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Estimates for Monge–Ampère operators acting on positive plurisubharmonic currents

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Abstract. Let $\widetilde{\Omega}$ be an open subset of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$, let T be a positive plurisubharmonic (psh: meaning $dd^cT \geqslant 0$) current of bidegree (k, k) on $\widetilde{\Omega}$ and let U be the Lelong-Skoda potential current associated to the d-closed positive current dd^cT. We denote $(z, t) \in \mathbb{C}^n \times \mathbb{C}^m$ and consider φ: $(z,t) \mapsto \varphi(z)$ a C^2 positive semi-exhaustive plurisubharmonic (psh: meaning $dd^c\varphi \geqslant 0$) function on $\widetilde{\Omega}$ such that $\log\varphi$ is also plurisubharmonic on the open set $\{\varphi > 0\}$. For $p \in \mathbb{N}$ such that $1 \leqslant p \leqslant n - k$, we generalize some properties of the current $U \wedge (dd^c\omega)^p$ where $\omega = \log\varphi$, known when $\omega(z) = \log|z|$ (see [12]). Finally we want to define the current $T \wedge (dd^c\omega)^p$ and as an application, we prove a version of the chern-Levine-Nirenberg for a positive or negative psh current which is defined out side a pluripolar set $A \subset \widetilde{\Omega}$.

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1. Introduction

Let $\Omega \subset\subset \widetilde{\Omega}$ be two open subsets of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$, let T be a positive plurisubharmonic current of bidegree (k,k) on $\widetilde{\Omega}$ where $1 \leqslant k \leqslant n$, we refer to Lelong's paper [14] for the basic properties of positive currents.

We denote $(z, t) \in \mathbb{C}^n \times \mathbb{C}^m$ and consider a C^2 positive plurisubharmonic function φ: $(z,t) \mapsto \varphi(z)$ on $\widetilde{\Omega}$, semi-exhaustive on $\widetilde{\Omega} \cap (\mathbb{C}^n \times \{0\})$ i.e. $\varphi : \widetilde{\Omega} \cap (\mathbb{C}^n \times \{0\}) \to [-\infty, +\infty[$ satisfies the following condition: there exists a real number R such that for all $c < R$, we have:

$$\{z,\ \varphi(z) < c\} \subset\subset \widetilde{\Omega} \cap (\mathbb{C}^n \times \{0\}).$$

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We suppose also that $\log\varphi$ is also plurisubharmonic on the open set $\{\varphi > 0\}$. Let U be The potential associated to the positive d-closed current dd^cT.

For $p \in \mathbb{N}$ such that $1 \leqslant p \leqslant n-k$, we generalize some properties of the current $U \wedge (dd^c\omega)^p$ where $\omega = \log\varphi$, known when $\omega(z) = \log|z|$ (see [12]). Let $(\chi_j)_{j\in\mathbb{N}}$ be a family of smooth regularizing kernels which only depend on $|(z,t)|$ where

$$(z = (z_1, \dots z_n), t = (t_1, \dots t_m)) \in \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^N; \quad |(z,t)|^2 = |z|^2 + |t|^2$$

with

$$|z|^2 = \sum_{j=1}^{n} z_j\overline{z_j}; \quad |t|^2 = \sum_{j=1}^{m} t_j\overline{t_j}.$$

We denote $U_j = U*_{\chi j}$, $T_j = T*_{\chi j}$ and let $(\omega_j)_{j\in\mathbb{N}}$ be a decreasing sequence of smooth plurisubharmonic functions converging pointwise to $\log\varphi$, then we have:

Theorem 1. *Let $p \in \mathbb{N}$ such that $1 \leqslant p \leqslant n-k$, then: $U_j \wedge (dd^c\log\varphi)^p$ converges weakly (i.e. in the sense of currents) in Ω to a negative current denoted by $U \wedge (dd^c\log\varphi)^p$.*

Theorem 2. *Under the hypothesis of Theorem* 1 *and with the following additional condition: $(\bigwedge_{2(N-p-k)}(\{\varphi = 0\}) = 0$, where $\bigwedge_{2q}$ is the Hausdorff 2q-dimensional measure) we have: $T_j \wedge (dd^c\omega_j)^p$ converges weakly in Ω to a current denoted by $T \wedge (dd^c\log\varphi)^p$.*

This theorem generalizes the results of Siu [17], Alessandrini and Bassanelli [1] who noticed that in some cases the family $T_j \wedge dd^c\omega_j$ is bounded in mass. Similarly these results of the previous theorems are a generalizations of Feki's result where $\varphi(z) = |z|$ (see [12]). The case of the d-closed positive current T and $\varphi(z) = |z|$ was proved by Ben Messaoud-El Mir (see [5]). If ω is a plurisubharmonic non bounded function with Hausdorff measure condition and T a d-closed positive current then the current $T \wedge (dd^c\omega)^p$ is defined by Demailly (see [10]) and Cegrell (see [6]).

As an application, we prove a version of the chern-Levine–Nirenberg (see [7]) for a positive or negative psh current which is defined out side a pluripolar set $A \subset \widetilde{\Omega}$ and this result is given by the following theorem.

Theorem 3. *Let A be a closed complete pluripolar subset of an open $\widetilde{\Omega}$ of $\mathbb{C}^N$ and let T be a positive or negative plurisubharmonic (psh) current of bidimension (p,p) on $\widetilde{\Omega} \setminus A$ such that $1 \leqslant p \leqslant N$.*

Let $v_1, \dots, v_q$ be a plurisubharmonic C^2 functions and locally bounded in $\widetilde{\Omega} \setminus A$ with $1 \leqslant q \leqslant p$. If K, and L are two compact subsets of $\widetilde{\Omega}$ such that $K \subset L$, then there exists a positive constant $C_{K,L}$ independent of T and v_j such that:

$$\|T \wedge dd^c v_1 \wedge \dots \wedge dd^c v_q\|_{K\setminus A} \leqslant C_{K,L}\Big(\|T\|_{L\setminus A} + \|dd^cT\|_{L\setminus A}\Big)\Pi_{j=1}^q \|v_j\|_{\infty(L\setminus A)}.$$

Under the hypothesis of Theorem 3, H.Ben Messaoud, M.Toujani [3] when $A = \emptyset$, have proved In 2004 the following inequality:

$$\|T \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_q\|_K \leqslant C_{K,L}\|T\|_L \Pi_{j=1}^q \|v_j\|_{\infty(L)}.$$

In 2007 by using another technique Sibony and Dinh [16] have also proved the inequality of Theorem 3 when $A = \emptyset$, T is a positive current and $dd^c T$ has order zero.

2. Basic definitions

Let Ω be an open subset of $\mathbb{C}^N$ and $0 \leqslant p,q \leqslant N$. A continuous linear functional on the space $D_{N-p,N-q}(\Omega)$ of the smooth $(N-p, N-q)$ differential forms with compact support in Ω is called current of bidimension (p,q) or of bidegree $(N-p, N-q)$. We can also say: a (p,q) current on Ω is a (p,q) differential form whose coefficients are distributions on Ω.

Let T be a current of bidimension (p,p), T is called positive if the distribution $T \wedge i\psi_1 \wedge \overline{\psi}_1 \wedge \ldots \wedge i\psi_p \wedge \overline{\psi}_p$ is a positive measure on Ω for all $\psi_1, \ldots, \psi_p \in D_{(1,0)}(\Omega)$, T is called negative if $-T$ is positive, T is called plurisubharmonic (resp harmonic) if $dd^c T$ is positive (resp $= 0$). In ([2]) Bassanelli introduced the notion of current C-flat as follows: T is said $\mathbb{C}$-flat if there exist a currents F, G, H where their coefficients belong to $L^1_{loc}(\Omega)$ such that $T = F + \partial G + \overline{\partial} H$. Let T be a current of order zero (its coefficients are measures) of bidimension (p,p), the mass of T over Ω_1 (where Ω_1 is an open subset of Ω) which is noted $\|T\|_{\Omega_1}$ is defined by:

$$\|T\|_{\Omega_1} = \sup\{|T(\psi)|,\ \psi \in D_{(p,p)}(\Omega_1),\ \|\psi\| \leqslant 1\}$$

and if T is positive we have the following result:

$$T \wedge \frac{(dd^c|z|^2)^p}{2^p p!}(\Omega_1) \leqslant \|T\|_{\Omega_1} \leqslant cT \wedge (dd^c|z|^2)^p(\Omega_1)$$

(where $z \in \mathbb{C}^N$ and $c > 0$ is a constant). If K is a compact subset of Ω and T is a positive or negative current of bidimension (p,p) on Ω, we often identify the mass $\|T\|_K$ and $|\int_K T \wedge (dd^c|z|^2)^p|$. Let A be a closed subset of Ω, let (χ_n) be a smooth bounded sequence such that $0 \leqslant \chi_n \leqslant 1$ which vanishes in a neighborhood of A and $\lim_{n\to+\infty}(\chi_n) = 1_{\Omega\setminus A}$ (where $1_{\Omega\setminus A}$ is the characteristic function of $\Omega\setminus A$). Let T be a current of bidimension (p,p) defined on $\Omega\setminus A$. If $\chi_n T$ has a limit which does not depend on (χ_n), then the limit of $\chi_n T$ is called the trivial extension of T by zero across A and is noted by $\widetilde{T}$, moreover for $\psi \in D_{(p,p)}(\Omega)$ we define

$$\langle \widetilde{T}, \psi\rangle = \lim_{n\to+\infty} \langle \chi_n T, \psi\rangle.$$

If A is a closed locally complete pluripolar set (i.e. $\forall z \in \Omega$, $\exists r > 0$ such that $B(z,r) = \{z \in \Omega; |z| < r\} \subset \Omega, \exists \phi : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$ a plurisubharmonic function, $\phi \not\equiv -\infty$ on Ω such that $A \cap B(z,r) = \{z \in \Omega,\ \phi(z) = -\infty\}$), then we can find a sequence of smooth psh functions u_n, $0 \leqslant u_n \leqslant 1$, which vanish near A, converges to $1_{\Omega\setminus A}$ and such that $\lim_{n\to+\infty} u_n T = \widetilde{T}$ weakly (see [16]).

On the other hand, $\widetilde{T}$ exists if and only if the mass of T is locally finite on Ω moreover we have:

$$\widetilde{T} \textit{ exists} \Longleftrightarrow \forall \textit{ compact } K \subset \Omega, \|T\|_{K\setminus A} < +\infty.$$

In the same way, we have the following result: Let T be a positive current of bidimension (p,p) on Ω of $\mathbb{C}^N$ and if $\{T_v\}_{v\in\mathbb{N}}$ is a sequence of positive currents of Ω such that $\lim_{v\to\infty} T_v = T$ weakly in Ω, then

$$\lim_{v\to\infty}\int_V T_v \wedge \psi = \int_V T\wedge\psi \tag{1}$$

for every smooth differential form ψ on Ω and for every open relatively compact V of Ω such that $\|T\|(\partial V) = 0$ (where ∂V is the boundary of V).

Let T be a positive plurisubharmonic current of bidimension (p,p) in the open subset $\widetilde{\Omega}$ of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$, let $\eta \in D(\widetilde{\Omega})$, $0 \leqslant \eta \leqslant 1$ such that $\eta = 1$ in a neighborhood of $\overline{\Omega}$ where Ω is an open such that $\Omega \subset\subset \widetilde{\Omega}$, let $\beta(z-x) = dd^c(|z-x|^2)$; $\beta^{N-1}(z-x) = \underbrace{\beta(z-x)\wedge\beta(z-x)\wedge\ldots\ldots\wedge\beta(z-x)}_{(N-1)times}$, then the potential $U = U(\widetilde{\Omega}, dd^cT)$ associated to the positive d-closed current dd^cT is the current of bidimension $(p+1, p+1)$ on $\mathbb{C}^N$ defined by

$$U(z) = -c_N\int_{x\in\mathbb{C}^N}\eta(x)T(x)\wedge\frac{\beta^{N-1}(z-x)}{|z-x|^{2N-2}},\quad N\geqslant 2,\quad c_N = \frac{1}{(N-1)(4\pi)^N}.$$

U is a negative current on $\mathbb{C}^N$ and $dd^cU - dd^cT \in C^\infty(\Omega)$ (see [4] or [18]).

Now, we shall use the following notations:

We denote $(z,t)\in\mathbb{C}^n\times\mathbb{C}^m$ and consider a C^2 positive psh function on $\widetilde{\Omega}$ φ: $(z,t)\mapsto\varphi(z)$ such that $\log\varphi$ is also psh on the open set $\{\varphi > 0\}$.

We suppose $\varphi : \widetilde{\Omega}\cap(\mathbb{C}^n\times\{0\}) \to [-\infty,+\infty[$ semi-exhaustive i.e., there exists a real number R such that for all $c < R$, we have

$$\{z,\ \varphi(z) < c\} \subset\subset \widetilde{\Omega}\cap(\mathbb{C}^n\times\{0\})$$

For simplicity we shall use the following notations:

If $z = (z_1,\ldots z_n)\in\mathbb{C}^n$ and $t = (t_1,\ldots t_m)\in\mathbb{C}^m$ we denote:

$$|z|^2 = \sum_{j=1}^n z_j\overline{z_j};\quad |t|^2 = \sum_{j=1}^m t_j\overline{t_j};\quad \beta(z) = dd^c(|z|^2); \beta(t) = dd^c(|t|^2);$$

$\beta = \beta(z) + \beta(t)$ is the Euclidean form on $\mathbb{C}^N = \mathbb{C}^n\times\mathbb{C}^m$.

$$\beta^p = \underbrace{\beta\wedge\beta\wedge\ldots\wedge\beta}_{p\ times}.$$

Recall that; if $z\in\mathbb{C}^n$ we have:

$$\partial = \sum_{j=1}^n\frac{\partial}{\partial z_j}dz_j;\quad \bar\partial = \sum_{j=1}^n\frac{\partial}{\partial\bar z_j}d\bar z_j; d = \partial+\bar\partial;\quad d^c = i(\bar\partial-\partial); dd^c = 2i\partial\bar\partial.$$

If w is a locally bounded function in $\widetilde{\Omega}\subset\mathbb{C}^N$ and if $K\subset\widetilde{\Omega}$ is a compact, we denote:

$$\|w\|_{\infty(K)} = \sup_{(z,t)\in K}|w(z,t)|.$$

3. Monge–Ampère operator

We first prove the following theorem:

Theorem 1. *Let T be a positive psh current of bidegree (k,k) in the open subset $\widetilde{\Omega}$ of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ with $1 \leqslant k \leqslant n$, let $U = U(\widetilde{\Omega}, dd^cT)$ be the potential of the positive d-closed current dd^cT in Ω where Ω is an open such that $\Omega \subset\subset \widetilde{\Omega}$ and consider a C^2 positive semi-exhaustive plurisubharmonic function*

$\varphi: (z,t) \mapsto \varphi(z)$ on $\widetilde{\Omega}$, such that $\log\varphi$ is also psh on the open set $\{\varphi > 0\}$.

Let $(\chi)_{j\in\mathbb{N}}$ be a family of smooth regularizing kernels. If $U_j = U*\chi_j$ and $1 \leqslant p \leqslant n-k$, then we have:

$U_j \wedge (dd^c\log\varphi)^p$ converges weakly (i.e. in the sense of currents) in Ω to a negative current denoted by $U \wedge (dd^c\log\varphi)^p$.

Proof. If $T_j = T*\chi_j$, then in the paper of M.Toujani (see [19]) we have proved that:

$$\sup_j \int_V T_j \wedge (dd^c \log\varphi(z))^p \wedge \beta^{N-k-p} < +\infty, \tag{2}$$

where V is an open such that $V \subset\subset \Omega$.

As the problem of convergence is local, therefore according to the formula (2), the sequence $(T_j \wedge (dd^c\log\varphi)^p)_j$ is bounded in mass. Since, $dd^cU = dd^cT + R$ (see [5]) with R is a C^∞ form of bidegree $(k+1, k+1)$ in Ω and as the notion of U is local, we can assume that R is a negative form, therefore the current $T-U$ is a positive and plurisubharmonic, hence by using the formula (2), the sequence $(T-U)_j \wedge (dd^c\log\varphi)^p$ is bounded in mass. As the sequence $T_j \wedge (dd^c\log\varphi)^p$ is bounded in mass, then the sequence $U_j \wedge (dd^c\log\varphi)^p$ is also bounded in mass. Now, the function h which is defined by $h(t) = \frac{-1}{(N-1)(4\pi)^N |t|^{2N-2}}$ is subharmonic, therefore according to Ben Messaoud–El Mir [5], the family of currents $(U_j)_j$ which is defined by

$$U_j(z) = \int_{x\in\mathbb{C}^N} \eta(x)(h*\chi_j)(z-x)T(x) \wedge \beta^{N-1}(z-x)$$

is a decreasing sequence of negative currents. Hence, $U_j \wedge (dd^c\log\varphi)^p$ converges weakly in Ω to a current denoted by $U \wedge (dd^c\log\varphi)^p$. square

The Main Theorem of this paper is given by

Theorem 2. *Let $\Omega \subset\subset \widetilde{\Omega}$ be two open subsets of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$, let T be a positive plurisubharmonic current of bidegree (k,k) on $\widetilde{\Omega}$ where $1 \leqslant k \leqslant n$, consider a C^2 positive semi-exhaustive plurisubharmonic function $\varphi: (z,t) \mapsto \varphi(z)$ on $\widetilde{\Omega}$, such that $\log\varphi$ is also plurisubharmonic on the open set $\{\varphi > 0\}$, we suppose that $\bigwedge_{2(N-k-p)}(\{\varphi = 0\}) = 0$ (where $\bigwedge_{2q}$ is the Hausdorff 2q-dimensional measure). Let $(\omega_j)_{j\in\mathbb{N}}$ be a decreasing sequence of smooth plurisubharmonic functions converging pointwise to $\log\varphi$, let $(\chi)_{j\in\mathbb{N}}$ be a family of smooth regularizing kernels. If $T_j = T*\chi_j$ and $1 \leqslant p \leqslant n-k$, then we have:*

- $T_j \wedge (dd^c \log \varphi)^p$ converges weakly (i.e. in the sense of currents) in Ω to a positive current denoted by $T \wedge (dd^c \log \varphi)^p$.
- $T \wedge (dd^c \omega_j)^p$ converges weakly in Ω to $T \wedge (dd^c \log \varphi)^p$.
- $T_j \wedge (dd^c \omega_j)^p$ converges weakly in Ω to $T \wedge (dd^c \log \varphi)^p$.

Proof. The problem of convergence is useful when $\log \varphi$ is unbounded about of neighborhood of $\{\varphi = 0\}$. As in the previous theorem, we will prove that the sequence $T_j \wedge (dd^c \log \varphi)^p$ is bounded in mass, (moreover, this sequence is defined as follows:

$$\langle T_j \wedge (dd^c \log \varphi)^p, \psi \rangle = \int_{\widetilde{\Omega}} T_j \wedge (dd^c \log \varphi)^p \wedge \psi$$

when ψ is a smooth $(N - k - p, N - k - p)$ differential form with compact support in $\widetilde{\Omega}$).

(1) We suppose that T is pluriharmonic (i.e. $dd^c T = 0$.)Using the techniques of I.Feki-H.El Mir (see [12]) as follows:As $T_j \wedge (dd^c \log \varphi)^p$ is bounded in mass, we can consider two weak limits μ_1 and μ_2 of the sequence of currents $T_j \wedge (dd^c \log(\varphi))^p$, the current $\mu = \mu_1 - \mu_2$ which is of order zero and of bidegree $(k + p, k + p)$ is supported in $\{\varphi = 0\}$ and it is pluriharmonic (i.e. $dd^c \mu = 0$).

- If $p < n - k$, then we have $N - (k + p) = n + m - k - p > m$, the subset $K := \{\varphi = 0\} = \{\log \varphi = -\infty\}$ is a locally complete pluripolar because $\log \varphi$ is plurisubharmonic and it has Lebesgue measure zero in $\mathbb{C}^N$. As φ is semi-exhaustive then $K = \{\varphi = 0\} \subset\subset \widetilde{\Omega} \cap (\mathbb{C}^n \times \{0\})$, therefore K is a compact of $\widetilde{\Omega} \cap (\mathbb{C}^n \times \{0\})$. We know that $\bigwedge_{2(N-k-p)}(supp \mu) = 0$ (because the current $\mu = \mu_1 - \mu_2$ is supported in $\{\varphi = 0\}$), from the theorem of support for the current $\mathbb{C}$-flat (see [2]) $\mu_1 = \mu_2$.
- If $p = n - k$, consider $0 < r < r' < 1$ such that $K \subset \{|z| < r\}$ and $\{|z| < r'\} \times \{|t| < r'\}$ is relatively compact in $\widetilde{\Omega}$. Let $g(z)$ be a positive smooth function supported in $\{|z| < r'\}$ such that $g \equiv 1$ on $\{|z| < r\}$ and $h(t)$ be a positive smooth function supported in $\{|t| < r'\}$, we have:

$$\begin{aligned}
&\langle T_j \wedge (dd^c \log \varphi)^{n-k}, g(z)h(t)\beta(t)^m \rangle \\
&= \int_{\widetilde{\Omega}} \log \varphi (dd^c \log \varphi)^{n-k-1} \wedge dd^c(g(z)T_j) \wedge h(t)\beta(t)^m \\
&= \int_{\widetilde{\Omega}} \log \varphi (dd^c \log \varphi)^{n-k-1} \wedge dd^c g \wedge T_j \wedge h(t)\beta(t)^m \\
&\quad - \int_{\widetilde{\Omega}} \log \varphi (dd^c \log \varphi)^{n-k-1} \wedge d^c g \wedge dT_j \wedge h(t)\beta(t)^m \\
&\quad + \int_{\widetilde{\Omega}} \log \varphi (dd^c \log \varphi)^{n-k-1} \wedge dg \wedge d^c T_j \wedge h(t)\beta(t)^m \\
&\quad + \int_{\widetilde{\Omega}} \log \varphi (dd^c \log \varphi)^{n-k-1} \wedge dd^c T_j \wedge g(z)h(t)\beta(t)^m.
\end{aligned}$$

We have $dd^cT = 0$ therefore $dd^cT_j = 0$. As the functions d^cg, dg, and dd^cg vanish on an open neighborhood of $K := \{\varphi = 0\}$, then the first three integrals converge, the last integral vanishes since $dd^cT_j = 0$, hence the sequence$\langle T_j \wedge (dd^c \log\varphi)^{n-k}, g(z)h(t)\beta(t)^m\rangle$ converges. We will have therefore

$$\forall h \in D(\{|t| < r'\}), \int_{\varphi=0} \mu h(t)\beta(t)^m = 0, \quad \text{hence } \mu \equiv 0.$$

(2) We suppose that $dd^cT \geqslant 0$, let U be the potential associated to the positive d-closed current dd^cT, since $dd^cU = dd^cT + R$ (see [5]) with R is a C^∞ form of bidegree $(k+1, k+1)$ in $\widetilde{\Omega}$. According to Hörmander [13] there exists a C^∞ positive form R_1 of bidegree (k,k) such that $dd^cR_1 = R$, hence $\widetilde{R} := T - U + R_1$ is a positive pluriharmonic current of bidegree (k,k), therefore the sequence $\widetilde{R}_j \wedge (dd^c \log\varphi)^p$ is convergent. Since R_1 is a C^∞ positive form and from the previous theorem we know that the sequence $U_j \wedge (dd^c \log\varphi)^p$ converges in Ω, we deduce then the convergence of the sequence $T_j \wedge (dd^c \log\varphi)^p$ in Ω. The convergence of the sequences $T \wedge (dd^c\omega_j)^p$ and $T_j \wedge (dd^c\omega_j)^p$ can be proved in the same way. □

4. Applications

As an application we prove the following result:

Theorem 3. em Let A be a closed complete pluripolar subset of an open set $\widetilde{\Omega}$ of $\mathbb{C}^N$ and let T be a positive or negative plurisubharmonic (psh) current of bidimension (p,p) on $\widetilde{\Omega} \setminus A$, such that $1 \leqslant p \leqslant N$.

Let $v_1, \dots v_q$ be a psh C^2 functions and locally bounded in $\widetilde{\Omega} \setminus A$ with $1 \leqslant q \leqslant p$. If K, L are two compact subsets of $\widetilde{\Omega}$, such that $K \subset L$, then there exists a positive constant $C_{K,L}$ independent of T and v_j such that:

$$\|T \wedge dd^cv_1 \wedge \dots \wedge dd^cv_q\|_{K\setminus A} \leqslant C_{K,L}\Big(\|T\|_{L\setminus A} + \|dd^cT\|_{L\setminus A}\Big)\Pi_{j=1}^q \|v_j\|_{\infty(L\setminus A)}.$$

Proof. By induction, it is sufficient to prove the result for $q = 1$ and $v_1 = v$, in all the proof of theorem, we can assume $T \geqslant 0$.

- If $\|T\|$ or $\|dd^cT\|$ is infinite near A, then the result of theorem is true.
- If the mass of T and dd^cT is locally finite near A, then the current dd^cT is a positive d-closed current on $\widetilde{\Omega} \setminus A$ and it has locally finite mass near A, according to El Mir [11], the trivial extension of dd^cT by zero across A noted by $\widetilde{dd^cT}$ exists and it is a positive d-closed current. In the same way the trivial extension of T by zero across A noted by $\widetilde{T}$ exists and it is a positive current, by means of Elkhadhra–Dabbek–El Mir [8] we have $dd^c\widetilde{T} = \widetilde{dd^cT} - S$ where S is a positive d-closed current supported on A. The function v is plurisubharmonic (psh) of class C^2 bounded in neighborhood of every point of A, therefore v admit an extension $\tilde{v}$ on $\widetilde{\Omega}$, which is psh continuous definite by v on $\widetilde{\Omega} \setminus A$ and if $a \in A$ we have: $\tilde{v}(a) = \limsup_{x\in\Omega\setminus A}(v(x))$ when $x \to a$.

We may write also: $\tilde{v}(a) = \lim_{\varepsilon\to 0} v_\varepsilon(a)$ with $v_\varepsilon = v * \rho_\varepsilon$ and ρ_ε be a family of smooth regularizing kernels. There is a covering of K by a family of balls $B'_j \subset\subset B_j \subset\subset \overset{\circ}{L}$, therefore it is sufficient to prove the theorem on two balls B,B' of center 0 such that $B \subset B'$. A is closed complete pluripolar, according to El Mir [11] or Zeriahi [20] there exists a psh negative function ϕ on B, smooth on $B\backslash A$ such that $A = \{\phi = -\infty\}$. Consider a decreasing sequence of smooth plurisubharmonic functions $\tilde{v}_j = \tilde{v} * \rho_{\frac{1}{j}}$, converging pointwise to $\tilde{v}$. Following standard lemma (which is used by Demailly [9]).

Lemma 1. *Let f_k be a decreasing sequence of upper semi-continuous functions converging to a function f on an open set $\widetilde{\Omega} \subset \mathbb{C}^N$, let μ_k be a sequence of positive measures converging weakly to a measure μ on $\widetilde{\Omega}$, then every weak limit v of $f_k\mu_k$ satisfies $v \leqslant f\mu$.*

Remark 1. Let f_k be a increasing sequence of lower semi-continuous functions converging to f, under the assumptions of the previous lemma, we have $v \geqslant f\mu$.

Continuation of the proof of Theorem 3

According to Sibony (see p. 8 in [15]) there is a sequence of smooth psh positive functions u_k, $0 \leqslant u_k \leqslant 1$ on $B\backslash A$, vanishing near A and increasing to $1_{B\backslash A}$, thus we have $u_k T \to \widetilde{T}$ when $k \to +\infty$. As $\widetilde{T}$ is a positive current and $dd^c\widetilde{T}$ has order zero (its coefficients are measures), hence by means of Elk–Dab–El Mir (see [8]) we have: $dd^c\widetilde{T} = \widetilde{dd^cT} - S$ with the current S is d-closed positive supported on A and $\widetilde{dd^cT}$ is the trivial extension by zero of dd^cT, thanks to El Mir (see [11]), the current $\widetilde{dd^cT}$ is d-closed positive. By using the Main Theorem, the family $(\mu_j = T \wedge dd^c\tilde{v}_j \wedge \beta^{p-1})_j$ is a sequence of positive measures converging weakly to $\mu = T \wedge dd^c\tilde{v} \wedge \beta^{p-1}$. By means of the previous remark, we obtain then

$$\begin{aligned}\int_{B\backslash A} T \wedge dd^c v \wedge \beta^{p-1} &= \int_B 1_{B\backslash A} T \wedge dd^c v \wedge \beta^{p-1} \leqslant \liminf_{j\to+\infty} \int_B u_j T \wedge dd^c\tilde{v}_j \wedge \beta^{p-1} \\ &= \int_B \widetilde{T} \wedge dd^c\tilde{v} \wedge \beta^{p-1}.\end{aligned}$$

According to Sibony–Dinh (see pp. 359–360 in [16]) we will have:

$$\int_B \widetilde{T} \wedge dd^c\tilde{v}_j \wedge \beta^{p-1} \leqslant C_{B,B'}\left(\|\widetilde{T}\|_{B''} + \|dd^c\widetilde{T}\|_{B''}\right)\|\tilde{v}\|_\infty(B'').$$

B'' is a neighborhood of B in B', $C_{B,B'}$ is a positive constant depending only on B and B', since $dd^c\widetilde{T} = \widetilde{dd^cT} - S$ with $S := \lim_{k\to+\infty} T \wedge dd^c u_k$, then

$$\|dd^c\widetilde{T}\| \leqslant \|\widetilde{dd^cT}\| + \|S\|,$$

hence

$$\int_B \widetilde{T} \wedge dd^c\tilde{v} \wedge \beta^{p-1} \leqslant C_{B,B'}\left(\|\widetilde{T}\|_{B''} + \|\widetilde{dd^cT}\|_{B''} + \|S\|_{B\cap A}\right)\|\tilde{v}\|_\infty(B'').$$

Definition 1. Let T be a current of bidimension (p,p) defined on an open set $\widetilde{\Omega}$ of $\mathbb{C}^N$, we say that T is a normal current if T and dT are of order zero (its coefficients are measures). We say that T is a pluripositive current if T is a normal positive or negative current and dd^cT is positive.

If T is of bidimension (p,p), To proceed further, we use the current $T\wedge\beta^{p-1}$, which is a current of bidimension $(1,1)$. In order, we use the following lemma (which is proved by Sibony (see [15]) when T is a positive pluripositive current).

Lemma 2. *Let A be a closed complete pluripolar subset of an open $\widetilde{\Omega}$ of $\mathbb{C}^N$ and T be a positive or negative plurisubharmonic (psh) current of bidimension (1,1) on $\widetilde{\Omega}\setminus A$. Assume that $\|T\|_{\widetilde{\Omega}\setminus A}$ and $\|dd^cT\|_{\widetilde{\Omega}\setminus A}$ are finite. Let K' be a compact in $\widetilde{\Omega}$, then there exists a constant $C_{K',\widetilde{\Omega}}>0$, such that for every smooth bounded psh function u on $\widetilde{\Omega}$ vanishing near A and for every a, b, 0 < a < b < 1, we have the following estimate:*

$$\int_{(a\leqslant|u|\leqslant b)\cap K'} T\wedge du\wedge d^cu\leqslant C_{K',\widetilde{\Omega}}(b-a)(\|T\|_{\widetilde{\Omega}\setminus A}+\|dd^cT\|_{\widetilde{\Omega}\setminus A}).$$

The proof for the lemma when T is a pluripositive current still holds true in this case (i.e. if T is a positive psh current) (see p. 31 in [15]).

By using the last lemma and consider the positive functions u_k which previously defined, we will have:

$$\int_{(a\leqslant u_k\leqslant b)\cap K'} T\wedge du_k\wedge d^cu_k\leqslant C_{K',\widetilde{\Omega}}(b-a)(\|\widetilde{T}\|_{\widetilde{\Omega}}+\|\widetilde{dd^cT}\|_{\widetilde{\Omega}}),$$

write

$$dd^cu_k\wedge T:=u_kdd^cT-dd^c(u_kT)+d(d^cu_k\wedge T)-d^c(du_k\wedge T).$$

Let χ is a real-valued smooth function, $0\leqslant\chi\leqslant 1$ supported on K', therefore on $supp(\chi)$ we can write:

$$\begin{aligned}\|dd^cu_k\wedge T\|_{supp(\chi)}&\leqslant\int_{supp(\chi)}\chi dd^cu_k\wedge T\leqslant\int_{K'}\chi dd^cu_k\wedge T\\&=\int_{K'}\chi u_kdd^cT-\int_{K'}dd^c\chi\wedge u_kT-\int_{K'}d\chi\wedge d^cu_k\wedge T+\int_{K'}d^c\chi\\&\wedge du_k\wedge T.\end{aligned}$$

By using a gain the lemma of Sibony–Dinh (see pp. 359–360 in [16]) we then obtain:

$$\begin{aligned}\int_{(a\leqslant u_k\leqslant b)\cap K'}\lim_{k\to+\infty}T\wedge dd^cu_k\wedge\beta^{p-1}&\leqslant\lim_{k\to+\infty}\int_{(a\leqslant u_k\leqslant b)\cap K'}T\wedge dd^cu_k\wedge\beta^{p-1}\\&\leqslant C'_{K',\widetilde{\Omega}}(\|\widetilde{T}\|_{\widetilde{\Omega}}+\|\widetilde{dd^cT}\|_{\widetilde{\Omega}}).\end{aligned}$$

Therefore for every compact K' of $\widetilde{\Omega}$ we have:

$$\|S\|_{K'\cap B\cap A}\leqslant C''_{K',\widetilde{\Omega}}(\|\widetilde{T}\|_{B''}+\|\widetilde{dd^cT}\|_{B''}).$$

Hence

$$\int_{B\setminus A} T \wedge dd^c v \wedge \beta^{p-1} \leqslant C''_{B,B'} \left(\|\widetilde{T}\|_{B''} + \|\widetilde{dd^c T}\|_{B''} \right) \|\tilde{v}\|_\infty (B'').$$

Finally, we get:

$$\|T \wedge dd^c v\|_{K\setminus A} \leqslant C_{K,L} \left(\|T\|_{L\setminus A} + \|dd^c T\|_{L\setminus A} \right) \|v\|_\infty (L \setminus A). \qquad \square$$

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