Exact travelling wave solutions of nonlinear evolution equations by using the $\left(\frac{G'}{G}\right)$-expansion method

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Abstract   In this work, we established abundant travelling wave solutions for some nonlinear evolution equations. The $\left(\frac{G'}{G}\right)$-expansion method was used to construct travelling wave solutions of nonlinear evolution equations. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. This method presents a wider applicability for handling nonlinear wave equations.

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1. Introduction

The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerfull methods, such as inverse scattering method [1,19], Hirota bilinear tranformation [11,23], the tanh–sech method [13,22,25,17], sine–cosine method [21,4] and Exp-function method [5,14,28,9] were used to develop nonlinear dispersive and dissipative problems.

The pioneer work Wang et al. [20] introduced the $\left(\frac{G'}{G}\right)$-expansion method for a reliable treatment of the nonlinear wave equations. The useful $\left(\frac{G'}{G}\right)$-expansion method is widely used by many such as in [29,3,30,2,26 and by the reference therein].

Our first interest in the present work is in implementing the $\left(\frac{G'}{G}\right)$-expansion method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. In Section 2, we describe this method for finding exact travelling wave solutions of nonlinear evolution equations. In Sections 3–5, we illustrate this method in detail with the celebrated the foam drainage, $(2+1)$-dimensional breaking soliton and $(3+1)$-dimensional Kadomstev–Petviashvili equations. In Section 6, some conclusions are given.

2. The $\left(\frac{G'}{G}\right)$-expansion method

Wang has summarized the $\left(\frac{G'}{G}\right)$-expansion method.

 Step 1. A PDE

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0. \tag{2.1}$$

can be converted to on ODE

$$Q(U, U', U'', U''', \ldots) = 0. \tag{2.2}$$

upon using a wave variable $u(x,t) = U(\xi), \xi = x - ct$. Eq. (2.2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

 Step 2. Suppose that the solution of ODE (2.2) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{2.3}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{2.4}$$

where $G' = \frac{dG(\xi)}{d\xi}$, $G'' = \frac{dh^2 G(\xi)}{d\xi^2}$. $a_0, \ldots, a_m$, λ and μ are constants to be determined later, $a_m \neq 0$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and highest order nonlinear terms appearing in ODE (2.2).

Step 3. By substituting (2.3) into (2.4) and using second order of LODE (2.4), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq. (2.2) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_0, \ldots, a_m$, λ, c and μ by using Maple.

Step 4. Assuming that the constants $a_0, \ldots, a_m, \lambda, c$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (2.4) have been well known for us, then substituting $a_0, \ldots, a_m, c$ and the general solutions of Eq. (2.4) into (2.3) we have more travelling wave solutions of the nonlinear evolution Eq. (2.1).

3. The foam drainage equation

Consider the foam drainage equation [8]

$$u_t + \left(u^2 - \frac{\sqrt{u}}{2} u_x\right)_x = 0, \tag{3.1}$$

where x and t are scaled position and time coordinates, respectively. In this paper, we show the effectiveness and convenience of the method by obtaining the exact solution of Eq. (3.1). Foam is central to a number of everyday activities, both natural and industrial. As such foam has been of great interest for academic research. In the process industries, foam can be a desirable and even essential element of a process. An example is in the case of the froth flotation separation of minerals and coal [18]. Foaming occurs in many distillation and absorption processes. Foams are very important in many technological processes and applications. Their properties are subject to intensive investigational efforts from both practical developers and scientific researchers [10].

Using the wave variable $\xi = k(x + ct)$, the Eq. (3.1) is carried to an ODE

$$cku' + k\left(u^2 - \frac{k}{2}\sqrt{u}u'\right)' = 0. \tag{3.2}$$

Integrating (3.2) with respect to ξ and considering the zero constants for intergation we obtain

$$cku + k\left(u^2 - \frac{k}{2}\sqrt{u}u'\right) = 0, \tag{3.3}$$

then we use the transformation

$$u(\xi) = v^2(\xi), \tag{3.4}$$

that will convert Eq. (3.3) to

$$kcv^2 + kv^4 - k^2v^2v' = 0, \tag{3.5}$$

or equivalently

$$c + v^2 - kv' = 0, \tag{3.6}$$

where the prime denotes differentiation with respect to ξ. Balancing v' with v^2 in (3.6) gives

$$m + 1 = 2m, \tag{3.7}$$

so that

$$m = 1. \tag{3.8}$$

Suppose that the solutions of (3.6) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$v(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{3.9}$$

By using Eq. (2.4), from Eq. (3.9) we have

$$v'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu, \tag{3.10}$$

and

$$v^2(\xi) = a_1^2\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + a_0^2. \tag{3.11}$$

Substituting Eqs. (3.10) and (3.11) into Eq. (3.6), collecting the coefficients of $\left(\frac{G'}{G}\right)^i (i = 0,\ldots,2)$ and set it to zero we obtain the system

$$ka_1 + a_1^2 = 0,$$
$$2a_0a_1 + ka_1\lambda = 0, \tag{3.12}$$
$$a_0^2 + c + ka_1\mu = 0,$$

Solving this system by Maple gives

$$a_0 = -\frac{\lambda k}{2}, \quad a_1 = -k, \quad c = \frac{k^2}{4}(4\mu - \lambda^2). \tag{3.13}$$

where λ and μ are arbitrary constants.

By using Eq. (3.13), expression (3.9) can be written as

$$v(\xi) = -\frac{\lambda k}{2} - k\left(\frac{G'}{G}\right), \tag{3.14}$$

where

$$\xi = k\left[x + \frac{k^2}{4}(4\mu - \lambda^2)t\right].$$

Substituting the general solutions of Eq. (2.4) into Eq. (3.14) we have three types of travelling wave solutions of the nonlinear wave equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$v_1(\xi) = -k\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda k}{2}, \tag{3.15}$$

and

$$u_1(\xi) = \left[-k\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda k}{2}\right]^2, \tag{3.16}$$

where

$$\xi = k\left[x + \frac{k^2}{4}(4\mu - \lambda^2)t\right].$$

When $\lambda^2 - 4\mu < 0$,

$$v_2(\xi) = -k\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) - \frac{\lambda k}{2}, \tag{3.17}$$

and

$$u_2(\xi) = \left[-k\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) - \frac{\lambda k}{2}\right]^2, \tag{3.18}$$

where

$$\xi = k\left[x + \frac{k^2}{4}(4\mu - \lambda^2)t\right].$$

When $\lambda^2 - 4\mu = 0$,

$$v_3(\xi) = -\frac{kc_2}{c_1 + c_2 kx},\tag{3.19}$$

and

$$u_3(\xi) = \frac{k^2 c_2^2}{(c_1 + c_2 kx)^2}.\tag{3.20}$$

Comparing our results and Darvishi et al.'s results [8] with Helal and Mehanna's results [10] then it can be seen that the results are same.

It is noted that the rational solutions (3.19) and (3.20) derived here do not appear in Darvishi et al. [8] and Helal and Mehanna [10].

4. The $(2+1)$-dimensional breaking soliton equations

Let us consider the $(2+1)$-dimensional breaking soliton equations [12]:

$$\begin{aligned}
u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha u_x v &= 0, \\
u_y &= v_x,
\end{aligned}\tag{4.1}$$

where α is known constant. Eq. (4.1) describes the $(2+1)$-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. In the past years, many authors have studied Eq. (4.1). For instance, Zhang has successfully extended the generalized auxiliary equation method to the $(2+1)$-dimensional breaking soliton equations in [27]. Some soliton-like solutions were obtained by the generalized expansion method of Riccati equation in [7]. Recently, a class of periodic wave solutions were obtained by the mapping method in [15]. Two classes of new exact solutions were obtained by the singular manifold method in [16].

Using the wave variable $\xi = x + y - ct$ and proceeding as before we find

$$\begin{aligned}
-cu' + \alpha u''' + 4\alpha u v' + 4\alpha u' v &= 0, \\
u' &= v'.
\end{aligned}\tag{4.2}$$

Integrating the second equation in the system and neglecting constants of integration we find

$$u = v.\tag{4.3}$$

Substituting (4.3) into the first equation of the system and integrating we find

$$-cu + 4\alpha u^2 + \alpha u'' = 0.\tag{4.4}$$

By the same procedure as illustrated in Section 3, we can determine the value of m by balancing u^2 and u'' in Eq. (4.4). We find $m = 2$. We can suppose that the solutions of Eq. (4.4) is of the form

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \tag{4.5}$$

By using Eqs. (4.5) and (2.4) it is derived that

$$u^2(\xi) = a_2^2 \left(\frac{G'}{G}\right)^4 + 2a_2a_1 \left(\frac{G'}{G}\right)^3 + (a_1^2 + 2a_0a_2)\left(\frac{G'}{G}\right)^2 + 2a_0a_1 \left(\frac{G'}{G}\right) + a_0^2, \tag{4.6}$$

and

$$u''(\xi) = 6a_2 \left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda)\left(\frac{G'}{G}\right)^3 + (8a_2\mu + 3a_1\lambda$$
$$+ 4a_2\lambda^2)\left(\frac{G'}{G}\right)^2 + (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu. \tag{4.7}$$

Substituting Eq. (4.5)–(4.7) into Eq. (4.4), collecting the coefficients of $\left(\frac{G'}{G}\right)^i (i = 0, \ldots, 4)$ and set it to zero we obtain the system

$$6\alpha a_2 + 4\alpha a_2^2 = 0,$$
$$8\alpha a_1 a_2 + 2\alpha a_1 + 10\alpha a_2\lambda = 0,$$
$$3\alpha a_1\lambda + 4\alpha a_2\lambda^2 - ca_2 + 8\alpha a_2\mu + 8\alpha a_0 a_2 + 4\alpha a_1^2 = 0, \tag{4.8}$$
$$8\alpha a_0 a_1 + 2\alpha a_1\mu + \alpha a_1\lambda^2 - ca_1 + 6\alpha a_2\lambda\mu = 0,$$
$$4\alpha a_0^2 - ca_0 + 2\alpha a_2\mu^2 + \alpha a_1\lambda\mu = 0.$$

Solving this system by Maple gives

$$a_0 = -\frac{3}{2}\mu, \quad a_1 = -\frac{3}{2}\lambda, \quad a_2 = -\frac{3}{2}, \quad c = \alpha(\lambda^2 - 4\mu), \tag{4.9}$$

or

$$a_0 = -\frac{1}{2}\left(\mu + \frac{\lambda^2}{2}\right), \quad a_1 = -\frac{3}{2}\lambda, \quad a_2 = -\frac{3}{2}, \quad c = -\alpha(\lambda^2 - 4\mu), \tag{4.10}$$

where λ and μ are arbitrary constants.

Substituting Eq. (4.9) or Eq. (4.10) into Eq. (4.5) yields

$$u(\xi) = -\frac{3}{2}\mu - \frac{3}{2}\lambda\left(\frac{G'}{G}\right) - \frac{3}{2}\left(\frac{G'}{G}\right)^2, \tag{4.11}$$

where $\xi = x + y - \alpha(\lambda^2 - 4\mu)t$, or

$$u(\xi) = -\frac{1}{2}\left(\mu + \frac{\lambda^2}{2}\right) - \frac{3}{2}\lambda\left(\frac{G'}{G}\right) - \frac{3}{2}\left(\frac{G'}{G}\right)^2, \tag{4.12}$$

where $\xi = x + y + \alpha(\lambda^2 - 4\mu)t$.

Substituting the general solutions of Eq. (2.4) into Eqs. (4.11) and (4.12) we have six types of travelling wave solutions of the $(2+1)$-dimensional breaking soliton equations as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{3}{8}(4\mu - \lambda^2)\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right)^2 + \frac{3\lambda^2}{8} - \frac{3}{2}\mu,$$

$$v_1(\xi) = \frac{3}{8}(4\mu - \lambda^2)\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right)^2 + \frac{3\lambda^2}{8} - \frac{3}{2}\mu,$$

$$(4.13)$$

where $\xi = x + y - \alpha(\lambda^2 - 4\mu)t$, or

$$u_2(\xi) = \frac{3}{8}(4\mu - \lambda^2)\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right)^2 + \frac{\lambda^2}{8} - \frac{1}{2}\mu,$$

$$v_2(\xi) = \frac{3}{8}(4\mu - \lambda^2)\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right)^2 + \frac{\lambda^2}{8} - \frac{1}{2}\mu,$$

$$(4.14)$$

where $\xi = x + y + \alpha(\lambda^2 - 4\mu)t$.

When $\lambda^2 - 4\mu < 0$,

$$u_3(\xi) = \frac{3}{8}(\lambda^2 - 4\mu)\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right)^2 + \frac{3\lambda^2}{8} - \frac{3}{2}\mu,$$

$$v_3(\xi) = \frac{3}{8}(\lambda^2 - 4\mu)\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right)^2 + \frac{3\lambda^2}{8} - \frac{3}{2}\mu,$$

$$(4.15)$$

where $\xi = x + y - \alpha(\lambda^2 - 4\mu)t$, or

$$u_4(\xi) = \frac{3}{8}(\lambda^2 - 4\mu)\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right)^2 + \frac{\lambda^2}{8} - \frac{1}{2}\mu,$$

$$v_4(\xi) = \frac{3}{8}(\lambda^2 - 4\mu)\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right)^2 + \frac{\lambda^2}{8} - \frac{1}{2}\mu,$$

$$(4.16)$$

where $\xi = x + y + \alpha(\lambda^2 - 4\mu)t$.

When $\lambda^2 - 4\mu = 0$,

$$u_5(\xi) = -\frac{3c_2^2}{2[c_1 + c_2(x+y)]^2} + \frac{3\lambda^2}{8} - \frac{3}{2}\mu,$$

$$v_5(\xi) = -\frac{3c_2^2}{2[c_1 + c_2(x+y)]^2} + \frac{3\lambda^2}{8} - \frac{3}{2}\mu,$$

$$\text{(4.17)}$$

or

$$u_6(\xi) = -\frac{3c_2^2}{2[c_1 + c_2(x+y)]^2} + \frac{\lambda^2}{8} - \frac{1}{2}\mu,$$

$$v_6(\xi) = -\frac{3c_2^2}{2[c_1 + c_2(x+y)]^2} + \frac{\lambda^2}{8} - \frac{1}{2}\mu.$$

$$\text{(4.18)}$$

Comparing our results and Zhang's results [27], Cheng and Li's results [7] with Peng's results [15,16] it can be seen that the results are same. If proper λ,μ values are chosen, then it can be seen that the results are same.

5. The $(3+1)$-dimensional KP equation

We next consider the $(3+1)$-dimensional KP equation

$$u_{xt} + 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \tag{5.1}$$

Xie et al. [24] obtained non-travelling wave solutions by the improved tanh function method, in which they introduced a generalized Riccati equation and gained its 27 new solutions. In this paper, we will construct new non-travelling wave solutions of Eq. (2.1). As a result, new non-travelling wave solutions including soliton-like solutions and periodic solutions of Eq. (2.1) are obtained. A generalized variable-coefficient algebraic method with computerized symbolic computation is developed to deal with the $(3+1)$-dimensional KP equation with variable coefficients in [31]. Chen et al. [6] study the $(3+1)$-dimensional KP equation by using the new generalized transformation in homogeneous balance method.

Using the wave variable $\xi = x + y + z - ct$, the Eq. (5.1) is carried to an ODE of the form

$$-(c+2)u'' + 6(u')^2 + 6uu'' - u'''' = 0. \tag{5.2}$$

Integrating twice and setting the constants of integration to zero, we obtain

$$-(c+2)u + 3u^2 - u'' = 0. \tag{5.3}$$

Balancing u'' with u^2 in (5.3) gives

$$m + 2 = 2m, \tag{5.4}$$

so that

$$m = 2. \tag{5.5}$$

Suppose that the solutions of (5.3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0, \tag{5.6}$$

By using Eqs. (5.6) and (2.4) it is derived that

$$u^2(\xi) = a_2^2\left(\frac{G'}{G}\right)^4 + 2a_2a_1\left(\frac{G'}{G}\right)^3 + (a_1^2 + 2a_0a_2)\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + a_0^2, \tag{5.7}$$

and

$$u''(\xi) = 6a_2\left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda)\left(\frac{G'}{G}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2)\left(\frac{G'}{G}\right)^2$$
$$+ (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu, \tag{5.8}$$

Substituting Eqs. (5.6)–(5.8) into Eq. (5.3), collecting the coefficients of $\left(\frac{G'}{G}\right)^i (i = 0, \ldots, 4)$ and set it to zero we obtain the system

$$-6a_2 + 3a_2^2 = 0,$$
$$-10a_2\lambda + 6a_1a_2 - 2a_1 = 0,$$
$$-2a_2 - ca_2 - 3a_1\lambda - 4a_2\lambda^2 + 3a_1^2 - 8a_2\mu + 6a_0a_2 = 0, \tag{5.9}$$
$$6a_0a_1 - 2a_1 - ca_1 - 2a_1\mu - a_1\lambda^2 - 6a_2\lambda\mu = 0,$$
$$-ca_0 + 3a_0^2 - 2a_0 - 2a_2\mu^2 - a_1\lambda\mu = 0.$$

Solving this system by Maple gives

$$a_0 = 2\mu, \quad a_1 = 2\lambda, \quad a_2 = 2, \quad c = -\lambda^2 + 4\mu - 2, \tag{5.10}$$

or

$$a_0 = \frac{1}{3}(\lambda^2 + 2\mu), \quad a_1 = 2\lambda, \quad a_2 = 2, \quad c = \lambda^2 - 4\mu - 2 \tag{5.11}$$

where λ and μ are arbitrary constants.

Substituting Eq. (5.10) or Eq. (5.11) into Eq. (5.6) yields

$$u(\xi) = 2\mu + 2\lambda\left(\frac{G'}{G}\right) + 2\left(\frac{G'}{G}\right)^2, \tag{5.12}$$

where $\xi = x + y + z + (\lambda^2 - 4\mu + 2)t$ or

$$u(\xi) = \frac{1}{3}(\lambda^2 + 2\mu) + 2\lambda\left(\frac{G'}{G}\right) + 2\left(\frac{G'}{G}\right)^2, \tag{5.13}$$

where $\xi = x + y + z - (\lambda^2 - 4\mu - 2)t$ or

Substituting the general solutions of Eq. (2.4) into Eq. (5.12) we have six types of travelling wave solutions of the $(3+1)$-dimensional KP equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{\lambda^2 - 4\mu}{2}\left[\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2 - 1\right],$$

(5.14)

where $\xi = x + y + z + (\lambda^2 - 4\mu + 2)t$, or

$$u_2(\xi) = \frac{\lambda^2 - 4\mu}{2}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2$$
$$+ \frac{5\lambda^2}{6} + \frac{2\mu}{3},$$

(5.15)

where $\xi = x + y + z + (\lambda^2 - 4\mu - 2)t$.

When $\lambda^2 - 4\mu < 0$,

$$u_3(\xi) = \frac{4\mu - \lambda^2}{2}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cos\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cos\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sin\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2 - \frac{\lambda^2}{2}$$
$$+ 2\mu,$$

(5.16)

where $\xi = x + y + z + (\lambda^2 - 4\mu + 2)t$, or

$$u_4(\xi) = \frac{4\mu - \lambda^2}{2}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cos\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cos\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sin\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2 + \frac{5\lambda^2}{6} + \frac{2\mu}{3},$$

(5.17)

where $\xi = x + y + z + (\lambda^2 - 4\mu - 2)t$.

When $\lambda^2 - 4\mu = 0$,

$$u_5(\xi) = \frac{2c_2^2}{(c_1 + c_2\xi)^2} - \frac{\lambda^2}{2} + 2\mu,$$

(5.18)

$$u_6(\xi) = \frac{2c_2^2}{(c_1 + c_2\xi)^2} + \frac{5\lambda^2}{6} + \frac{2\mu}{3},$$

(5.19)

where $\xi = x + y + z + 2t$.

Comparing our results and Xie et al.'s results [24], Zhao and Bai's results [31] with Chen et al.'s results [6] it can be seen that the results are same. All the solutions reported in this paper have been verified with Maple by putting them back into the original Eq. (5.1), which can not be obtained by the methods.

6. Conclusion

The $\left(\frac{G'}{G}\right)$-expansion method was succesfully used to establish travelling wave solutions. Many well known nonlinear wave equations were handled by this method. The performance of this method is reliable and effective. This method has many advantages: it is direct and concise. It is shown that the algorithm can be also applied to other NLPDEs in mathematical physics. The availability of computer systems like *Mathematica* or *Maple* facilitates the tedious algebraic calculations. The method which we have proposed in this letter is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation.

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