limit of* $S'(u^\varepsilon)g_\varepsilon(u^\varepsilon)|\nabla u^\varepsilon|^p$. The pointwise convergence of $S'(u^\varepsilon)$ to $S'(u)$ as ε tends to 0, the bounded character of S and (3.69) of Lemma 3.4 allow to conclude that $S'(u^\varepsilon)g_\varepsilon(T_K(u^\varepsilon))|\nabla T_K(u^\varepsilon)|^p$ converges to $S'(u)g(T_K(u))|\nabla T_K(u)|^p$ strongly in $L^1(Q)$, as ε tends to 0.

☆ *Limit of* $f^\varepsilon S'(u^\varepsilon)$. Due to (3.4) and (3.18), we have $f^\varepsilon\ S'(u^\varepsilon)$ converges to $f\ S'(u)$ strongly in $L^1(Q)$, as ε tends to 0.

As a consequence of the above convergence result, we are in a position to pass to the limit as ε tends to 0 in Eq. (3.72) and to conclude that u satisfies (2.14).

It remains to show that $b_S(x,u)$ satisfies the initial condition (2.15). To this end, firstly remark that, S' has a compact support, we have $b_S^\varepsilon(x,u^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (3.72) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type Lemma (see e.g., [28, Corollary 4]) implies that $b_S^\varepsilon(x,u^\varepsilon)$ lies in a compact set of $C^0([0,T];W^{-1,s}(\Omega))$ for any $s<inf\left(p',\frac{N}{N-1}\right)$. It follows that, on one hand, $b_S^\varepsilon(x,u^\varepsilon)(t=0)$converges to $b_S(x,u)(t=0)$ strongly in $W^{-1,s}(\Omega)$. On the order hand, the smoothness of S implies that $b_S^\varepsilon(x,u^\varepsilon)(t=0)$ converges to $b_S(x,u)(t=0)$ strongly in $L^q(\Omega)$ for all $q<+\infty$. Due to (3.5), we conclude that $b_S^\varepsilon(x,u^\varepsilon)(t=0)=b_S^\varepsilon(x,u_0^\varepsilon)$ converges to $b_S(x,u)(t=0)$ strongly in $L^q(\Omega)$. Then we conclude that $b_S(x,u)(t=0)=b_S(x,u_0)$ in Ω.

As a conclusion of Steps 3, 7 and 8, the proof of Theorem 3.1 is complete. □

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