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Double Hopf bifurcation in delay differential equations

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Abstract. The paper addresses the computation of elements of double Hopf bifurcation for retarded functional differential equations (FDEs) with parameters. We present an efficient method for computing, simultaneously, the coefficients of center manifolds and normal forms, in terms of the original FDEs, associated with the double Hopf singularity up to an arbitrary order. Finally, we apply our results to a nonlinear model with periodic delay. This shows the applicability of the methodology in the study of delay models arising in either natural or technological problems.

Keywords: Double Hopf; Delay; Bifurcation; Functional differential equation; Center manifold; Normal forms; Regenerative cutting tool

1. Introduction

The double Hopf bifurcation, which is readily located via linear stability analysis as the codimension-two point at which two pairs of complex conjugate eigenvalues have their real part simultaneously change sign, has associated very rich nonlinear dynamics in its neighborhood. Depending on the particulars of the system under consideration, there are around thirty different dynamical scenarios, divided into simple and difficult cases. Center manifold theory is of fundamental importance in the study of nonlinear dynamical systems when analyzing bifurcations of such a type. In fact, this theory allows us to reduce the study of a differential equation with delay near to a non-hyperbolic equilibrium point to that of an ordinary differential equation on a finite-dimensional invariant manifold. This

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approach is particularly interesting when the starting point is an infinite-dimensional problem, such as a functional, partial or an integro differential equation. In fact, the reduction forms also a qualitative simplification. Although center manifolds exist, they need not be unique, there are some points which must always be on any center manifold: all the center manifolds have the same Taylor expansion near the equilibrium up to their order of regularity; such an expansion may give an idea of the local dynamics near the steady state. Center manifolds and normal forms of such bifurcations have been discussed in [5,9] for ordinary differential equations. Center manifolds for Hopf and fold-Hopf bifurcations in FDEs with discrete delays have been studied in [6,7]. In [1,2], Faria and Magalhaès have considered the computation of coefficients of normal forms for functional differential equations (FDEs) of both Hopf and Bogdanov singularities. However, it is difficult to apply the method to compute the explicit expressions for these coefficients since it demands much more computation efforts for high-order normal forms.

The attention of this paper will be focused on the development of methodology and software for computing the center manifolds and normal forms of double Hopf bifurcation for general FDEs with parameters. According to the structure of linearized equation of a retarded system evaluated at an equilibrium, the case considered in this paper corresponds to two different pairs of purely imaginary eigenvalues.

The aim is twofold: the primary objective of this paper is to prove that the terms of a parameterized center manifold associated with double Hopf singularity up to the order of regularity for FDEs satisfy an efficient and explicit recursive formulas, and to give an efficient algorithm. The purpose of the second fold is to obtain, for the general situation of double Hopf bifurcation for FDEs, explicit formulas giving the coefficients of normal forms in terms of the coefficients of the original equation. This allows a simpler scheme that greatly saves computational time and computation memory. Furthermore, the obtained method combines center manifolds and normal form schemes into one step. To show the applicability and the efficiency of the method, we apply our algorithmic scheme to obtain normal forms of a nonlinear mechanical model with periodic delay. We achieve this by augmenting the explicit-time dependent delay terms as new state variables to the original equations of motions with appropriate initial conditions and applying the method of computing the center manifolds and normal forms obtained in the present paper. For the notation, background about the theory of FDEs and all needed results in the remainder of this paper, we follow [9]. However, we use $C_n = C([-r, 0], \mathbb{R}^n), r \geqslant 0$ since we need to work in realization spaces with different dimensions, depending on whether the parameters are incorporated or not incorporated in the realization space variables.

The paper is organized as follows: a computational scheme for a center manifold associated with double Hopf with parameters are given in Section 2. Normal forms for FDEs are derived, in terms of the original equation and the center manifold coefficients, in the same section. To illustrate our results, we study the bifurcation of a model arising from the mechanics in Section 3. Conclusions are drawn in Section 4.

2. Main results

In this section, we present our result concerning the computation of terms of center manifolds and normal forms for FDEs of the form

$$\dot{x}(t) = L(\alpha)x_t + F(x_t, \alpha), \tag{2.1}$$

where $\alpha \in \mathbb{R}^p, \alpha \mapsto L(\alpha)$ is a C^∞ function with values in the space of bounded linear operators from $C_n := C([-r, 0], \mathbb{R}^n)$ to $\mathbb{R}^n$. $F : C_n \times \mathbb{R}^p \to \mathbb{R}^n$ is assumed, without loss of generality, to be sufficiently smooth ($F \in C^\infty$) with $F(0, \alpha) = 0$ and $DF(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}^p$.

We denote $L^0 = L(0)$. In the sequel of this paper, we assume that the following hypothesis is satisfied:

(H): The linear equation $\dot{x}(t) = L^0 x_t$ has two purely imaginary pairs $(\pm i\omega_1)$ and $(\pm i\omega_2)$ as eigenvalues and no other characteristic values with zero real part.

One way of considering center manifolds for differential equations with parameters is to reduce the situation to the case of differential equations without parameters by considering the system

$$\dot{x}(t) = L^0 x_t + [L(\alpha) - L^0]x_t + F(x_t, \alpha), \tag{2.2}$$

$$\dot{\alpha}(t) = 0. \tag{2.3}$$

The solutions of this system are of the form $\tilde{x}(t) := (x(t), \alpha(t)) \in \mathbb{R}^{n+p}$, the phase space is $\widetilde{C} := C_{n+p}$, and the system can be written as

$$\dot{\tilde{x}}(t) = \widetilde{L}\tilde{x}_t + \widetilde{F}(\tilde{x}_t), \tag{2.4}$$

where $\widetilde{L}(u, v) := (L^0 u, 0)$ and $\widetilde{F}(u, v) := ([L(v(0)) - L^0]u + F(u, v(0)), 0)$ with $u \in C_n, v \in C_p$. Let A and $\widetilde{A}$ denote the infinitesimal generators associated with the equations $\dot{x}(t) = L^0 x_t$ and $\dot{x}(t) = \widetilde{L}x_t$, respectively. The equation $\dot{\alpha}(t) = 0$ has a unique characteristic value $\lambda = 0$ with multiplicity p. The associated generalized eigenspace consists of the elements of C_p which are constant functions, and it is denoted here also by $\mathbb{R}^p$. Let $\Lambda := \sigma(A) \cap i\mathbb{R}$ and consider the spectral decomposition $C_n = X_c \oplus X_s$ as in [9]. In particular, we consider bases for X_c and X_c^* denoted by $\Phi = (\phi_1, \cdots, \phi_4) and \Psi = \mathrm{col}(\psi_1, \cdots, \psi_4)$, respectively, and satisfying $(\Psi, \Phi) = I$. We define $\widetilde{\Lambda} = \Lambda \bigvee \{0\}, \widetilde{X}_c = X_c \times \mathbb{R}^p, and\ \widetilde{X}_s = X_s \times R$ where $R = \{v \in C_p : v(0) = 0\}$. As bases of $\widetilde{X}_c$ and $\widetilde{X}_c^*$, we consider, respectively, the columns of the matrix $\widetilde{\Phi}$ and the rows of the matrix $\widetilde{\Psi}$,

$$\widetilde{\Phi} = \begin{bmatrix} \Phi & 0 \\ 0 & I_p \end{bmatrix}, \quad \widetilde{\Psi} = \begin{bmatrix} \Psi & 0 \\ 0 & I_p \end{bmatrix},$$

which satisfies $\langle \widetilde{\Psi}, \widetilde{\Phi} \rangle = I_{n+p}$, where $\langle \cdot, \cdot \rangle$ is the bilinear form in $\widetilde{C}^* \times \widetilde{C}$ introduced in [9] and defined by

$$< \psi, \phi >= \psi(0)\phi(0) - \int_{-r}^{0} \int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi.$$

and $\dot{\widetilde{\Phi}} = \widetilde{\Phi}\widetilde{B}$ with

$$\widetilde{B} = \begin{bmatrix} -i\omega_1 & 0 & 0 & 0 & 0 \\ 0 & i\omega_1 & 0 & 0 & 0 \\ 0 & 0 & -i\omega_2 & 0 & 0 \\ 0 & 0 & 0 & i\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0_p \end{bmatrix}.$$

Then, we obtain the decomposition $\widetilde{C} = \widetilde{X}_c \oplus \widetilde{X}_s$, where $\widetilde{X}_c$ is the invariant space of $\widetilde{A}$ associated with $\widetilde{\Lambda}$.

In the sequel, we recall the definition of a local center manifold associated to Eq. (2.4).

Definition 2.1 [9]. Given a C^1 map $\tilde{h}$ from $\mathbb{R}^{4+p}$ into $\widetilde{X}_s$, the graph of $\tilde{h}$ is said to be a local center manifold associated to Eq. (2.4) if and only if $\tilde{h}(0) = 0, D\tilde{h}(0) = 0$, and there exists a neighborhood V of zero in $\mathbb{R}^{4+p}$ such that, for each $\xi \in V$, there exist $\delta = \delta(\xi) > 0$ and a function x defined on $]-\delta - r, \delta[$ such that $x_0 = \widetilde{\Phi}\xi + \tilde{h}(\xi)$ and x verify Eq. (2.4) on $]-\delta, \delta[$ and satisfy the identity

$$x_t = \widetilde{\Phi}\tilde{z}(t) + \tilde{h}(\tilde{z}(t)), \text{for } t \in [0, \delta[,$$

where $\tilde{z}(t)$ is the unique solution of the ordinary differential equation (ODE)

$$\begin{cases} \frac{d}{dt}\tilde{z}(t) = \widetilde{B}\tilde{z}(t) + \widetilde{\Psi}(0)\widetilde{F}(\widetilde{\Phi}\tilde{z}(t) + \tilde{h}(\tilde{z}(t))), \\ \tilde{z}(0) = \xi, \qquad \xi \in \mathbb{R}^{4+p}. \end{cases} \tag{2.5}$$

Remark 2.2. (i) If we write $\tilde{h}(\tilde{z}) = (h(\tilde{z}), h_0(\tilde{z})), \tilde{z} = (z, \alpha)$ for $z \in \mathbb{R}^4$ and $\alpha \in \mathbb{R}^p$, then Eq. (2.5) is equivalent to

$$\frac{d}{dt}\begin{pmatrix} z \\ \alpha \end{pmatrix} = \begin{pmatrix} Bz \\ 0 \end{pmatrix} + \begin{pmatrix} \Psi(0)[(L(\alpha(0)+h_0(z,\alpha)(0)) - L^0)(\Phi z + h(z,\alpha)) + F(\Phi z + h(z,\alpha), \alpha(0) + h_0(z,\alpha)(0))] \\ 0 \end{pmatrix}$$

with

$$B = \begin{bmatrix} -i\omega_1 & 0 & 0 & 0 \\ 0 & i\omega_1 & 0 & 0 \\ 0 & 0 & -i\omega_2 & 0 \\ 0 & 0 & 0 & i\omega_2 \end{bmatrix}$$

and $z(0) = \xi \in \mathbb{R}^4$. Noting that $h_0(z(t), \alpha)(0) = 0$ and dropping the auxiliary equations introduced for handling the parameter, we get the reduced equation on the center manifold – represented as a graph over the ξ and α variables of $h(\xi, \alpha)$ for sufficiently small ξ and α – associated with the parameterized Eq. (2.1) as

$$\begin{cases} \frac{d}{dt}z(t) = Bz(t) + \Psi(0)[(L(\alpha) - L^0)(\Phi z(t) + h(z(t),\alpha)) + F(\Phi z(t) + h(z(t),\alpha),\alpha)], \\ z(0) = \xi, \quad \xi \in \mathbb{R}^4. \end{cases} \tag{2.6}$$

(ii) It is noted that the invariance properties of center manifolds guarantee that any small solutions bifurcating from $(0, 0, 0)$ must lie in any center manifold and thus we may follow the local evolution of bifurcating families of solutions in this suspended family of center manifolds (see [18] for more details).

2.1. Calculation of center manifolds

In the following, we give an analytic characterization of a center manifold associated to Eq. (2.1).

Theorem 2.3. *Given a C^1 map h from $\mathbb{R}^{4+p}$ into X_s with $h(0) = 0$ and $Dh(0) = 0$, a necessary condition for the graph of h to be a local center manifold of Eq. (2.1) is that there exists a neighborhood V of zero in $\mathbb{R}^{4+p}$ such that, for each $(\xi, \alpha) \in V$,*

$$\begin{aligned} \frac{\partial}{\partial\theta}(h(\xi,\alpha))(\theta) = & -i\omega_1 \frac{\partial h(\xi,\alpha)}{\partial \xi_1}(\theta)\xi_1 + i\omega_1 \frac{\partial h(\xi,\alpha)}{\partial \xi_2}(\theta)\xi_2 - i\omega_2 \frac{\partial h(\xi,\alpha)}{\partial \xi_3}(\theta)\xi_3 \\ & + i\omega_2 \frac{\partial h(\xi,\alpha)}{\partial \xi_4}(\theta)\xi_4 + \frac{\partial h(\xi,\alpha)}{\partial \xi}(\theta)\Psi(0)[(L(\alpha) - L(0))(\Phi\xi + h(\xi,\alpha)] \\ & + \frac{\partial h(\xi,\alpha)}{\partial \xi}(\theta)\Psi(0)F(\Phi\xi + h(\xi,\alpha),\alpha) + \Phi(\theta)\Psi(0)[(L(\alpha) \\ & - L(0))(\Phi\xi + h(\xi,\alpha)] + \Phi(\theta)\Psi(0)F(\Phi\xi + h(\xi,\alpha),\alpha), \end{aligned} \tag{2.7}$$

$$\frac{\partial}{\partial\theta}(h(\xi,\alpha))(0) = L^0 h(\xi,\alpha) + (L(\alpha) - L^0)(\Phi\xi + h(\xi,\alpha)) + F(\Phi\xi + h(\xi,\alpha),\alpha). \tag{2.8}$$

Proof. Let h be a graph of a local center manifold of Eq. (2.1). From Definition 2.1, there exists a neighborhood V of zero in $\mathbb{R}^{4+p}$ such that, for each $(\xi, \alpha) \in V$, there exists $\delta > 0$ such that the solution of (2.1) with initial data $\Phi\xi + h(\xi, \alpha)$ exists in the interval $]-\delta - r, \delta[$ and is given by

$$x_t = \Phi z(t) + h(z(t),\alpha), \text{for } t \in]-\delta, \delta[,$$

such that $z(t)$ is a solution of the equation

$$\begin{cases} \frac{d}{dt}z(t) = Bz(t) + \Psi(0)[(L(\alpha) - L^0)(\Phi z(t) + h(z(t),\alpha)) + F(\Phi z(t) + h(z(t),\alpha),\alpha)], \\ z(0) = \xi, \quad \xi \in \mathbb{R}^4, \end{cases}$$

with $B = \begin{bmatrix} -i\omega_1 & 0 & 0 & 0 \\ 0 & i\omega_1 & 0 & 0 \\ 0 & 0 & -i\omega_2 & 0 \\ 0 & 0 & 0 & i\omega_2 \end{bmatrix}$.

The variation of constants formula of Eq. (2.1) can be written as

$$x_t = T(t)\phi + \int_0^t T(t-s)X_0[(L(\alpha) - L^0)x_s + F(x_s, \alpha)]ds, \ \text{t} \geqslant 0,$$

where $(T(t))_{t\geqslant 0}$ is the semi-group solution of the linear equation $\dot{x}(t) = L^0 x_t$. It follows from the decomposition of the phase space $\Lambda : C_n = X_c \oplus X_s$ that the function h satisfies

$$h(z(t), \alpha) = T(t)h(z(t), \alpha) + \int_0^t T(t-\sigma)X_0^s F(\Phi z(\sigma) + h(z(\sigma), \alpha), \alpha)d\sigma.$$

Then

$$\begin{aligned}\frac{1}{t}[T(t)h(\xi,\alpha) - h(\xi,\alpha)] &= \frac{1}{t}[h(z(t),\alpha) - h(z(0),\alpha)] \\ &- \frac{1}{t}\int_0^t T(t-\sigma)X_0^s[(L(\alpha) - L^0)(\Phi z(\sigma) + h(z(\sigma),\alpha)) + F(\Phi z(\sigma) + h(z(\sigma),\alpha),\alpha)]d\sigma,\end{aligned}$$

which implies, from the fact that h and F are smooth and $T(\cdot)$ is a strongly continuous semi-group on the Banach space C_n, that $h(\xi)$ is in the domain of A, and

$$\begin{aligned}Ah(\xi,\alpha) = \left(\frac{\partial h(\xi,\alpha)}{\partial \xi}\right)&\{B\xi + \Psi(0)[(L(\alpha) - L^0)(\Phi\xi + h(\xi,\alpha)) + F(\Phi\xi \\ &+ h(\xi,\alpha),\alpha)]\} - X_0^s[(L(\alpha) - L^0)(\Phi\xi + h(\xi,\alpha)) + F(\Phi\xi + h(\xi,\alpha),\alpha)].\end{aligned}$$

Consequently, by evaluating the above equation at $\theta \neq 0$, we have

$$\begin{aligned}\frac{\partial}{\partial\theta}(h(\xi,\alpha))(\theta) = \left(\frac{\partial h(\xi,\alpha)}{\partial\xi}(\theta)\right)&\{B\xi + \Psi(0)[(L(\alpha) - L^0)(\Phi\xi + h(\xi,\alpha)) \\ &+ F(\Phi\xi + h(\xi,\alpha),\alpha)]\} + \Phi(\theta)\Psi(0)[(L(\alpha) - L^0)(\Phi\xi \\ &+ h(\xi,\alpha)) + F(\Phi\xi + h(\xi,\alpha),\alpha)],\end{aligned} \tag{2.9}$$

which is the formula (2.7) of theorem.

On the other hand, it results from the fact that the semi-flow $t \mapsto x_t = \Phi z(t) + h(z(t), \alpha)$ exists on the open $]-\delta, \delta[$ such that for $\theta \in]-\delta, 0]$,

$$\begin{aligned}\frac{d}{d\theta}(\Phi(\theta)\xi + h(\xi,\alpha)(\theta)) &= \frac{d}{d\theta}x_0(\theta) \\ &= \frac{d}{d\theta}x(\theta) \\ &= L(\alpha)(\Phi z(\theta) + h(z(\theta),\alpha)) + F(\Phi z(\theta) + h(z(\theta),\alpha),\alpha).\end{aligned}$$

Consequently, by the fact that $\frac{d}{d\theta}\Phi(0)\xi = \Phi(0)B\xi = L^0(\Phi\xi)$, we obtain

$$\frac{\partial}{\partial\theta}(h(\xi,\alpha))(0) = L^0h(\xi) + (L(\alpha) - L^0)(\Phi\xi + h(\xi,\alpha)) + F(\Phi\xi + h(\xi,\alpha),\alpha), \quad (2.10)$$

which completes the proof of theorem. □

Let us recall that the function h which represents the center manifold for Eq. (2.1) has the same regularity as the nonlinearity F. From this fact and in view of the assumed smoothness on F, for all $m \in \mathbb{N}$, we can write

$$h(\xi,\alpha) = \sum_{k=2}^{m} h_k(\xi,\alpha) + \chi(\xi,\alpha) \text{ for } \xi \in V, \quad (2.11)$$

where h_k is the homogeneous part of degree k and $\chi(\xi,\alpha) = o(|(\xi,\alpha)|^m)$.

Let $k \in \mathbb{N},\ k \geqslant 2$. The homogeneous parts of degree k of Eqs. (2.7) and (2.8) are respectively given by

$$\frac{\partial}{\partial\theta}(h_k(\xi,\alpha))(\theta) = -i\omega_1\frac{\partial h_k(\xi,\alpha)}{\partial\xi_1}\xi_1 + i\omega_1\frac{\partial h_k(\xi,\alpha)}{\partial\xi_2}\xi_2 - i\omega_2\frac{\partial h_k(\xi,\alpha)}{\partial\xi_3}\xi_3 + i\omega_2\frac{\partial h_k(\xi,\alpha)}{\partial\xi_4}\xi_4 + \mathcal{N}^{k-1}(\xi,\alpha), \quad (2.12)$$

$$\frac{\partial}{\partial\theta}(h_k(\xi,\alpha))(0) = L^0h_k(\xi,\alpha) + R^{k-1}(\xi,\alpha), \quad (2.13)$$

where

$$\mathcal{N}^{k-1}(\xi,\alpha) = \Phi\Psi(0)R^{k-1}(\xi,\alpha) + \sum_{j=2}^{k-1}\frac{\partial h_{k-j+1}(\xi,\alpha)}{\partial\xi}\Psi(0)R^{j-1},$$

and R^{i-1} is the homogeneous part of degree i of $R(\xi,\alpha) = (L(\alpha) - L^0)\ (\Phi\xi + h(\xi,\alpha)) + F(\Phi\xi + h(\xi,\alpha),\alpha)$. In particular, R^1 is the homogeneous part of degree 2 of $(L(\alpha) - L^0)\Phi\xi + F(\Phi\xi,\alpha)$ which is independent from the terms of a center manifold.

If $\xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{R}^4, q = (q_1,q_2,q_3,q_4), \alpha = (\alpha_1,\cdots,\alpha_p), \alpha^l = \alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_p^{l_p}$ for $l = (l_1,\cdots l_p) \in \mathbb{N}^p$, and $|l| = \sum_{i=1}^{p} l_i$, then we can write

$$\begin{aligned} h_k(\xi,\alpha) &= \sum_{(q,l)\in D_k} h^k_{(q,l)}\xi_1^{q_1}\xi_2^{q_2}\xi_3^{q_3}\xi_4^{q_4}\alpha^l, \quad \text{for some } \mathrm{h}^k_{(q,l)} \in X_s, \\ \mathcal{N}^{k-1}(\xi,\alpha)(\theta) &= \sum_{(q,l)\in D_k} N^{k-1}_{(q,l)}\xi_1^{q_1}\xi_2^{q_2}\xi_3^{q_3}\xi_4^{q_4}\alpha^l, \quad \text{for some } \mathrm{N}^k_{(q,l)} \in X_s, \end{aligned} \quad (2.14)$$

and

$$R^{k-1}(\xi,\alpha) = \sum_{(q,l)\in D_k} R^{k-1}_{(q,l)}\xi_1^{q_1}\xi_2^{q_2}\xi_3^{q_3}\xi_4^{q_4}\alpha^l, \quad \text{for some } R^{k-1}_{(q,l)} \in \mathbb{R}^n, \quad (2.15)$$

where $D_k = \{(q,l) \in \mathbb{N}^4 \times \mathbb{N}^p : |(q,l)| = k\}$.

For simplicity, we put $W_q = -\omega_1(q_1 - q_2) - \omega_2(q_3 - q_4)$.

Theorem 2.4. *Assume that (H) holds. Then the vector of the coefficients of the homogeneous part of degree k of a local center manifold associated with Eq. (2.1) is given in a unique way by the following recursive formulas:*

For $k = 2, (q_1, q_2, q_3, q_4, l) \in D_2$*:*

$$\begin{aligned}
&h^2_{(q_1,q_2,q_3,q_4,l)}(\theta)\\
&= \Big\{ e^{iW_q\theta} h^2_{(q_1,q_2,q_3,q_4,l)}(0) + \frac{1}{i(W_q+\omega_1)}(e^{iW_q\theta} - e^{-i\omega_1\theta})\phi_1(0)\psi_1(0)\\
&\quad + \frac{1}{i(W_q-\omega_1)}(e^{iW_q\theta} - e^{i\omega_1\theta})\phi_2(0)\psi_2(0) + \frac{1}{i(W_q+\omega_2)}(e^{iW_q\theta} - e^{-i\omega_2\theta})\phi_3(0)\psi_3(0)\\
&\quad + \frac{1}{i(W_q-\omega_1)}(e^{iW_q\theta} - e^{i\omega_1\theta})\phi_2(0)\psi_2(0) + \frac{1}{i(W_q+\omega_2)}(e^{iW_q\theta} - e^{-i\omega_2\theta})\phi_3(0)\psi_3(0)\\
&\quad + \frac{1}{i(W_q-\omega_2)}(e^{iW_q\theta} - e^{i\omega_2\theta})\phi_4(0)\psi_4(0)\Big\} R^1_{(q_1,q_2,q_3,q_4,l)}.
\end{aligned} \tag{2.16}$$

For $k > 2$ and $(q_1, q_2, q_3, q_4, l) \in D_k$*:*

$$h^k_{(q_1,q_2,q_3,q_4,l)}(\theta) = e^{iW_q\theta} h^k_{(q_1,q_2,q_3,q_4,l)}(0) + \int_0^\theta e^{iW_q(\theta-s)} N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(s)ds, \tag{2.17}$$

where the vectors $h^k_{(q_1,q_2,q_3,q_4,l)}(0), (q_1, q_2, q_3, q_4, l) \in D_k$ *are given by solving the following systems:*

$$\text{for } W_q \notin \{\pm\omega_1, \pm\omega_2\}, \qquad h^k_{(q_1,q_2,q_3,q_4l)}(0) = [\Delta(iW_q)]^{-1} E^{k-1}_{(q_1,q_2,q_3,q_4,l)}, \tag{2.18}$$

$$\text{for } W_q = -\omega_1, \qquad M_1 \begin{pmatrix} h^k_{(q_1,q_2,q_3,l)}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,l)} \\ v^{k-1}_{(q_1,q_2,q_3,l)} \end{pmatrix}, \tag{2.19}$$

$$\text{for } W_q = \omega_1, \qquad \overline{M}_1 \begin{pmatrix} h^k_{(q_1,q_2,q_3,q_4,l)}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \\ v^{k-1}_{(q_1,q_2,q_3,q_4,l)} \end{pmatrix}, \tag{2.20}$$

$$\text{for } W_q = -\omega_2, \qquad M_2 \begin{pmatrix} h^k_{(q_1,q_2,q_3,q_4,l)}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \\ v^{k-1}_{(q_1,q_2,q_3,q_4,l)} \end{pmatrix}, \tag{2.21}$$

and

$$\text{for } W_q = \omega_2, \qquad \overline{M}_2 \begin{pmatrix} h^k_{(q_1,q_2,q_3,q_4,l)}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \\ v^{k-1}_{(q_1,q_2,q_3,q_4,l)} \end{pmatrix}. \tag{2.22}$$

M_1 *and* M_2 *are* $(n+1) \times (n+1)$ *matrices defined by*

$$M_1 = \begin{pmatrix} \Delta(-\omega_1 i) & \psi_1^\top(0) \\ \langle \psi_1, e^{-\omega_1 i \cdot} \rangle & 0 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} \Delta(-\omega_2 i) & \psi_3^\top(0) \\ \langle \psi_3, e^{-\omega_2 i \cdot} \rangle & 0 \end{pmatrix}.$$

$E^{k-1}_{(q_1,q_2,q_3,l)}$ *and* $v^{k-1}_{(q_1,q_2,q_3,l)}$ *are vectors given by means of the coefficients of the center manifolds already computed (see (2.26) and (2.28) in the proof below).*

Proof. Let $(q_1, q_2, q_3, q_4, l) \in D_k$, from Eq. (2.12), we have

$$\frac{dh^k_{(q_1,q_2,q_3,q_4,l)}(\theta)}{d\theta} = [-i\omega_1(q_1-q_2) - i\omega_2(q_3-q_4)]h^k_{(q_1,q_2,q_3,q_4,l)}(\theta) + N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(\theta), \tag{2.23}$$

or equivalently

$$h^k_{(q_1,q_2,q_3,q_4,l)}(\theta) = e^{iW_q\theta}h^k_{(q_1,q_2,q_3,q_4,l)}(0) + \int_0^\theta e^{i(\theta-s)W_q}N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(s)ds. \tag{2.24}$$

However, for $k = 2$, we have that

$$\begin{aligned} N^1(\xi,\alpha)(\theta) &= \Phi(\theta)\Psi(0)R^1(\xi,\alpha) \\ &= \left[e^{-i\omega_1\theta}\phi_1(0)\psi_1(0) + e^{i\omega_1\theta}\phi_2(0)\psi_2(0) + e^{-i\omega_2\theta}\phi_3(0)\psi_3(0) + e^{i\omega_2\theta}\phi_4(0)\psi_4(0)\right]R^1(\xi,\alpha). \end{aligned}$$

It follows that

$$\begin{aligned} &\int_0^\theta e^{iW_q(\theta-s)}N^1_{(q_1,q_2,q_3,q_4,l)}(s)ds \\ &= e^{iW_q\theta}\left\{\int_0^\theta e^{-i(W_q+\omega_1)s}ds\phi_1(0)\psi_1(0) + \int_0^\theta e^{-i(W_q-\omega_1)s}ds\phi_2(0)\psi_2(0)\right. \\ &\quad \left. + \int_0^\theta e^{-i(W_q+\omega_2)s}ds\phi_3(0)\psi_3(0) + \int_0^\theta e^{-i(W_q-\omega_2)s}ds\phi_4(0)\psi_4(0)\right\}R^1_{(q_1,q_2,q_3,q_4,l)}, \end{aligned}$$

which implies, by discussing the cases whether $W_q \pm \omega_{1,2} = 0$ or not, the value of $h^2_{(q_1,q_2,q_3,q_4,l)}(\theta)$ is given in theorem.

The Eq. (2.13) is equivalent to

$$\frac{dh^k_{(q_1,q_2,q_3,q_4,l)}}{d\theta}(0) = L^0h^k_{(q_1,q_2,q_3,q_4,l)} + R^{k-1}_{(q_1,q_2,q_3,q_4,l)},$$

which gives by evaluating the relation (2.23) at $\theta = 0$ that

$$L^0h^k_{(q_1,q_2,q_3,q_4,l)} + R^{k-1}_{(q_1,q_2,q_3,q_4,l)} = iW_qh^k_{(q_1,q_2,q_3,q_4,l)}(0) + N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(0).$$

On the other hand, by taking the values of the linear operator L^0 on both sides of Eq. (2.24), we have

$$L^0h^k_{(q_1,q_2,q_3,q_4,l)} = L^0(e^{iW_q\cdot})h^k_{(q_1,q_2,q_3,q_4,l)}(0) + L^0\left(\int_0^\cdot e^{i(\cdot-s)W_q}N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(s)ds\right).$$

Consequently, we obtain from the expression of the characteristic equation that

$$\Delta(iW_q)h^k_{(q_1,q_2,q_3,q_4,l)}(0) = E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \quad \text{for } (q_1,q_2,q_3,q_4,l) \in D_k, \tag{2.25}$$

where

$$E^{k-1}_{(q_1,q_2,q_3,q_4,l)} = R^{k-1}_{(q_1,q_2,q_3,q_4,l)} + L^0\left(\int_0^{\cdot} e^{i(\cdot-s)W_q} N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(s)ds\right) - N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(0). \tag{2.26}$$

We will also use the fact that all center manifolds have rank in the subspace X_s, which implies in particular that $\langle \psi_1, h^k_{(q_1,q_2,q_3,q_4,l)}(\cdot)\rangle = 0$ for all $(q_1,q_2,q_3,q_4,l) \in D_k$ and $k > 1$. Consequently, we obtain from Eq. (2.24) that

$$\langle \psi_1, e^{iW_q\cdot}\rangle h^k_{(q_1,q_2,q_3,q_4,l)}(0) = v^{k-1}_{(q_1,q_2,q_3,q_4,l)}, \quad \text{for } (q_1,q_2,q_3,q_4,l) \in D_k, \tag{2.27}$$

where the vector $v^{k-1}_{(q_1,q_2,q_3,q_4,l)}$ is given by

$$v^{k-1}_{(q_1,q_2,q_3,q_4,l)} = -\left\langle \psi_1, \int_0^{\cdot} e^{i(\cdot-s)W_q} N^{k-1}_{(q_1,q_2,q_3,q_4,l)}(s)ds\right\rangle. \tag{2.28}$$

We introduce the $(n+1)\times(n+1)$ matrices M_1 and M_2 defined by

$$M_1 = \begin{pmatrix} \Delta(-i\omega_1) & \psi_1^\top(0) \\ \langle \psi_1, e^{-i\omega_1\cdot}\rangle & 0 \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} \Delta(-i\omega_2) & \psi_3^\top(0) \\ \langle \psi_3, e^{-i\omega_2\cdot}\rangle & 0 \end{pmatrix}.$$

Then it follows from the Eqs. (2.25) and (2.27) that the vectors defined by

$$\hat{h}^k_{(q_1,q_2,q_3,q_4,l)}(0) = \begin{pmatrix} h^k_{(q_1,q_2,q_3,q_4,l)}(0) \\ 0 \end{pmatrix}$$

satisfy

$$M_1\hat{h}^k_{(q_1,q_2,q_3,q_4,l)}(0) = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \\ v^{k-1}_{(q_1,q_2,q_3,q_4,l)} \end{pmatrix} \qquad \text{for } W_q = -\omega_1,$$

$$\overline{M}_1\hat{h}^k_{(q_1,q_2,q_3,q_4,l)}(0) = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \\ v^{k-1}_{(q_1,q_2,q_3,q_4,l)} \end{pmatrix} \qquad \text{for } W_q = \omega_1,$$

$$M_2\hat{h}^k_{(q_1,q_2,q_3,q_4,l)}(0) = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \\ v^{k-1}_{(q_1,q_2,q_3,q_4,l)} \end{pmatrix} \qquad \text{for } W_q = -\omega_2$$

and

$$\overline{M}_2\hat{h}^k_{(q_1,q_2,q_3,q_4,l)}(0) = \begin{pmatrix} E^{k-1}_{(q_1,q_2,q_3,q_4,l)} \\ v^{k-1}_{(q_1,q_2,q_3,q_4,l)} \end{pmatrix} \qquad \text{for } W_q = \omega_2.$$

Now to complete the proof of the theorem, we need to show the uniqueness of $h^k_{(q_1,q_2,q_3,q_4,l)}(0)$, which is a consequence of the next lemma: $\square$

Lemma 1. *The* $(n+1)\times(n+1)$ *matrices* M_1 *and* M_2 *are invertible.*

Proof. Let $x \in \mathbb{R}^n, \eta \in \mathbb{R}$ such that

$$M_1\begin{pmatrix} x \\ \eta \end{pmatrix} = 0.$$

Then

$$\begin{cases} \Delta(-i\omega_1)x + \eta\psi_1^\top(0) = 0, \\ \langle \psi_1, e^{-i\omega_1 \cdot}\rangle x = 0. \end{cases}$$

From the fact that $\psi_1(0)\Delta(-i\omega_1) = 0$, we have $\eta = 0$, and the above system becomes

$$\begin{cases} \Delta(-i\omega_1)x = 0 \\ \langle \psi_1, e^{-i\omega_1 \cdot}\rangle x = 0. \end{cases}$$

From the fact that $\dim\ker\Delta(-i\omega_1) = 1$ and since we have $\Delta(-i\omega_1)\phi_1(0) = 0$, the first equation implies that $x \in span\{\phi_1(0)\}$, which follows that there exists $c \in \mathbb{R}$ such that $x = c\phi_1(0)$. From the second equation, we have $c\langle\psi_1, \phi_1\rangle = 0$ which yields $c = 0$ and finally we obtain $x = 0$. Consequently M_1 is invertible. By the same manner as before, we prove that M_2 is invertible. □

2.2. Normal forms calculation

For $j \in \mathbb{N}$ and Y a normed space, $V_j^m(Y)$ denotes the space of homogeneous polynomials of degree j and m variables $z = (z_1, z_2, \ldots, z_m)$ with coefficients in Y. In other words

$$V_j^m(Y) = \left\{ \sum_{|q|=j} c_q z^q : q \in \mathbb{N}^m, c_q \in Y \right\}$$

where $|q| = \sum_{i=1}^m q_i$.

Let

$$\widetilde{H}(z,\alpha) := \Psi(0)[(L(\alpha) - L^0)(\Phi z + h(z,\alpha)) + F(\Phi z + h(z,\alpha),\alpha)].$$

If we suppose that the homogeneous polynomials $(h_i)_{i\geqslant 2}$ are already computed, the ODE (2.6) is known at any order. In other words, if $\widetilde{H} = \sum_{i\geqslant 2}\widetilde{H}_i$, then $\widetilde{H}_i$ is known for all $i \geqslant 2$ once $(h_i)_{i\geqslant 2}$ are computed.

Consider a nonlinear transformation of the form

$$z = x + V(x,\alpha), \tag{2.29}$$

where V is mapping from $\mathbb{R}^m$ into $\mathbb{R}^m$ such that $V(0) = 0$ and $DV(0) = 0$. Assume that the effects of the above change of variables on the reduced system (2.6) is of the form

$$\dot{x} = Bx + N(x,\alpha). \tag{2.30}$$

Let

$$V = \sum_{i\geqslant 2} V_i, N = \sum_{i\geqslant 2} N_i$$

and

$$(L(\alpha) - L^0)(\phi) + F(\phi, \alpha) = \sum_{i \geqslant 2} f_j(\phi, \alpha). \tag{2.31}$$

Definition 2.5. Let $j \geqslant 2$. We denote by M_j^1 the operator defined by

$$\left(M_j^1 p\right)(x) = Dp(x)Bx - Bp(x),$$

for all $p \in V_j^m(\mathbb{R}^m)$.

The operators M_j^1, defined above, are exactly the Lie brackets that appear in computing normal forms for finite dimensional ODEs. It is well known that the space $V_j^m(\mathbb{R}^m)$ may be decomposed as

$$V_j^m(\mathbb{R}^m) = \mathrm{Im}(M_j^1) \oplus \mathrm{Im}\left(M_j^1\right)^c, \tag{2.32}$$

where the complementary space $Im\left(M_j^1\right)^c$ is not uniquely determined.

The following theorem summarizes the result for the recursive efficient approach for computing the kth order normal form and the kth order nonlinear transformation V_k in terms of center manifold coefficients $(h_i)_{2 \leqslant i \leqslant k}$ as well as the terms $(N_i)_{2 \leqslant i \leqslant k-1}$ already computed.

Theorem 2.6. *Fix $k \geqslant 2$. Then, the recursive formula for computing the coefficients of the normal form associated to the decomposition (2.32) is given by*

$$\begin{aligned} N_k(x,\alpha) = {} & \widetilde{H}_k(x,\alpha) - M_k^1 V_k(x,\alpha) \\ & + \sum_{i=2}^{k-1} \left\{ D\widetilde{H}_{k+1-i}(x,\alpha) V_i(x,\alpha) - DV_i(x,\alpha) N_{k+1-i}(x,\alpha) \right\} \\ & + \sum_{j=2}^{[\frac{k}{2}]} \frac{1}{j!} \sum_{i=j}^{k-j} D^j \widetilde{H}_i(x,\alpha) \left\{ \sum_{l_1+l_2+\cdots+l_j = k-i+j} V_{l_1} V_{l_2} \cdots V_{l_j} \right\}(x,\alpha), \end{aligned} \tag{2.33}$$

and for all $i \geqslant 2$,

$$\begin{aligned} \widetilde{H}_i(x,\alpha) = {} & \Psi(0) f_i(\Phi x, \alpha) + \sum_{j=2}^{i-1} \Psi(0) Df_j(\Phi x, \alpha) h_{i+1-j}(x) \\ & + \Psi(0) \sum_{j=2}^{[\frac{i}{2}]} \frac{1}{j!} \sum_{p=j}^{i-j} D^j f_p(\Phi x, \alpha) \left\{ \sum_{l_1+l_2+\cdots+l_j = k-i+j} h_{l_1} h_{l_2} \cdots h_{l_j} \right\}(x,\alpha). \end{aligned}$$

Proof. The Taylor expansion of $\widetilde{H}(x,\alpha)$ near $h = 0$ is given by

$$\widetilde{H}(x,\alpha) = \psi(0) f(\Phi x, \alpha) + \psi(0) Df(\Phi x, \alpha) h(x,\alpha) + \psi(0) \sum_{i \geqslant 2} \frac{1}{i!} D^i f(\Phi x, \alpha) [h(x,\alpha)]^i.$$

So, according to the representations (2.11) and (2.31), the above equation becomes

$$\sum_{i\geqslant 2}\widetilde{H}_i = \psi(0)\sum_{i\geqslant 2}f_i(\Phi x,\alpha) + \Psi(0)\sum_{j\geqslant 2}\sum_{i\geqslant 2}Df_i(\Phi x,\alpha)h_j(x,\alpha)$$
$$+ \Psi(0)\sum_{j\geqslant 2}\frac{1}{j!}\sum_{i\geqslant 2}D^jf_i(\Phi x,\alpha)\left[\sum_{l\geqslant 2}h_l(x,\alpha)\right]^i. \tag{2.34}$$

Comparing the same order terms of (2.34) yields the second formula of the theorem.

On the other hand, substituting the change of variables of (2.29) into (2.6), we have

$$Bz + \sum_{i\geqslant 2}\widetilde{H}_i(z,\alpha) = [I + DV(x,\alpha)]\left[Bx + \sum_{k\geqslant 2}N_k(x,\alpha)\right],$$

which can be rearranged as

$$\sum_{k\geqslant 2}N_k(x,\alpha) = \sum_{k\geqslant 2}\widetilde{H}_k(x + V(x,\alpha),\alpha) + \sum_{k\geqslant 2}BV_k(x,\alpha) - \sum_{k\geqslant 2}DV_k(x,\alpha)Bx$$
$$- \left[\sum_{k\geqslant 2}DV_k(x,\alpha)\right]\left[\sum_{k\geqslant 2}N_k(x,\alpha)\right].$$

Then we use Taylor expansion of $\widetilde{H}(x + V(x,\alpha),\alpha)$ near $V = 0$ to rewrite the above equation as

$$\sum_{i\geqslant 2}N_i(x,\alpha) = \sum_{i\geqslant 2}\widetilde{H}_i(x,\alpha) + \sum_{j\geqslant 2}\sum_{i\geqslant 2}D\widetilde{H}_i(x,\alpha)V_j(x,\alpha) + \sum_{j\geqslant 2}\left(M_j^1V_j\right)(x,\alpha)$$
$$- \left[\sum_{j\geqslant 2}DV_j(x,\alpha)\right]\left[\sum_{j\geqslant 2}N_j(x,\alpha)\right] + \sum_{j\geqslant 2}\frac{1}{j!}\sum_{i\geqslant 2}D^j\widetilde{H}_i(x,\alpha)\left[\sum_{l\geqslant 2}V_l(x,\alpha)\right]^i.$$

As for (2.34), we compare the same order terms, which gives the formula (2.33) in theorem. □

The formula (2.33) given in the above theorem can be rewritten in a compact form

$$N_k = H_k - M_k^1V_k, \tag{2.35}$$

where

$$H_k(x,\alpha) = \widetilde{H}_k(x,\alpha) + \sum_{i=2}^{k-1}\left\{D\widetilde{H}_{k+1-i}(x,\alpha)V_i(x,\alpha) - DV_i(x,\alpha)N_{k+1-i}(x,\alpha)\right\}$$
$$+ \sum_{j=2}^{\left[\frac{k}{2}\right]}\frac{1}{j!}\sum_{i=j}^{k-j}D^j\widetilde{H}_i(x,\alpha)\left\{\sum_{l_1+l_2+\cdots+l_j=k-i+j}V_{l_1}V_{l_2}\cdots V_{l_j}\right\}(x,\alpha).$$

Now consider the formula (2.33). On one hand, by the aid of (2.32), we can compute an adequate nonlinear transformation V_k such that

$$M_k^1V_k = P^kH_k, \tag{2.36}$$

which leads to

$$N_k = \left(I - p^K\right)H_k. \tag{2.37}$$

In other words, we can find a change of variables that affects the reduced system by taking away nonlinear terms (called *non-resonant terms*) that are in the range subspaces $\mathrm{Im}(M_k^1)$ and conserving only terms (called *resonant terms*) that are in the complementary subspace $\mathrm{Im}(M_k^1)^c$.

Remark 1. This theorem is an important approach because it combines center manifolds and normal form schemes into one step. In fact, the kth order N_k of the normal form depends upon the known $(N_i)_{2\leqslant i\leqslant k-1}$ and the results $(V_i)_{2\leqslant i\leqslant k}$ as well as the terms $(h_i)_{2\leqslant i\leqslant k-2}$ of center manifolds. We will see in Section 3 that the above theorem gives a useful and efficient tool to obtain the normal forms for FDEs in a compact form.

3. Application

The mathematical model which describes the regenerative cutting tool chatter in turning was introduced in [12,17] and given by

$$y''(t) + A(\alpha)y'(t) + B(\alpha)y(t) + C(\alpha)y(t - r_0 - \epsilon\rho(t)) = f(\alpha, y(t), y(t - r_0 - \epsilon\rho(t)), \tag{3.1}$$

where $y(t)$ is roughly the displacement normal to the machined surface at time t, α is a bifurcation parameter, ϵ is a small perturbation parameter and f is a nonlinear function given by

$$\begin{aligned} f(\alpha, y(t), y(t - r_0 - \epsilon\rho(t)) \\ &= c_{20}(\alpha)y^2(t) + c_{11}(\alpha)y(t)y(t - r_0 - \epsilon\rho(t)) + c_{02}(\alpha)y^2(t - r_0 - \epsilon\rho(t)) \\ &\quad + c_{30}(\alpha)y^3(t) + c_{21}(\alpha)y^2(t)y(t - r_0 - \epsilon\rho(t)) \\ &\quad + c_{12}(\alpha)y(t)y^2(t - r_0 - \epsilon\rho(t)) + c_{03}(\alpha)y^3(t - r_0 - \epsilon\rho(t)). \end{aligned}$$

The regenerative effect enters the equation of motion through chip thickness

$$\kappa = y(t) - y(t - \tau), \quad \tau = r_0 + \epsilon\rho(t) = \frac{1}{\Omega(t, \epsilon)}, \ \rho(t) = \mu\cos(vt)$$

and Ω is the spindle speed. Here v is the frequency of the periodic fluctuations and μ is the amplitude. The width-of-cut κ, and time delay r_0, (which corresponds to the inverse of the mean value of the spindle speed Ω) are natural control or bifurcation parameters in the machine cutting process.

We consider the following hypothesis on the linear part of system (3.1):

(H'): In the absence of periodic perturbation, i.e., $\epsilon = 0$, FDE exhibits a Hopf bifurcation at $\alpha = 0$, with a simple pair of pure imaginary eigenvalues on the imaginary axis.

3.1. Augmented autonomous system

The explicit time-dependent delay terms are replaced by state-dependent delay terms. To this end, we define the additional coordinations:

$$x_1(t) = y(t),\ x_2(t) = \dot{x}_1(t),\ x_3(t) = \mu\cos(\nu t),\ x_4(t) = \mu\sin(\nu t).$$

Then the relationship between $x_3(t)$ and $x_4(t)$ can be given as follows:

$$\dot{x}_3(t) = -\nu x_4(t),\ \dot{x}_4(t) = \nu x_3(t),$$

where the parameter μ will be identified later as an amplitude. By augmenting $x_3(t)$ and $x_4(t)$ to the problem, we get the autonomous system

$$\dot{X}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -B(\alpha) & -A(\alpha) & 0 & 0 \\ 0 & 0 & 0 & -\nu \\ 0 & 0 & \nu & 0 \end{pmatrix} X(t) + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -C(\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} X(t - r_0 - \epsilon x_4(t)) + \begin{pmatrix} 0 \\ \bar{f} \\ 0 \\ 0 \end{pmatrix} \tag{3.2}$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$$

and

$$\begin{aligned}\bar{f} = {} & c_{20}(\alpha)x_1^2(t) + c_{11}(\alpha)x_1(t)x_1(t - r_0 - \epsilon x_4(t)) + c_{02}(\alpha)x_1^2(t - r_0 - \epsilon x_4(t)) \\ & + c_{30}(\alpha)x_1^3(t) + c_{21}(\alpha)x_1^2(t)x_1(t - r_0 - \epsilon x_4(t)) + c_{12}(\alpha)x_1(t)x_1^2(t - r_0 - \epsilon x_4(t)) \\ & + c_{03}(\alpha)x_1^3(t - r_0 - \epsilon x_4(t)).\end{aligned}$$

Here, it is assumed that $|\epsilon|$ is small enough. Using Taylor expansion of the term $X(t - r_0 - \epsilon x_3(t))$ and substituting the obtained formula into Eq. (3.2), we obtain the following system with constant delay

$$\dot{X}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -B(\alpha) & -A(\alpha) & 0 & 0 \\ 0 & 0 & 0 & -\nu \\ 0 & 0 & \nu & 0 \end{pmatrix} X(t) + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -C(\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} X(t - r_0) + \begin{pmatrix} 0 \\ f(x_t, \alpha) \\ 0 \\ 0 \end{pmatrix} \tag{3.3}$$

where

$$\begin{aligned} f(x_t,\alpha) &= \sum_{|k|=2} c_{k_1k_2}(\alpha)x_1^{k_1}(t)x_1^{k_2}(t-r_0) + C(\alpha)x_2(t-r_0)x_4(t) \\ &+ \sum_{|k|=3} c_{k_1k_2}(\alpha)x_1^{k_1}(t)x_1^{k_2}(t-r_0) \\ &- \sum_{|k|=2} k_2 c_{k_1k_2}(\alpha)x_1^{k_1}(t)x_1^{k_2-1}(t-r_0)x_2(t-r_0)x_4(t) \\ &+ \frac{C(\alpha)}{2}[B(\alpha)x_1(t-r_0) + A(\alpha)x_2(t-r_0)] + O(|x|^4) \end{aligned}$$

As a FDE in $C := C([-r_0,0],\mathbb{R}^4)$, Eq. (3.3) becomes Eq. (2.1), where

$$L(\alpha)\phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -B(\alpha) & -A(\alpha) & 0 & 0 \\ 0 & 0 & 0 & -v \\ 0 & 0 & v & 0 \end{pmatrix}\phi(0) + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -C(\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\phi(-r_0),$$

and

$$F(\phi,\alpha) = \begin{pmatrix} 0 \\ f(\phi,\alpha) \\ 0 \\ 0 \end{pmatrix}.$$

Define the following constants

$$K_1 := (-C_1e^{-ir_0} - iA_1 - B_1)/N, \quad N = A_0 + 2i - r_0C_0e^{-ir_0},$$

$$K_2(v) := \frac{C_0}{2N}e^{-ir_0}\left(1 - \frac{2C_0(1+v)e^{-i(1+v)r_0}}{T(v)} + \frac{2C_0(1-v)e^{-i(1-v)r_0}}{T(-v)}\right)$$

$$T(v) = B_0 - (1+v)^2 + C_0e^{-i(1+v)r_0} + i(1+v)A_0$$

$$\begin{aligned} K_3 &= \left(c_{12} + \frac{c_{11}(2c_{02}+c_{11})}{B_0+C_0} + \frac{2c_{11}c_{02}+c_{11}c_{20}+2c_{02}c_{20}}{T(1)}\right)\frac{e^{-2ir_0}}{N} \\ &+ \left(c_{12} + \frac{c_{11}(2c_{20}+c_{11})}{B_0+C_0} + \frac{c_{11}c_{20}}{T(1)}\right)\frac{e^{ir_0}}{N} \\ &+ \left(3c_{03} + 2c_{21} + \frac{4c_{02}^2+c_{11}^2+4c_{02}c_{20}+4c_{11}c_{20}+2c_{11}c_{02}}{B_0+C_0}\right. \\ &\left. + \frac{2c_{02}c_{20}+2c_{11}c_{20}+c_{11}c_{02}}{T(1)}\right)\frac{(3c_{30}+2c_{12})e^{-ir_0}}{N^2} \\ &+ \frac{4c_{20}^2+c_{11}^2+4c_{02}c_{20}+2c_{11}c_{20}+4c_{11}c_{02}}{N(B_0+C_0)} \\ &+ \frac{2c_{20}^2+c_{11}^2+2c_{11}c_{02}e^{-4ir_0}+c_{11}^2e^{-3ir_0}+2c_{02}^2e^{-3ir_0}}{NT(1)}. \end{aligned}$$

Using the main result of Section 2.1, we obtain the following theorem:

Theorem 3.1. *Consider the class of equations with periodic delay (3.1). Then the normal form up to third order, in polar coordinates, associated to this class of equations is given by*

$$\dot{r}(t) = [K_1^{\mathcal{R}}\alpha + K_2^{\mathcal{R}}(v)|\mu|^2 + K_3^{\mathcal{R}}r^2(t)]r(t)$$
$$\dot{\phi}(t) = 1 + \alpha K_1^{\mathcal{I}} + K_2^{\mathcal{I}}(v)|\mu|^2 + K_3^{\mathcal{I}}r^2(t),$$

where $K_i^{\mathcal{R}}$ and $K_i^{\mathcal{I}}$ are respectively, the real and imaginary parts of $K_i, i = 1, 2, 3$.

Proof. The associated characteristic equation is given by

$$\begin{cases} \lambda^2 + A(\alpha)\lambda + B(\alpha) + C(\alpha)e^{-\lambda r_0} = 0 \\ \lambda^2 + v^2 = 0. \end{cases}$$

It is obvious that the eigenvalues of the augmented oscillator is $\lambda = \pm iv$. The transcendental equation, at $\alpha = 0$, has a pair of pure imaginary roots, which we normalize to one, that is $\lambda = \pm i$. This normalization of chatter frequency gives the following relation between the mean delay and the linear critical coefficients

$$C_0 \cos(r_0) = 1 - B_0 \quad \text{and} \quad C_0 \sin(r_0) = A_0,$$

where $A_0 := A(0), B_0 := B(0)$ and $C_0 := C(0)$. Thus, at $\alpha = 0$, Eq. (3.3) has two pairs of eigenvalues on the imaginary axis and all other eigenvalues have negative real parts. Let

$$\Lambda = \{+i, \pm iv\}$$

and X_c the generalized eigenspace associated with the eigenvalues of Λ with the basis $\Phi = [\phi_1, \phi_2, \phi_3, \phi_4]$ given by

$$\Phi(\theta) = \begin{bmatrix} e^{-i\theta} & e^{i\theta} & 0 & 0 \\ -ie^{-i\theta} & ie^{i\theta} & 0 & 0 \\ 0 & 0 & e^{-iv\theta} & e^{iv\theta} \\ 0 & 0 & -ie^{-iv\theta} & ie^{iv\theta} \end{bmatrix}, \quad \text{for } \theta \in [-r_0, 0].$$

Let Ψ be the basis for the generalized eigenspace (dual space X_c^*) of the transposed equation associated with Λ such that $\langle \Psi, \Phi \rangle = I$. We have

$$\Psi(s) = \begin{bmatrix} \psi_1(s) \\ \psi_2(s) \\ \psi_3(s) \\ \psi_4(s) \end{bmatrix} = \begin{bmatrix} \frac{(A_0-i)}{N}e^{is} & \frac{e^{is}}{N} & 0 & 0 \\ \frac{(A_0+i)}{\overline{N}}e^{-is} & \frac{e^{-is}}{\overline{N}} & 0 & 0 \\ 0 & 0 & \frac{e^{ivs}}{2} & i\frac{e^{ivs}}{2} \\ 0 & 0 & \frac{e^{-ivs}}{2} & -i\frac{e^{-ivs}}{2} \end{bmatrix}, \text{for } s \in [0, r_0].$$

The long term behavior of solutions of the original FDE is described by the solutions of the four-dimensional reduced system on a local center manifold

$$y'(t) = By + R^1(y,\alpha) + R^2(y,\alpha) + h.o.t., \tag{3.4}$$

where $R^1(y,\alpha)$ and $R^2(y,\alpha)$ are respectively the homogeneous parts of degree two and three in (y,α) given in Section 2. The coefficients $\left(R^1_{(q_1,q_2,q_3,q_4,l)}, (q_1,q_2,q_3,q_4,l) \in D_2\right)$ of $R^1(y,\alpha)$, are given by

$$\begin{aligned}
R^1_{(2,0,0,0,0)} &= c_{20} + c_{11}e^{ir_0} + c_{02}e^{2ir_0}, & R^1_{(0,2,0,0,0)} &= \overline{R^1_{(2,0,0,0,0)}},\\
R^1_{(1,1,0,0,0)} &= 2c_{20} + 2c_{11}\cos(r_0) + 2c_{02}, & R^1_{(0,2,0,0,0)} &= \overline{R^1_{(2,0,0,0,0)}},\\
R^1_{(1,0,1,0,0)} &= -C_0e^{ir_0}, & R_{(0,1,0,1,0)} &= \overline{R^1_{(1,0,1,0,0)}},\\
R^1_{(1,0,0,1,0)} &= C_0e^{ir_0}, & R^1_{(0,1,0,0,1)} &= \overline{R^1_{(1,0,0,0,1)}},\\
R^1_{(1,0,0,0,1)} &= -C_1e^{-ir_0} - iA_1 - B_1, & R^1_{(q,l)} &= 0, \text{otherwise.}
\end{aligned}$$

Then, by applying directly the schemes for computing the terms of center manifolds obtained in Section 2, we obtain all the terms of center manifolds up to second order as:

$$h^2_{(q,l)}(\theta) = \begin{bmatrix} \left(\frac{e^{W_q\theta}}{\Delta_{(q,l)}} - \left(\frac{e^{-i\theta}}{\overline{N}(i+W_q)} - \frac{e^{i\theta}}{N(i-W_q)}\right)\right) R^1_{(q,l)} \\ \left(\frac{W_q e^{W_q\theta}}{\Delta_{(q,l)}} + i\left(\frac{e^{-i\theta}}{\overline{N}(i+W_q)} + \frac{e^{i\theta}}{N(i-W_q)}\right)\right) R^1_{(q,l)} \\ 0 \\ 0 \end{bmatrix}$$

where $\Delta_q := (1 + (W_q)^2) + C_0\left[e^{-W_q r_0} - \frac{1}{2i}(i - W_q)e^{ir_0} - \frac{1}{2i}(i + W_q)e^{-ir_0}\right]$.

Now, let us compute the normal forms of the reduced system. One can see that

$$R^2(y,\alpha) = \Psi(0)DF_2(\phi y, 0)h_2(y,0) + \Psi(0)F_3(\Phi y, 0) + O(\alpha|y|^2, \alpha^2|y|).$$

However, the terms $O(\alpha|y|^2, \alpha^2|y|)$ are irrelevant to determine the generic double Hopf bifurcation. Hence, for simplicity, we will omit these terms in the calculations of normal forms of order 3.

Consider a nonlinear transformation of the form

$$y = z + V_2(z)$$

where $V_2(z)$ is a homogeneous polynomial of degree two. The effects of the above change of variables on the reduced system is

$$z' = Bz + N(z,\alpha),$$

with $N(z,\alpha) = \sum_{k\geqslant 2} N_k(z,\alpha)$.

By using the formulas given by Theorem 2.6, we obtain

$$N_j = \widetilde{H}_j - M^1_j V_j, \qquad j = 2,3, \tag{3.5}$$

where

$$\widetilde{H}_2(y,\alpha) = \Psi(0)(L(\alpha) - L(0))\Phi y + \Psi(0)F_2(\Phi y, 0) \tag{3.6}$$

and

$$\widetilde{H}(y,\alpha) = \Psi(0)F_3(\phi y,0) + \Psi(0)DF_2(\Phi y,0)[h_2(z,\alpha) + \Phi V_2(z,\alpha)].$$

Note that

$$\ker(M_3^1) = span\left\{\begin{pmatrix} z_1^2 z_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1 z_3 z_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2 z_3 z_4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_3^2 z_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1 z_2 z_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_3 z_4^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1 z_2 z_4 \end{pmatrix}\right\}. \tag{3.7}$$

After simple calculations, one can obtain

$$\tilde{H}_2(z,\alpha) = \begin{bmatrix} \frac{1}{N} \sum_{(q,l)\in D_2} R^1_{(q,l)} z^q \alpha^l \\ \frac{1}{\overline{N}} \sum_{(q,l)\in D_2} R^1_{(q,l)} z^q \alpha^l \\ 0 \\ 0 \end{bmatrix}.$$

So, by virtue of (3.5) and (3.6) it follows that

$$V_2(z,\alpha) = \begin{bmatrix} \frac{1}{N} \sum_{(q,l)\in D_2} \frac{R^2_{(q,l)}}{(W_q - i)} z^q \alpha^l \\ \frac{1}{\overline{N}} \sum_{(q,l)\in D_2} \frac{R^1_{(q,l)}}{(W_q - i)} z^q \alpha^l \\ 0 \\ 0 \end{bmatrix}$$

and

$$N_2(z,\alpha) = \begin{bmatrix} R^1_{(1,0,0,0,1)} z_1 \alpha \\ R^1_{(0,1,0,0,1)} z_2 \alpha \\ 0 \\ 0 \end{bmatrix}.$$

On the other hand, we have

$$\psi(0)F_3(\phi z,0) = \begin{bmatrix} \frac{1}{N} \sum_{(q,l)\in D_3} \beta_{(q,l)} z^q \alpha^l \\ \frac{1}{\overline{N}} \sum_{(q,l)\in D_3} \beta_{(q,l)} z^q \alpha^l \\ 0 \\ 0 \end{bmatrix},$$

where

$$\beta_{(3,0,0,0,0)} = c_{30} + c_{21}e^{-ir_0} + c_{12}e^{-2ir_0} + c_{03}e^{-3ir_0},$$
$$\beta_{(2,1,0,0,0)} = 3c_{30} + c_{21}(2e^{-ir_0} + e^{ir_0}) + c_{12}(2 + e^{-2ir_0}) + 3c_{03}e^{-3ir_0},$$
$$\beta_{(2,0,1,0,0)} = -c_{11}e^{-ir_0} - 2c_{02}e^{-2ir_0}, \quad \beta_{(2,0,0,1,0)} = c_{11}e^{-ir_0} + 2c_{02}e^{-2ir_0},$$
$$\beta_{(1,2,0,0,0)} = \overline{\beta_{(2,1,0,0,0)}}, \quad \beta_{(1,1,1,0,0)} = \overline{\beta_{(1,1,0,1,0)}}, \quad \beta_{(0,3,0,0,0)} = \overline{\beta_{(3,0,0,0,0)}},$$
$$\beta_{(0,2,1,0,0)} = \overline{\beta_{(2,0,0,1,0)}}, \quad \beta_{(0,2,0,1,0)} = \overline{\beta_{(2,0,1,0,0)}}, \quad \beta_{(1,1,0,1,0)} = -c_{11}(e^{-ir_0} - e^{ir_0}),$$
$$\beta_{(q,l)} = 0, \text{otherwise},$$

and

$$\Psi(0)DF_2(\Phi y, 0)[h_2(z,0) + \Phi V_2(z,0)] = \begin{bmatrix} \frac{1}{N}\sum_{(q,l)\in D_3} \gamma_{(q,l)} z^q \alpha^l \\ \frac{1}{\overline{N}}\sum_{(q,l)\in D_3} \gamma_{(q,l)} z^q \alpha^l \\ 0 \\ 0 \end{bmatrix}$$

where $\gamma_{(q,l)}$ are complex constants. According to (3.7), the specific coefficients $\gamma_{(q,l)}$ that are needed in the third order normal forms are $\gamma_{(2,1,0,0,0)}, \gamma_{(1,2,0,0,0)}, \gamma_{(1,0,1,1,0)}$ and $\gamma_{(0,1,1,1,0)}$. We have

$$\begin{aligned}\gamma_{(2,1,0,0,0)} = {} & \frac{iR^1_{(2,1,0,0,0)}}{N}\left(\frac{R^1_{(3,0,0,0,0)}}{N} - \frac{R^1_{(2,1,0,0,0)}}{N}\right) - \frac{i2R^1_{(1,2,0,0,0)}}{|N|^2} + \left[h^2_{(2,0,0,0,0)}(0)\right]_1 \\ & \times (2c_{20} + c_{11}e^{ir_0}) + \left[h^2_{(2,0,0,0,0)}(-r_0)\right]_1 (c_{11} + 2c_{02}e^{ir_0}) + \left[h^2_{(1,1,0,0,0)}(0)\right]_1 \\ & \times (2c_{20} + c_{11}e^{-ir_0}) + \left[h^2_{(1,1,0,0,0)}(-r_0)\right]_1 (c_{11} + 2c_{02}e^{-ir_0}),\end{aligned}$$

$$\begin{aligned}\gamma_{(1,0,1,1,0)} = {} & \frac{iR^1_{(1,1,1,0,0)}R^1_{(2,0,0,1,0)}}{|N|^2(2-v)} - \frac{iR^1_{(1,1,0,1,0)}R^1_{(2,0,1,0,0)}}{|N|^2(2-v)} + \left[h^2_{(0,0,1,1,0)}(0)\right]_1 (2c_{20} + c_{11}e^{-ir_0}) \\ & + \left[h^2_{(0,0,1,1,0)}(0)\right]_1 (2c_{11} + c_{02}e^{-ir_0}) + iC_0\left(\left[h^2_{(1,0,0,1,0)}(-r_0)\right]_2 - \left[h^2_{(1,0,1,0,0)}(-r_0)\right]_2\right).\end{aligned}$$

and $\gamma_{(1,2,0,0,0)} = \overline{\gamma_{(2,1,0,0,0)}}, \gamma_{(0,1,1,1,0)} = \overline{\gamma_{(1,0,1,1,0)}}$.

This leads to

$$N_3(z,\alpha) = \begin{bmatrix} (\beta_{(2,1,0,0,0)} + \gamma_{(2,1,0,0,0)})z_1^2 z_2 + \beta_{(1,0,1,1,0)} + \gamma_{(1,0,1,1,0)})z_1 z_3 z_4 \\ (\overline{\beta_{(2,1,0,0,0)}} + \overline{\gamma_{(2,1,0,0,0)}})z_1 z_2^2 + \overline{\beta_{(1,0,1,1,0)}} + \overline{\gamma_{(1,0,1,1,0)}})z_2 z_3 z_4 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, the truncated normal forms of the reduced Eq. (3.4) up to third order reads

$$\begin{cases} z_1'(t) = (-i + R^1_{(1,0,0,0,1)}\alpha)z_1(t) + (\beta_{(2,1,0,0,0)} + \gamma_{(2,1,0,0,0)})z_1^2 z_2 \\ \qquad\quad +(\beta_{(1,0,1,1,0)} + \gamma_{(1,0,1,1,0)})z_1 z_3 z_4 \\ z_2'(t) = (i + R^1_{(0,1,0,0,1)}\alpha)z_2(t) + (\beta_{(1,2,0,0,0)} + \gamma_{(1,2,0,0,0)})z_2^2 z_1 \\ \qquad\quad +(\beta_{(0,1,1,1,0)} + \gamma_{(0,1,1,1,0)})z_2 z_3 z_4 \\ z_3'(t) = -ivz_3(t) \\ z_4'(t) = -ivz_4(t) \end{cases}$$

which, by transforming the above equation in polar coordinates, completes the proof of theorem. □

4. Conclusion

Methodology for simultaneous computation of center manifolds and normal forms for double Hopf bifurcation, in delay differential equations, has been developed. The calculations and formulas are given in an explicit iterative procedure, and thus, are easy to be implemented on a symbolic computation system. It has been shown by an example that the method is computationally efficient, particularly, suitable for the computations of complicated systems and higher-order center manifolds and normal forms.

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